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EXPONENT OF CLASS GROUP OF CERTAIN IMAGINARY QUADRATIC FIELDS

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To Prof. Wenpeng Zhang on his 60th birthday with great respect and friendship.

Abstract. Let $n > 1$ be an odd integer. We prove that there are infinitely many imaginary quadratic fields of the form $\mathbb{Q}\big(\sqrt{x^2 - 2y^n}\big)$ whose ideal class group has an element of order n. This family gives a counterexample to a conjecture by H. Wada (1970) on the structure of ideal class groups.

Keywords: quadratic field; discriminant; class group; Wada's conjecture

MSC 2020: 11R29, 11R11

1. Introduction

Let x, y and n be positive integers. We consider the family of imaginary quadratic fields

$$K_{x,y,n,\mu} = \mathbb{Q}\big(\sqrt{x^2 - \mu y^n}\big)$$

with the conditions: $\gcd(x, y) = 1$, $y > 1$, $\mu \in \{1, 2, 4\}$ and $x^2 < \mu y^n$. Let $\mathcal{H}(K_{x,y,n,\mu})$ and $\mathcal{C}(K_{x,y,n,\mu})$ denote the class number and (ideal) class group of $K_{x,y,n,\mu}$, respectively. For $\mu \in \{1, 4\}$, there are many results concerning the divisibility of $\mathcal{H}(K_{x,y,n,\mu})$. We highlight some important results for these values of μ in the next two paragraphs.

In 1922, Nagell in [17] proved that $\mathcal{H}(K_{x,y,n,1})$ is divisible by n if both n and y are odd, and $l \mid x$, but $l^2 \nmid x$ for all prime divisors l of n. Let s be the square factor of $x^2 - y^n$, that is

$$x^2 - y^n = -s^2 D,$$

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where $D > 0$ is the square-free part of $x^2 - y^n$. For $s = 1$, Ankeny and Chowla in [1] proved that $\mathcal{H}(K_{x,3,n,1})$ is divisible by n if both n and x are even, and $x < (2 \times 3^{n-1})^{1/2}$. In 1998, Murty in [15] considered the divisibility of $\mathcal{H}(K_{1,y,n,1})$ by n, when $s = 1$ and $n \geqslant 5$ is odd. In the same paper, he further discussed this result when $s < y^{n/4}/2^{3/2}$. He also discussed a more general case, that is the divisibility of $\mathcal{H}(K_{x,y,n,1})$ in [16]. Soundararajan in [18] or Ito in [10] studied the divisibility of $\mathcal{H}(K_{x,y,n,1})$ by n under the condition $s < \sqrt{(y^n - x^2)/(y^{n/2} - 1)}$ or the condition that each prime divisor of s divides also D, respectively. Furthermore, Kishi in [13] (or Ito in [11] or Zhu and Wang in [21]) studied the divisibility by n of $\mathcal{H}(K_{2^k,3,n,1})$ or $\mathcal{H}(K_{2^k,p,n,1})$ with p prime or of $\mathcal{H}(K_{2^k,y,n,1})$ with y odd integer, respectively. In [8], Hoque and Saikia investigated the divisibility of $\mathcal{H}(K_{x,p,n,1})$ for any positive integer x, odd integer n and any prime p satisfying certain assumptions. Recently, Chakraborty et al. in [2] discussed the divisibility by n of $\mathcal{H}(K_{p,q,n,1})$ when both p and q are odd primes, and n is an odd integer.

On the other hand, Gross and Rohrlich in [6] (or Cohn in [5] and Ishii in [9]) proved the divisibility by n of $\mathcal{H}(K_{1,y,n,4})$ for odd n or $\mathcal{H}(K_{1,2,n,4})$ except for $n = 4$, and $\mathcal{H}(K_{1,y,n,4})$ for even n, respectively. Further, Louboutin in [14] proved that $\mathcal{C}(K_{1,y,n,4})$ has an element of order n if at least one odd prime divisor of y is equal to 3 (mod 4). Recently, Ito in [12] discussed the divisibility of $\mathcal{H}(K_{3^e,y,n,4})$ by n under certain conditions.

More recently, Chakraborty and Hoque in [7] (and in [3]) proved the divisibility by 3 of $\mathcal{H}(K_{1,y,3,2})$ for any odd integer y (and of $\mathcal{H}(K_{l,k^n,3,2l})$ for certain odd integers l, n, k. One can consult the survey [4] for more information about the divisibility of class number of quadratic fields. In this paper, we show that $\mathcal{C}(K_{p,q,n,2})$ has an element of order n when both p and q are odd primes, and n is an odd integer. Namely, we prove:

Theorem 1. *Let p and q be distinct odd primes, and let $n \geqslant 3$ be an odd integer with $p^2 < 2q^n$ and $2q^n - p^2 \neq \square$. Assume that $3q^{n/3} \neq p + 2$ whenever $3 \mid n$. Then $\mathcal{C}(K_{p,q,n,2})$ has an element of order n.*

An immediate consequence of the above result is:

Corollary 1. *Let p, q and n as in Theorem 1. Then there are infinitely many imaginary quadratic fields with discriminants of the form $p^2 - 2q^n$ whose class number is divisible by n.*

The present family of imaginary quadratic fields provides a counterexample of a conjecture (namely, Conjecture 1 given by Wada in [20]). We define some notation before stating this conjecture. The class group of a number field can be expressed, by the structure theorem of abelian groups, as the direct product of cyclic groups of orders $h_1, h_2, \ldots, h_t$. We denote the direct product $C_{h_1} \times C_{h_2} \times \ldots \times C_{h_t}$ of cyclic groups by $[h_1, h_2, \ldots, h_t]$. By Gauss's genus theory, if there are r distinct rational primes that ramify in $\mathbb{Q}(\sqrt{-D})$ for some square-free integer $D > 1$, then the 2-rank of its class group is $r - 1$. In other words, the 2-Sylow subgroup of its class group has rank $r - 1$. It is noted that the 2-Sylow subgroup of class group tends to $r - 2$ elementary 2-groups and one large cyclic factor collecting the other powers of 2 in the class number so that the 2-Sylow subgroup of the subgroup of squares is cyclic. On the other hand, there are $r - 1$ even integers among $h_1, h_2, \ldots, h_t$. Sometimes, the structure of the class group of $\mathbb{Q}(\sqrt{-D})$ can be trivially determined by r and the class number, $h = h_1 h_2 \ldots h_t$. In this case, the group is cyclic when $r = 1$ or $r = 2$ or it is of the type $(h_1, h_2, 2^{r_1}, \ldots, 2^{r_k})$ when $r \geqslant 3$. In this context, Wada in [20] gave the following conjecture.

Conjecture 2 (Wada [20]). *All the class groups of imaginary quadratic fields are either cyclic or of the type* $(h_1, h_2, 2^{r_1}, 2^{r_2}, \ldots, 2^{r_k})$.

In Table 2, we find a class group of the type $(h_1, h_2, h_3, 2^{r_1}, 2^{r_2}, \ldots, 2^{r_k})$ (see the $**$ mark) which is not cyclic. This gives a counterexample to Conjecture 1.

2. Proof of Theorem 1

We begin the proof with the following crucial proposition.

Proposition 1. *Let p, q and n be as in Theorem 1, and let s be the positive integer such that*

$$p^2 - 2q^n = -s^2 D, \tag{2.1}$$

where D is a square-free positive integer. Then for $\alpha = p + s\sqrt{-D}$ and for any prime divisor l of n, 2α is not an lth power of an element in the ring of integers of $K_{p,q,n,\mu}$.

Proof. Let l be a prime such that $l \mid n$. Then l is odd since n is odd. From (2.1), we see that $-D \equiv 3 \pmod 4$, and thus if 2α is an lth power of an element in the ring of integers $\mathcal{O}_{K_{p,q,n,2}}$ in $K_{p,q,n,2}$, then we can write

$$2\alpha = \left(u + v\sqrt{-D}\right)^l \tag{2.2}$$

for some $u, v \in \mathcal{O}_{K_{p,q,n,2}}$. Taking the norm, we obtain

$$8q^n = (u^2 + Dv^2)^l. \tag{2.3}$$

This shows that $l = 3$ with $3 \mid n$. Thus (2.3) reduces to

$$2q^m = u^2 + Dv^2, \tag{2.4}$$

where $n = 3m$ for some integer $m > 0$. Now (2.4) implies that u and v are either both odd or both even since D is odd. If both u and v are even, then $2 \mid q$ which is a contradiction. It remains to treat the case when both u and v are odd. We compare the real and imaginary parts on both sides of (2.2), and get

$$2p = u^3 - 3uv^2D, \tag{2.5}$$

$$2s = 3u^2v - v^3D. \tag{2.6}$$

We see that (2.5) implies $u \mid 2p$. As u is an odd integer and p is an odd prime, we must have $u = \pm 1$ or $u = \pm p$.

If $u = 1$, then (2.5) would imply $2p = 1 - 3v^2D$, which is a contradiction. Similarly, if $u = p$ then (2.5) gives

$$2 = p^2 - 3v^2D. \tag{2.7}$$

Reading (2.7) modulo 3, we see that $p^2 \equiv 2 \pmod 3$ which again leads to a contradiction.

Again when $u = -1$, the relations (2.5) and (2.4) give

$$3Dv^2 - 1 = 2p, \tag{2.8}$$

$$Dv^2 + 1 = 2q^m. \tag{2.9}$$

We now add (2.8) and (2.9), and that gives

$$2Dv^2 = p + q^m. \tag{2.10}$$

Now subtracting (2.9) from (2.8) leads us to

$$Dv^2 = 1 + p - q^m. \tag{2.11}$$

In this case finally (2.10) and (2.11) together give $3q^m = p + 2$. This contradicts the assumption.

We are now left with the case when $u = -p$. In this case (2.4), (2.5) and (2.6) become,

$$Dv^2 + p^2 = 2q^m, \tag{2.12}$$

$$3Dv^2 - p^2 = 2, \tag{2.13}$$

$$Dv^3 + 2s = 3p^2v. \tag{2.14}$$

Multiplying (2.12) by 3 and then subtracting the resulting equation from (2.13), we get

$$2p^2 + 1 = 3q^m. \tag{2.15}$$

Equation (2.14) also shows that $v \mid s$, and thus we can write $s = vt$ for some odd integer t, since both v and s are odd. Therefore (2.14) can be rewritten as

$$Dv^2 + 2t = 3p^2. \tag{2.16}$$

Since $s = vt$, (2.1) becomes

$$Dv^2t^2 + p^2 = 2q^n.$$

Using (2.16), we obtain

$$3p^2t^2 - 2t^3 + p^2 = 2q^n.$$

Since $n = 3m$, by using (2.15) the above equation reduces to

$$81p^2t^2 - 54t^3 + 27t^2 = 2(2p^2 + 1)^3.$$

Finally reading this equation modulo 4, we arrive at

$$0 \equiv 2 \pmod 4,$$

which is not possible. This completes the proof. □

Proof of Theorem 1. Let s be the positive integer such that

$$p^2 - 2q^n = -s^2D,$$

where D is a square-free positive integer. Let $\alpha := p + s\sqrt{-D}$. Then

$$N(\alpha) = \alpha\overline{\alpha} = 2q^n,$$

where $\overline{\alpha}$ is the conjugate of α, and they are coprime.

We see that q splits completely in $K_{p,q,n,2} = \mathbb{Q}(\sqrt{-D})$, and thus we have

$$(q) = \mathfrak{q}\mathfrak{q}',$$

where $\mathfrak{q}$ and its conjugate $\mathfrak{q}'$ are prime ideals in $\mathcal{O}_{K_{p,q,n,2}}$ different from each other. Moreover, $(2) = \mathfrak{p}^2$ with $\mathfrak{p} = (2, 1 + \sqrt{-D})$.

We see that (α) is not divisible by any rational integer other than ± 1, and thus we can consider the decomposition

$$(\alpha) = \mathfrak{p}\mathfrak{q}^m.$$

Then $N(\alpha) = 2q^m$ and hence $n = m$.

We now put $\mathfrak{A} = \mathfrak{p}\mathfrak{q}$. Then (since n is odd)

$$\mathfrak{A}^n = \mathfrak{p}^{n-1}(\mathfrak{p}\mathfrak{q}^n) = (2^{(n-1)/2}\alpha) \subseteq (2\alpha),$$

which is principal in $\mathcal{O}_{K_{p,q,n,2}}$. Thus if $[\mathfrak{A}]$ denotes the ideal class containing $\mathfrak{A}$ then by Proposition 1, we see that the order of $[\mathfrak{A}]$ is n. This completes the proof. □

3. Numerical examples

Here, we provide some numerical values to corroborate Theorem 1. All the computations in this paper were done using PARI/GP (version 2.7.6), see [19]. Table 1 gives the list of imaginary quadratic fields $K_{p,q,n,2}$ corresponding to the distinct primes p and q not larger than 17 and an odd integer $3 \leqslant n \leqslant 19$. We see that absolute discriminants do not exceed 5×10^{23}, and the corresponding class numbers are very large and can go up to (about) 5.5×10^{11}. It is noted that this list does not exhaust all the imaginary quadratic fields $K_{p,q,n,2}$ of discriminants not exceeding 5×10^{23}. In Table 1, we use the $*$ mark in the column for the class number to indicate the failure of the assumption "$3q^{n/3} \neq p+2$" of Theorem 1.

n	p	q	$p^2 - 2q^n$	$h(-D)$	n	p	q	$p^2 - 2q^n$	$h(-D)$
3	3	5	−241	12	3	3	7	−677	30
3	3	11	−2653	24	3	3	13	−4385	96
3	3	17	−9817	48	3	5	3	−29	6
3	5	7	−661	18	3	5	11	−2637	36
3	5	13	−4369	48	3	5	17	−9801	72
3	7	3	−5	2*	3	7	5	−201	12
3	7	11	−2613	24	3	7	13	−4345	48
3	7	17	−9777	60	3	11	5	−129	12
3	11	7	−565	12	3	11	13	−4273	24
3	11	17	−9705	72	3	13	5	−81	1*
3	13	7	−517	12	3	13	11	−2493	24

Table 1. Numerical examples of Theorem 1 (continued).

n	p	q	p^2-2q^n	$h(-D)$	n	p	q	p^2-2q^n	$h(-D)$
3	13	17	−9657	48	3	17	7	−397	6
3	17	11	−2373	24	3	17	13	−4105	48
5	3	5	−6241	40	5	3	7	−33605	240
5	3	11	−322093	150	5	3	13	−742577	800
5	3	17	−2839705	800	5	5	3	−461	30
5	5	7	−33589	150	5	5	11	−322077	280
5	5	13	−742561	500	5	5	17	−2839689	1760
5	7	3	−437	20	5	7	5	−6201	80
5	7	11	−322053	320	5	7	13	−742537	380
5	7	17	−2839665	1120	5	11	3	−365	20
5	11	5	−6129	60	5	11	7	−33493	70
5	11	13	−742465	480	5	11	17	−2839593	800
5	13	3	−317	10	5	13	5	−6081	60
5	13	7	−33445	80	5	13	11	−321933	440
5	13	17	−2839545	960	5	17	3	−197	10
5	17	5	−5961	60	5	17	7	−33325	120
5	17	11	−321813	240	5	17	13	−742297	480
7	3	5	−156241	168	7	3	7	−1647077	1260
7	3	11	−38974333	2926	7	3	13	−125497025	11648
7	3	17	−820677337	10724	7	5	3	−4349	42
7	5	7	−1647061	896	7	5	11	−38974317	2968
7	5	13	−125497009	5824	7	5	17	−820677321	26432
7	7	3	−4325	84	7	7	5	−156201	308
7	7	11	−38974293	3696	7	7	13	−125496985	5768
7	7	17	−820677297	19096	7	11	3	−4253	42
7	11	5	−156129	392	7	11	7	−1646965	588
7	11	13	−125496913	4704	7	11	17	−820677225	18144
7	13	3	−4205	56	7	13	5	−156081	364
7	13	7	−1646917	728	7	13	11	−38974173	3360
7	13	17	−820677177	16800	7	17	3	−4085	56
7	17	5	−155961	224	7	17	7	−1646797	658
7	17	11	−38974053	3780	7	17	13	−125496745	7392
9	3	5	−3906241	1440	9	3	7	−80707205	11448
9	3	11	−4715895373	29556	9	3	13	−21208998737	162432
9	3	17	−237175752985	337176	9	5	3	−39341	198
9	5	7	−80707189	7272	9	5	11	−4715895357	67716
9	5	13	−21208998721	62136	9	5	17	−237175752969	349164
9	7	3	−39317	162	9	7	5	−3906201	2448
9	7	11	−4715895333	28512	9	7	13	−21208998697	57744
9	7	17	−237175752945	463824	9	11	3	−39245	288
9	11	5	−3906129	1692	9	11	7	−80707093	3852

Table 1. Numerical examples of Theorem 1 (continued).

n	p	q	p^2-2q^n	$h(-D)$	n	p	q	p^2-2q^n	$h(-D)$
9	11	13	−21208998625	79200	9	11	17	−237175752873	284256
9	13	3	−39197	108	9	13	5	−3906081	1512
9	13	7	−80707045	4608	9	13	11	−4715895213	33300
9	13	17	−237175752825	228096	9	17	3	−39077	180
9	17	5	−3905961	1368	9	17	7	−80706925	5184
9	17	11	−4715895093	35712	9	17	13	−21208998457	74376
11	3	5	−97656241	3608	11	3	7	−3954653477	46332
11	3	11	−570623341213	286770	11	3	13	−3584320788065	2956800
11	3	17	−68543792615257	2056120	11	5	3	−354269	704
11	5	7	−3954653461	36432	11	5	11	−570623341197	519200
11	5	13	−3584320788049	875072	11	5	17	−68543792615241	6392760
11	7	3	−354245	528	11	7	5	−97656201	6864
11	7	11	−570623341173	340032	11	7	13	−3584320788025	1146464
11	7	17	−68543792615217	5876112	11	11	3	−354173	528
11	11	5	−97656129	8712	11	11	7	−3954653365	39776
11	11	13	−3584320787953	797720	11	11	17	−68543792615145	3800544
11	13	3	−354125	660	11	13	5	−97656081	9944
11	13	7	−3954653317	37268	11	13	11	−570623341053	570240
11	13	17	−68543792615097	4511232	11	17	3	−354005	352
11	17	5	−97655961	7216	11	17	7	−3954653197	25872
11	17	11	−570623340933	353760	11	17	13	−3584320787785	1076416
13	3	5	−2441406241	29432	13	3	7	−193778020805	435968
13	3	11	−69045424287853	4435704	13	3	13	−605750213184497	30878952
13	3	17	−19809156065811865	59575360	13	5	3	−3188621	2028
13	5	7	−193778020789	357422	13	5	11	−69045424287837	4188392
13	5	13	−605750213184481	14891136	13	5	17	−19809156065811849	89333920
13	7	3	−3188597	1612	13	7	5	−2441406201	35984
13	7	11	−69045424287813	4845568	13	7	13	−605750213184457	10482888
13	7	17	−19809156065811825	104113152	13	11	3	−3188525	1560
13	11	5	−2441406129	35412	13	11	7	−193778020693	183716
13	11	13	−605750213184385	14582464	13	11	17	−19809156065811753	70607680
13	13	3	−3188477	1716	13	13	5	−2441406081	32552
13	13	7	−193778020645	228800	13	13	11	−69045424287693	4533152
13	13	17	−19809156065811705	117589680	13	17	3	−3188357	1560
13	17	5	−2441405961	22464	13	17	7	−193778020525	221000
13	17	11	−69045424287573	5311488	13	17	13	−605750213184217	13254488
15	3	5	−61035156241	133620	15	3	7	−9495123019877	3654000
15	3	11	−8354496338831293	44413440	15	3	13	−102371786028181505	389436480
15	3	17	−5724846103019631577	1053896220	15	5	3	−28697789	6330
15	5	7	−9495123019861	2182740	15	5	11	−8354496338831277	48860640
15	5	13	−102371786028181489	231737760	15	5	17	−5724846103019631561	2429285220

Table 1. Numerical examples of Theorem 1 (continued).

n	p	q	p^2-2q^n	$h(-D)$	n	p	q	p^2-2q^n	$h(-D)$
15	7	3	−28697765	7140	15	7	5	−61035156201	232380
15	7	11	−8354496338831253	58597320	15	7	13	−102371786028181465	184700760
15	7	17	−5724846103019631537	1804868640	15	11	3	−28697693	4740
15	11	5	−61035156129	251460	15	11	7	−9495123019765	1636920
15	11	13	−102371786028181393	129010680	15	11	17	−5724846103019631465	1339566720
15	13	3	−28697645	3960	15	13	5	−61035156081	177120
15	13	7	−9495123019717	1139880	15	13	11	−8354496338831133	43159500
15	13	17	−5724846103019631417	970122240	15	17	3	−28697525	7200
15	17	5	−61035155961	294480	15	17	7	−9495123019597	1418310
15	17	11	−8354496338831013	39988560	15	17	13	−102371786028181225	224843520
17	3	5	−1525878906241	902632	17	3	7	−465261027974405	21068168
17	3	11	−1010894056998587533	340866116	17	3	13	−17300831838762675857	3293665952
17	3	17	−1654480523772673528345	16865341776	17	5	3	−258280301	20672
17	5	7	−465261027974389	13394096	17	5	11	−1010894056998587517	483167744
17	5	13	−17300831838762675841	2677442880	17	5	17	−1654480523772673528329	45027534568
17	7	3	−258280277	17068	17	7	5	−1525878906201	887400
17	7	11	−1010894056998587493	513329824	17	7	13	−17300831838762675817	1808666448
17	7	17	−1654480523772673528305	31484143616	17	11	3	−258280205	16320
17	11	5	−1525878906129	891072	17	11	7	−465261027974293	7670808
17	11	13	−17300831838762675745	2180205664	17	11	17	−1654480523772673528233	18565719040
17	13	3	−258280157	9248	17	13	5	−1525878906081	1364692
17	13	7	−465261027974245	18608472	17	13	11	−1010894056998587373	805270688
17	13	17	−1654480523772673528185	17835810304	17	17	3	−258280037	9928
17	17	5	−1525878905961	601936	17	17	7	−465261027974125	8867200
17	17	11	−1010894056998587253	526706580	17	17	13	−17300831838762675577	2528985824
19	3	5	−38146972656241	2652552	19	3	7	−22797790370746277	153169260
19	3	11	−1223181808968290092573	5845471904	19	3	13	−2923840580750892221345	54859889096
19	3	17	−478144871370302649694297	246910222416	19	5	3	−2324522909	39976
19	5	7	−22797790370746261	174949568	19	5	11	−1223181808968290092557	5849478624
19	5	13	−2923840580750892221329	28846572768	19	5	17	−478144871370302649694281	548121230352
19	7	3	−2324522885	50388	19	7	5	−38146972656201	6622792
19	7	11	−1223181808968290092533	4729953024	19	7	13	−2923840580750892221305	36032330968
19	7	17	−478144871370302649694257	458154357332	19	11	3	−2324522813	37240
19	11	5	−38146972656129	4247868	19	11	7	−22797790370746165	76028880
19	11	13	−2923840580750892221233	26134377792	19	11	17	−478144871370302649694185	389010811712
19	13	3	−2324522765	40356	19	13	5	−38146972656081	5545188
19	13	7	−22797790370746117	55016248	19	13	11	−1223181808968290092413	6947197088
19	13	17	−478144871370302649694137	397061284992	19	17	3	−2324522645	36784
19	17	5	−38146972655961	3776896	19	17	7	−22797790370745997	77990972
19	17	11	−1223181808968290092293	8470575516	19	17	13	−2923840580750892221065	35488586232

Table 1. Numerical examples of Theorem 1.

4. Concluding remarks

We begin by observing that in Table 1 there are some values of p and q (see the $*$ mark) for which the class numbers of the corresponding imaginary quadratic fields are not divisible by a given odd integer $n \geqslant 3$. This is because of the failure of the assumption "$3q^{n/3} \neq p+2$ when $3 \mid n$". However, the class number of $K_{19,7,3,2}$ is 12 and satisfies the divisibility property even though this assumption does not hold. Thus this assumption is neither necessary nor sufficient. We have found only two pairs of values of p and q for which this assumption does not hold. Thus it may be possible to drop this assumption by adding some exceptions for the values of the pair (p, q).

In the light of the numerical evidence we are tempted to state the following conjecture:

Conjecture 3. *Let p and q be two distinct odd primes. For each positive odd integer n and for each positive integer m such that m is not an nth root of any rational integer, there are infinitely many imaginary quadratic fields of the form $\mathbb{Q}\big(\sqrt{p^2 - mq^n}\big)$ whose class number is divisible by n.*

For $m = 1, 4$, this conjecture is true (see [2] and references therein). Further Corollary 1 shows that the conjecture is true for any odd integer n when $m = 2$.

We now demonstrate the structures of class groups of $K_{p,q,n,2}$ for some values of p, q and n. In Table 2, by $(h_1, h_2, \ldots, h_t)$ we mean the group $\mathbb{Z}_{h_1} \times \mathbb{Z}_{h_2} \times \ldots \times \mathbb{Z}_{h_t}$.

$p^2 - 2q^n$	Structure of $\mathcal{C}(K_{p,q,n,2})$	2-parts	3-parts	5-parts	Remaining parts
$11^2 - 2 \times 17^5$	[20, 10, 2, 2]	(2,2,2,4)	–	(5,5)	–
$7^2 - 2 \times 17^{13}$	[1084512, 6, 2, 2, 2, 2]	(2,2,2,2,2,32)	(3)	–	(11,13,79)
$13^2 - 2 \times 11^{15}$	[479550, 30, 3]	(2,2)	(3,3,3)	(5,25)	(23,139)
$13^2 - 2 \times 17^{15}$	[10105440, 12, 2, 2, 2]	(2,2,2,4,32)	(3,3)	(5)	(37,569)
$3^2 - 2 \times 5^{21}$	[565992, 6, 2, 2, 2]	(2,2,2,2,8)	(3,9)	–	(7,1123)
$3^2 - 2 \times 13^{21}$	[7991268432, 6, 2, 2, 2, 2]	(2,2,2,2,2,16)	(3,3)	–	(7,23783537)
$17^2 - 2 \times 11^{21}$	[286454952, 12, 2, 2, 2, 2]	(2,2,2,2,4,8)	(3,9)	–	(7,568363)
$11^2 - 2 \times 7^{25}$	[292374800, 10, 2, 2, 2]	(2,2,2,2,16)	–	(5,25)	(101,7237)
$13^2 - 2 \times 17^{25}$	[60345039225000, 6, 2, 2, 2]	(2,2,2,2,8)	(3,3)	(3125)	(41,59,332617)
$5^2 - 2 \times 11^{27}$	[381006210618, 6, 6, 2, 2, 2]**	(2,2,2,2,2,2)	(3,3,81)	–	(11,211,92119)
$5^2 - 2 \times 13^{27}$	[9392738579820, 6, 2, 2, 2, 2]	(2,2,2,2,2,4)	(3,27)	(5)	(59,9011,32717)
$5^2 - 2 \times 17^{27}$	[627358330621332, 6, 2, 2, 2, 2]	(2,2,2,2,2,4)	(3,81)	–	(183167,10571179)
$17^2 - 2 \times 13^{27}$	[8751451767912, 6, 2, 2, 2, 2]	(2,2,2,2,2,8)	(3,27)	–	(12251,67493)
$17^2 - 2 \times 11^{29}$	[12592336322520, 2, 2, 2, 2, 2, 2]	(2,2,2,2,2,2,8)	(27)	(5)	(29,7411,54251)

Table 2. Structure of the class group of $K_{p,q,n,2}$.

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