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THE MODULE OF VECTOR-VALUED MODULAR FORMS IS COHEN-MACAULAY

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Abstract. Let H denote a finite index subgroup of the modular group Γ and let ϱ denote a finite-dimensional complex representation of H. Let $M(\varrho)$ denote the collection of holomorphic vector-valued modular forms for ϱ and let $M(H)$ denote the collection of modular forms on H. Then $M(\varrho)$ is a $\mathbb{Z}$-graded $M(H)$-module. It has been proven that $M(\varrho)$ may not be projective as a $M(H)$-module. We prove that $M(\varrho)$ is Cohen-Macaulay as a $M(H)$-module. We also explain how to apply this result to prove that if $M(H)$ is a polynomial ring, then $M(\varrho)$ is a free $M(H)$-module of rank $\dim \varrho$.

Keywords: vector-valued modular form; Cohen-Macaulay module

MSC 2020: 11F03, 13C14

1. Introduction

Let H denote a finite index subgroup of the modular group $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ and let ϱ denote a finite-dimensional complex representation of H. Let $k \in \mathbb{Z}$ and let $\mathfrak{H}$ denote the complex upper half plane. If $F\colon \mathfrak{H} \to \mathbb{C}^t$ is a holomorphic function and if $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$, then we define $F|_k\gamma$ by setting

$$F|_k\gamma(\tau) := (c\tau + d)^{-k} F((a\tau + b)/(c\tau + d)).$$

Definition. *A vector-valued modular form F of weight k with respect to ϱ is a holomorphic function $F\colon \mathfrak{H} \to \mathbb{C}^{\dim \varrho}$ which is also holomorphic at all of the cusps of $H \setminus (\mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q}))$ and such that for all $\gamma \in H$,*

$$F|_k\gamma = \varrho(\gamma)F. \tag{1.1}$$

The statement that F is holomorphic at all of the cusps of $H \setminus (\mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q}))$ is equivalent to the statement that for each $\gamma \in \Gamma$, each of the component functions

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of $F|_k\gamma$ has a holomorphic q-expansion. The notion of a holomorphic q-expansion is a bit more intricate in the vector-valued setting. A precise treatment of the notion of a holomorphic q-expansion can be found for Γ in [6] and for an arbitrary subgroup in [8].

The collection of all vector-valued modular forms of weight k for the representation ϱ forms a finite-dimensional complex vector-space, which we denote by $M_k(\varrho)$. We let $M_t(H)$ denote the collection of all modular forms of weight t on H. We define $M(\varrho) := \bigoplus_{k\in\mathbb{Z}} M_k(\varrho)$ and $M(H) := \bigoplus_{k\in\mathbb{Z}} M_k(H)$. If $F \in M_k(\varrho)$ and if $m \in M_t(H)$, then $mF \in M_{k+t}(\varrho)$. In this way, we view $M(\varrho)$ as a $\mathbb{Z}$-graded $M(H)$-module. If ϱ is a representation of Γ, then the module structure of $M(\varrho)$ is especially pleasing.

Theorem 1.1. *Let ϱ denote a representation of Γ. Then $M(\varrho)$ is a free $M(\Gamma)$-module of rank equal to the dimension of ϱ.*

Theorem 1.1 has been used to study the arithmetic of vector-valued modular forms for representations of Γ in [5], [9], [11]. There are multiple proofs of Theorem 1.1, and each one offers its own perspective and insights. Theorem 1.1 was proven by Marks and Mason in [10], by Gannon in [7], and by Candelori and Franc in [3].

Mason has shown that if H is equal to Γ^2, the unique subgroup of Γ of index two, then $M(\varrho)$ need not be a free module over $M(\Gamma^2)$. A proof of this fact, together with the result that $M(\varrho)$ need not even be projective over $M(\Gamma^2)$, appears in [4], Section 6. In view of this negative result, it is natural to ask if one may prove a positive result about the structure of $M(\varrho)$ as a $M(H)$-module. We prove the following:

Theorem 1.2. *$M(\varrho)$ is Cohen-Macaulay as a $M(H)$-module.*

We shall also explain how to apply Theorem 1.2 to prove the following theorem.

Theorem 1.3. *Let H denote a finite index subgroup of Γ such that there exist modular forms $X, Y \in M(H)$ for which X and Y are algebraically independent and $M(H) = \mathbb{C}[X, Y]$. Then $M(\varrho)$ is a free $M(H)$-module of rank* $\dim \varrho$.

We remark that Theorem 1.3 may also be obtained by applying the work of Candelori and Franc in [4]. A complete list of the finitely many subgroups H which satisfy the hypothesis of Theorem 1.3 is given in [1]. Two such subgroups are Γ and $\Gamma_0(2)$. The author employs Theorem 1.3 to study the arithmetic of vector-valued modular forms on $\Gamma_0(2)$ in [8].

In [4], Candelori and Franc study the commutative algebra properties of vector-valued modular forms in a geometric context. If H is a genus zero Fuchsian group of the first kind, with finite covolume and with finitely many cusps, then they define a collection of *geometrically weighted vector-valued modular forms*, $\mathrm{GM}(\varrho)$, which contains $M(\varrho)$, and a collection of geometrically weighted modular forms $S(H)$, which contains $M(H)$. They prove that $\mathrm{GM}(\varrho)$ is Cohen-Macaulay as a $S(H)$-module. The ideas in [4] involve the classification of vector bundles over orbifold curves of genus zero and are quite interesting. We emphasize that in our paper, Theorem 1.2 applies to the $M(H)$-module $M(\varrho)$ and holds for *all* finite index subgroups of Γ.

We refer the reader to Benson, see [2], for the relevant definitions and results from commutative algebra which we shall use in this paper.

2. Proofs

The following lemma will be used in the proof of Theorem 1.2 and Theorem 1.3. This lemma originates in a paper of Selberg, see [12].

Lemma 2.1. *Let $\mathrm{Ind}_H^\Gamma \varrho$ denote the induction of the representation ϱ from H to Γ. Then $M(\varrho)$ and $M(\mathrm{Ind}_H^\Gamma \varrho)$ are isomorphic as $\mathbb{Z}$-graded $M(\Gamma)$-modules.*

Proof. Let $\{g_i\colon 1 \leqslant i \leqslant [\Gamma : H]\}$ denote a complete set of left coset representatives of H in Γ, where g_1 denotes the identity matrix. Let $k \in \mathbb{Z}$, and let $F \in M_k(\varrho)$. We define $\Phi(F) := [F|_k g_1^{-1}, F|_k g_2^{-1}, \dots, F|_k g_{[\Gamma:H]}^{-1}]^\top$, where the superscript $\top$ denotes the transpose. We claim that $\Phi(F) \in M_k(\mathrm{Ind}_H^\Gamma \varrho)$. We first note that since $F \in M_k(\varrho)$, we must have that for all $g \in H$, the function $F|_k g$ is holomorphic in $\mathfrak{H}$ and that it has a holomorphic q-expansion. Thus $\Phi(F)$ is holomorphic in $\mathfrak{H}$ and $\Phi(F)$ is holomorphic at the cusp $\Gamma \setminus (\mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q}))$, since each of its component functions $F|_k g_i^{-1}$ has a holomorphic q-expansion. To prove that $\Phi(F) \in M_k(\mathrm{Ind}_H^\Gamma \varrho)$, it now suffices to show that for all $g \in \Gamma$,

$$\Phi(F)|_k g = \mathrm{Ind}_H^\Gamma \varrho(g)\Phi(F). \tag{2.1}$$

Let $n = [\Gamma : H]$. Let $\varrho^\bullet$ denote the function on Γ which is defined by the conditions that $\varrho^\bullet|_H = \varrho$ and $\varrho^\bullet(g) = 0$ if $g \notin H$. With respect to our choice of left coset representatives for H in Γ, the ith row and jth column block of the matrix $\mathrm{Ind}_H^\Gamma \varrho(g)$ is equal to $\varrho^\bullet(g_i^{-1} g g_j)$. Thus equation (2.1) is equivalent to the assertion that for each integer i with $1 \leqslant i \leqslant n$,

$$(F|_k g_i^{-1})|_k g = \sum_{t=1}^{n} \varrho^\bullet(g_i^{-1} g g_t) F|_k g_t^{-1}. \tag{2.2}$$

Fix an index i with $1 \leqslant i \leqslant n$ and fix $g \in \Gamma$. Then there exists a unique index j for which $g_i^{-1} g g_j \in H$. We then have that $F|_k g_i^{-1} g g_j = \varrho(g_i^{-1} g g_j) F$. Therefore

$$(2.3) \qquad \begin{aligned}(F|_k g_i^{-1})|_k g = F|_k g_i^{-1} g &= (F|_k g_i^{-1} g g_j)|_k g_j^{-1} = (\varrho(g_i^{-1} g g_j) F)|_k g_j^{-1} \\ &= \varrho(g_i^{-1} g g_j) F|_k g_j^{-1} = \sum_{t=1}^{n} \varrho^\bullet(g_i^{-1} g g_t) F|_k g_t^{-1}.\end{aligned}$$

We have thus proven that (2.2) holds and conclude that $\Phi(F) \in M_k(\mathrm{Ind}_H^\Gamma(\varrho))$. For each integer k, we have defined the map $\Phi\colon\ M_k(\varrho) \to M_k(\mathrm{Ind}_H^\Gamma(\varrho))$, and we extend it by linearity to a map $\Phi\colon\ M(\varrho) \to M(\mathrm{Ind}_H^\Gamma \varrho)$.

We now check that Φ is a map of $\mathbb{Z}$-graded $M(\Gamma)$-modules. Let $m \in M_t(\Gamma)$. We recall that $F \in M_k(\varrho)$. Then

$$(2.4) \qquad \begin{aligned} m\Phi(F) &= m[F|_k g_1^{-1}, F|_k g_2^{-1}, \ldots, F|_k g_n^{-1}]^\top \\ &= [mF|_{k+t} g_1^{-1}, mF|_{k+t} g_2^{-1}, \ldots, mF|_{k+t} g_n^{-1}]^\top = \Phi(mF).\end{aligned}$$

Finally, we check that Φ is a bijection. Let $X \in M_k(\mathrm{Ind}_H^\Gamma \varrho)$. The codomain of X is $\mathbb{C}^{\dim(\mathrm{Ind}_H^\Gamma \varrho)}$ and $\dim(\mathrm{Ind}_H^\Gamma \varrho) = n \dim \varrho$. Let $X_1, \ldots, X_n\colon\ \mathfrak{H} \to \mathbb{C}^{\dim \varrho}$ denote the holomorphic functions for which $X = [X_1, \ldots, X_n]^\top$. We define the map π by setting $\pi(X) = X_1$. We will show that $X_1 \in M_k(\varrho)$. Let $g \in H$. There exists a unique index j for which $g_1^{-1} g g_j \in H$. As g_1 is the identity matrix and $g \in H$, we must have that $g_j = g_1$. The fact that $X \in M_k(\mathrm{Ind}_H^\Gamma \varrho)$ implies that

$$(2.5) \qquad X_1|_k g = \sum_{t=1}^{n} \varrho^\bullet(g_1^{-1} g g_t) X_t = \varrho^\bullet(g_1^{-1} g g_1) X_1 = \varrho(g) X_1.$$

As $X = [X_1, \ldots, X_n]^\top$ is a holomorphic vector-valued modular form, X_1 is holomorphic in $\mathfrak{H}$ and X_1 has a holomorphic q-expansion. Thus for each $g \in \Gamma$, $X_1|_k g$ has a holomorphic q-expansion, since $X_1|_k g = \varrho(g) X_1$ and X_1 has a holomorphic q-series expansion. We have proven that X_1 is holomorphic at all of the cusps of $H \setminus (\mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q}))$ and conclude that $\pi(X) = X_1 \in M_k(\varrho)$. We also see that $\pi \circ \Phi(F) = \Phi(F)_1 = F|_k g_1^{-1} = F$ since g_1 is the identity matrix. As $\pi \circ \Phi$ is equal to the identity map, π must be surjective. All that remains is to prove that π is injective.

Let $X \in M_k(\mathrm{Ind}_H^\Gamma \varrho)$ such that $\pi(X) = 0$. We write $X = [X_1, X_2, \ldots, X_n]^\top$, where each X_i is a holomorphic function from $\mathfrak{H}$ to $\mathbb{C}^{\dim \varrho}$. We claim that $X = 0$. Suppose not. Then there exists some index i with $X_i \neq 0$. Let $g \in g_i H g_1^{-1}$. Then $g_i^{-1} g g_j \in H$ if and only if $g_1 = g_j$. Thus $X_i|_k g = \sum_{t=1}^{n} \varrho^\bullet(g_i^{-1} g g_t) X_t = \varrho(g_i^{-1} g g_1) X_1$. As $\pi(X) = X_1 = 0$, we have that $X_i|_k g = 0$. Hence $X_i = (X_i|_k g)|_k g^{-1} = 0$, a contradiction. Thus $X = 0$ and π is therefore injective. We have proven that π and hence Φ is a bijection and the lemma now follows. $\square$

Let $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$, $\sigma_3(n) = \sum\limits_{d|n} d^3$, and let $S_k(\Gamma)$ denote the space of weight k cusp forms on Γ. We recall the Eisenstein series $E_4 = 1 + 240 \sum\limits_{n=1}^{\infty} \sigma_3(n) q^n \in M_4(\Gamma)$ and the cusp form $\Delta = q(1-q^n)^{24} \in S_{12}(\Gamma)$. The following lemma is an extension of an argument of Marks and Mason in [10].

Lemma 2.2. *The sequence Δ, E_4 is a regular sequence for the $M(H)$-module $M(\varrho)$.*

Proof. It is clear that Δ is a nonzero-divisor for $M(\varrho)$ since Δ has no zeros in $\mathfrak{H}$. To prove that Δ is regular for $M(\varrho)$, it suffices to show that $M(\varrho) \neq \Delta M(\varrho)$. Suppose that $M(\varrho) = \Delta M(\varrho)$. Let X denote a nonzero element in $M(\varrho)$ of minimal weight w. Then $X = \Delta V$ for some $V \in M_{w-12}(\varrho)$. The weight of V is less than weight of X. This is a contradiction, and therefore $M(\varrho) \neq \Delta M(\varrho)$. We conclude that Δ is regular for $M(\varrho)$.

We will show that E_4 is regular for $M(\varrho)/\Delta M(\varrho)$. We have previously shown that $M(\varrho)/\Delta M(\varrho) \neq 0$. We now argue that E_4 is a nonzero-divisor for the module $M(\varrho)/\Delta M(\varrho)$. Suppose that $Y \in M(\varrho)$ and $E_4(Y + \Delta M(\varrho)) = \Delta M(\varrho)$. Then $E_4 Y \in \Delta M(\varrho)$. We write $E_4 Y = \Delta Z$ for some $Z \in M(\varrho)$. We wish to show that $Y \in \Delta M(\varrho)$, and it suffices to prove this when Y is a vector-valued modular form. Let k denote the weight of Y. Let y_i denote the ith component function of Y, let z_i denote the ith component function of Z, and let $\gamma \in \Gamma$. Therefore $E_4 y_i = \Delta z_i$ and $E_4(y_i|_k\gamma) = \Delta(z_i|_{k-8}\gamma)$. As $\Delta = q + O(q^2)$ and $E_4 = 1 + O(q)$, all the powers of q in $y_i|_k\gamma$ occur to at least the first power. We have thus shown that $\Delta^{-1}(y_i|_k\gamma)$ contains no negative powers of q and is therefore holomorphic at the cusp $\gamma \cdot \infty$. Hence Y/Δ is holomorphic at all of the cusps of $H \setminus (\mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q}))$. As Δ does not vanish in $\mathfrak{H}$, we have that Y/Δ is holomorphic in $\mathfrak{H}$. Hence $Y/\Delta \in M(\varrho)$ and thus $Y + \Delta M(\varrho) = \Delta M(\varrho)$. We have proven that E_4 is a nonzero-divisor for the module $M(\varrho)/\Delta M(\varrho)$.

Finally, we must show that $E_4(M(\varrho)/\Delta M(\varrho)) \neq M(\varrho)/\Delta M(\varrho)$. We recall that X denotes a nonzero element in $M(\varrho)$ of minimal weight w. If $M(\varrho)/\Delta M(\varrho) = E_4(M(\varrho)/\Delta M(\varrho))$, then there exists some $F \in M(\varrho)$ such that $X + \Delta M(\varrho) = E_4 F + \Delta M(\varrho)$. Let $G \in M(\varrho)$ such that $X = E_4 F + \Delta G$. We may write F and G uniquely as a sum of their homogeneous components. Let F_{w-4} and G_{w-12} denote the weight $w-4$ and the weight $w-12$ homogeneous components of F and G. Then $X = E_4 F_{w-4} + \Delta G_{w-12}$. We must have that $F_{w-4} \neq 0$ or $G_{w-12} \neq 0$ since $X \neq 0$. Thus F_{w-4} or G_{w-12} is a nonzero element of $M(\varrho)$ whose weight is less than the weight of X. This is a contradiction and we conclude that $M(\varrho)/\Delta M(\varrho) \neq E_4(M(\varrho)/\Delta M(\varrho))$. We have shown that E_4 is regular for $M(\varrho)/\Delta M(\varrho)$ and our proof is complete. $\square$

Lemma 2.3. *The Krull dimension of the $M(H)$-module $M(\varrho)$ is equal to two.*

Proof. We recall that the *Krull dimension* of the $M(H)$-module $M(\varrho)$ is defined to be the Krull dimension of the ring $M(H)/\mathrm{Ann}_{M(H)}(M(\varrho))$. As the zeros of a nonzero holomorphic function are isolated, $\mathrm{Ann}_{M(H)}M(\varrho)=0$. Therefore the Krull dimension of $M(\varrho)$ is equal to the Krull dimension of $M(H)$. It suffices to prove that the Krull dimension of $M(H)$ is equal to the Krull dimension of $M(\Gamma)=\mathbb{C}[E_4,E_6]$, which is equal to two. To do so, we use the fact (see [2], Corollary 1.4.5) that if $A\subset B$ are commutative rings and if B is an integral extension of A for which B is finitely generated as an A-algebra then the Krull dimensions of A and B are equal. It therefore suffices to show that $M(H)$ is an integral extension of $M(\Gamma)$ and that $M(H)$ is finitely generated as a $M(\Gamma)$-algebra.

Let $\{\gamma_i\colon 1\leqslant i\leqslant[\Gamma:H]\}$ denote a complete set of right coset representatives of H in Γ, where γ_1 denotes the identity matrix. If $f\in M_k(H)$, then $f|_k\gamma_1=f$ and thus f is a root of the monic polynomial $P(z):=\prod_{i=1}^{[\Gamma:H]}(z-f|_k\gamma_i)\in M(H)[z]$. If $\gamma\in\Gamma$, then let $P(z)|_k\gamma$ denote the polynomial obtained by replacing each monomial cz^t of P with the monomial $(c|_k\gamma)z^t$. The fact that $\{\gamma_i\gamma\colon 1\leqslant i\leqslant[\Gamma:H]\}$ is a complete set of right coset representatives of H in Γ, together with the fact that $f\in M_k(H)$, implies that

$$P(z)|_k\gamma=\prod_{i=1}^{[\Gamma:H]}(z-f|_k\gamma_i\gamma)=\prod_{i=1}^{[\Gamma:H]}(z-f|_k\gamma_i)=P(z). \tag{2.6}$$

Thus $P(z)\in M(\Gamma)[z]$. Hence $M(H)$ is an integral extension of $M(\Gamma)$.

Let $\alpha\colon H\to\mathbb{C}^\times$ denote the trivial representation of H. Theorem 1.1 implies that $M(\mathrm{Ind}_H^\Gamma\alpha)$ is a free $M(\Gamma)$-module whose rank equals $\dim(\mathrm{Ind}_H^\Gamma\alpha)=[\Gamma:H]$. Lemma 2.1 tells us that $M(\alpha)$ and $M(\mathrm{Ind}_H^\Gamma\alpha)$ are isomorphic as $M(\Gamma)$-modules. Hence $M(\alpha)=M(H)$ is a free $M(\Gamma)$-module of rank $[\Gamma:H]$. In particular, $M(H)$ is finitely generated as a $M(\Gamma)$-algebra. We conclude that the Krull dimensions of $M(H)$ and $M(\Gamma)$ are equal and the lemma now follows. $\square$

We now proceed with the proof of Theorem 1.2.

Proof. We have shown that the Krull dimension of the $M(H)$-module $M(\varrho)$ is equal to two and that $M(\varrho)$ has a regular sequence of length two. Therefore the depth of $M(\varrho)$ is at least two. Moreover, the depth is at most the Krull dimension (see [2], page 50), which is equal to two. Hence the depth and the Krull dimension of $M(\varrho)$ are both equal to two. $\square$

We shall use the following result from commutative algebra to prove Theorem 1.3. This result is stated and proven in Benson's book, see [2].

Theorem 2.1 ([2], Theorem 4.3.5.). *Let A denote a commutative Noetherian ring and let M denote a finitely generated A-module. Assume that $A = \bigoplus_{j=0}^{\infty} A_j$ and $M = \bigoplus_{j=-\infty}^{\infty} M_j$ are graded, $A_0 = K$ is a field, and A is finitely generated over K by elements of positive degree. Then the following statements are equivalent:*

(i) *M is Cohen-Macaulay.*

(ii) *If $x_1, \dots x_n \in A$ are homogenous elements generating a polynomial subring $K[x_1, \dots, x_n] \subset A/\mathrm{Ann}_A(M)$, over which M is finitely generated, then M is a free $K[x_1, \dots, x_n]$-module.*

We now give the proof of Theorem 1.3.

Proof. We first note that the hypotheses of Theorem 2.1 are satisfied if we take $A = M(H)$ and $M = M(\varrho)$. Thus statements (i) and (ii) in Theorem 2.1 are equivalent if $A = M(H)$ and $M = M(\varrho)$. We have proven that $M(\varrho)$ is Cohen-Macaulay as a $M(H)$-module. Thus statement (i) and hence statement (ii) in Theorem 2.1 must be true. In particular, if $X, Y \in M(H)$, which are algebraically independent, then $M(\varrho)$ is a free $\mathbb{C}[X, Y]$-module. The hypothesis of Theorem 1.3 asserts that there exist such modular forms X and Y for which $M(H) = \mathbb{C}[X, Y]$. Thus the hypothesis of Theorem 1.3 implies that $M(\varrho)$ is a free $M(H)$-module.

We now compute the rank r of $M(\varrho)$ as a $M(H)$-module. We have have shown in the proof of Lemma 2.3 that $M(H)$ is a free $M(\Gamma)$-module of rank $[\Gamma : H]$. Therefore $M(\varrho)$ is a free $M(\Gamma)$-module of rank $[\Gamma : H]r$. Theorem 1.1 tells us that $M(\mathrm{Ind}_H^\Gamma \varrho)$ is a free $M(\Gamma)$-module whose rank equals $\dim(\mathrm{Ind}_H^\Gamma \varrho) = [\Gamma : H] \dim \varrho$. Lemma 2.1 states that $M(\varrho)$ and $M(\mathrm{Ind}_H^\Gamma \varrho)$ are isomorphic as $M(\Gamma)$-modules. Thus $M(\varrho)$ is a free $M(\Gamma)$-module whose rank equals $[\Gamma : H] \dim \varrho$. Hence $[\Gamma : H]r = [\Gamma : H] \dim \varrho$ and $r = \dim \varrho$. $\square$

Remark. It seems that the proof of Theorem 1.2 can be extended from holomorphic vector-valued modular forms to vector-valued modular forms with arbitrary choices of exponents at each cusp of the subgroup H. This is important for applications to vertex operator algebras where the associated vector-valued modular forms will have poles at the cusps of H.

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