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A NOTE ON L-DUNFORD-PETTIS SETS IN A TOPOLOGICAL DUAL BANACH SPACE

Abderrahman Retbi, Kénitra

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Abstract. The present paper is devoted to some applications of the notion of L-Dunford-Pettis sets to several classes of operators on Banach lattices. More precisely, we establish some characterizations of weak Dunford-Pettis, Dunford-Pettis completely continuous, and weak almost Dunford-Pettis operators. Next, we study the relationships between L-Dunford-Pettis, and Dunford-Pettis (relatively compact) sets in topological dual Banach spaces.

Keywords: L-Dunford-Pettis set; weak almost Dunford-Pettis operator; weak Dunford-Pettis property; Banach lattice

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1. Introduction and notation

Recall that a subset A of a Banach space X is called a Dunford-Pettis set (DP set for short) whenever every weakly null sequence (f_n) in X' converges uniformly to zero on A, that is, $\lim\limits_{n\to\infty}\sup\limits_{x\in A}|f_n(x)| = 0$, see [1].

A norm bounded subset A of a topological dual Banach space X' is called

- an L-set, if every weakly null sequence (x_n) in X converges uniformly to zero on A, that is, $\lim\limits_{n\to\infty}\sup\limits_{f\in A}|f(x_n)| = 0$;
- an L-Dunford-Pettis set, if every weakly null sequence (x_n) which is a DP set in X converges uniformly to zero on A, that is, $\lim\limits_{n\to\infty}\sup\limits_{f\in A}|f(x_n)| = 0$, see [9].

In X' it is clear that:

$$\text{DP set} \Rightarrow \text{L-set} \Rightarrow \text{L-DP set}.$$

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Recently, the authors of [2] introduced a weak version of L-sets, the so called almost L-sets, that is, such that every disjoint weakly null sequence (x_n) in a Banach lattice E converges uniformly to zero on A, that is, $\lim\limits_{n\to\infty}\sup\limits_{f\in A}|f(x_n)| = 0$. Clearly, each L-set in a dual Banach lattice is an almost L-set.

Let us recall from [5] that a norm bounded subset A of a Banach lattice E is said to be almost Dunford-Pettis if every disjoint weakly null sequence (f_n) of E' converges uniformly on A, that is, $\lim\limits_{n\to\infty}\sup\limits_{x\in A}|f_n(x)| = 0$.

An operator T from a Banach space X into another Banach space Y is called

- ▷ Dunford-Pettis if T carries each relatively weakly compact set in X to relatively compact set in Y, equivalently, whenever $\|T(x_n)\| \to 0$ for every weakly null sequence (x_n) in X, see [1];
- ▷ weak Dunford-Pettis if T carries each relatively weakly compact set in X to a Dunford-Pettis set in Y, equivalently, whenever $f_n(T(x_n)) \to 0$, as $n \to \infty$ for every weakly null sequence (x_n) in X and every weakly null sequence (f_n) in Y', see [1];
- ▷ Dunford-Pettis completely continuous (DPcc for short) if T carries each Dunford-Pettis set in X to relatively compact set in Y, equivalently, whenever for each weakly null sequence (x_n) which is a Dunford-Pettis set in X, we have $\|T(x_n)\| \to 0$ as $n \to \infty$, see [10].

An operator T from a Banach lattice E into a Banach space Y is said to be almost Dunford-Pettis if $\|T(x_n)\| \to 0$ in Y for every weakly null sequence (x_n) consisting of pairwise disjoint elements in E, see [11]. Recall from [4] that an operator $T\colon X \to F$ from a Banach space X into a Banach lattice F is called weak almost Dunford-Pettis if T carries each relatively weakly compact set in X to an almost Dunford-Pettis set in F, equivalently, whenever $f_n(T(x_n)) \to 0$ for every weakly null sequence (x_n) in X and every disjoint weakly null sequence (f_n) in F'. A Banach space X has

- ▷ the Dunford-Pettis property (DP property for short), if $x_n \xrightarrow{w} 0$ in X and $f_n \xrightarrow{w} 0$ in X' imply $f_n(x_n) \to 0$, see [1];
- ▷ the relatively compact Dunford-Pettis property (DPrc property for short) if every weakly null sequence which is a Dunford-Pettis set in X is norm null, see [10];
- ▷ the Schur property, if every weakly null sequence in X is norm null.

Let us recall from [5] that a Banach lattice E has the weak Dunford-Pettis property (wDP property for short), if every relatively weakly compact set in E is almost Dunford-Pettis, equivalently, whenever $f_n(x_n) \to 0$ for every weakly null sequence (x_n) in E and for every disjoint weakly null sequence (f_n) in E'. Note that a Banach lattice E has the positive Schur property if each weakly null sequence with positive terms is norm null. It is pointed out that E has the positive Schur property if and only if each weakly null disjoint sequence in E converges to zero in norm.

Note that there is an L-Dunford-Pettis set which fails to be an almost L-set (L-set). In fact, the closed unit ball $B_{L_2[0,1]}$ is an L-Dunford-Pettis set in $L_2[0,1]$ but fails to be an L-set, as $L_2[0,1]$ has the DPrc property without the positive Schur property (respectively, Schur property) see Corollary 2.7 and [2], Corollary 3.9.

In this paper, the concept of an L-Dunford-Pettis set in a topological dual Banach space is used to characterize several classes of operators (weak Dunford-Pettis, weak almost Dunford-Pettis and Dunford-Pettis completely continuous) acting between Banach lattices (or mapping a Banach space into a Banach lattice) (see Theorem 2.1, Theorem 2.6 and Theorem 2.11). As consequences, we investigate new characterizations of various properties (Dunford-Pettis, weak Dunford-Pettis and relatively compact Dunford-Pettis) in Banach spaces or Banach lattices (see Corollary 2.2, Corollary 2.7 and Corollary 2.12). Note that each Dunford-Pettis (relatively compact) set in a dual Banach space is L-Dunford-Pettis, but the converse is not true in general. In fact, B_{ℓ^∞} is an L-Dunford-Pettis set because ℓ^1 has the DPrc property (see Corollary 2.7), but it is not Dunford-Pettis (respectively, relatively compact). In Theorem 2.8, we give an operator characterization of the class of L-Dunford-Pettis sets to coincide with Dunford-Pettis (respectively, relatively compact) in a topological dual Banach space.

To state our results, we need to fix some notation and recall some definitions. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leqslant |y|$, we have $\|x\| \leqslant \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice. We will use the term an operator $T\colon X \to Y$ between two Banach spaces to mean a bounded linear mapping, its dual operator T' is defined from Y' into X' by $T'(f)(x) = f(T(x))$ for each $f \in Y'$ and for each $x \in X$. An operator T between two Banach lattices E and F is positive if $T(x) \geqslant 0$ in F whenever $x \geqslant 0$ in E. A sequence (x_n) of a Banach lattice E is disjoint if $|x_n| \wedge |x_m| = 0$ for $n \neq m$. We refer the reader to [1] for unexplained terminology of the Banach lattice theory and positive operators.

2. Main results

We start by the following characterizations of weak Dunford-Pettis operator.

Theorem 2.1. *Let $T\colon X \to Y$ be an operator between two Banach spaces. The following statements are equivalent:*

(1) *T is a weak Dunford-Pettis operator;*

(2) *T' carries L-Dunford-Pettis sets in Y' to L-sets in X';*

(3) *for an arbitrary Banach space Z and for every DPcc operator $S\colon Y \to Z$, the product ST is Dunford-Pettis;*

(4) *for an arbitrary Banach space Z and for every weakly compact operator $S\colon Y \to Z$, the product ST is Dunford-Pettis;*

(5) *for an arbitrary Banach space Z and for every weakly compact operator $S\colon Z \to X$, the adjoint operator $(TS)'$ carries L-Dunford-Pettis sets in Y' to relatively compact sets in Z'.*

Proof. (1) $\Rightarrow$ (2) Let A be an L-Dunford-Pettis set in Y' and let (x_n) be a weakly null sequence in X, then by our hypothesis on T we have $(T(x_n))$ is a weakly null and Dunford-Pettis sequence in Y. Since

$$\lim_{n\to\infty} \sup_{f\in T'(A)} |f(x_n)| = \lim_{n\to\infty} \sup_{g\in A} |g(T(x_n))| = 0,$$

we see that $T'(A)$ is an L-set in X'.

(2) $\Rightarrow$ (3) Let Z be a Banach space and let $S\colon Y \to Z$ be a DPcc operator. Then $S'(B_{Z'})$ is an L-Dunford-Pettis set, and by our hypothesis we see that $T'(S'(B_{Z'}))$ is an L-set. Hence ST is a Dunford-Pettis operator.

(3) $\Rightarrow$ (4) Follows from [10], Corollary 1.1.

(4) $\Rightarrow$ (1) Let (x_n) be a weakly null sequence in X, and let (f_n) be a weakly null sequence in Y'.

Consider the operator $S\colon Y \to c_0$ defined by

$$S(x) = (f_n(x))_{n=1}^{\infty}.$$

Theorem 5.26 of [1] proves that S is weakly compact operator, and by our hypothesis ST is Dunford-Pettis. Since

$$|f_n(T(x_n))| \leqslant \|S(T(x_n))\|_{\infty},$$

we deduce that $f_n(T(x_n)) \to 0$ as $n \to \infty$, and we are done.

(2) $\Rightarrow$ (5) Let $S\colon Z \to X$ be a weakly compact operator and A an L-Dunford-Pettis set in Y', then by our hypothesis $T'(A)$ is an L-set in X', and by [7], Theorem 4.4 we have $(TS)'(A)$ is a relatively compact set in Z'.

(5) $\Rightarrow$ (1) Let (x_n) be a weakly null sequence in X, and let (f_n) be a weakly null sequence in Y'.

Consider the operator $S\colon \ell^1 \to X$ defined by

$$S((\lambda_n)_n) = \sum_{n=1}^{\infty} \lambda_n x_n.$$

Note that S is a weakly compact operator (see [1], Theorem 5.26) and its adjoint $S'\colon X' \to \ell^\infty$ is defined by

$$S'(f) = (f(x_n))_{n\geqslant 1},$$

and we have $S'(X') \subset c_0$. Now, we put $A = \{f_n\colon n \in \mathbb{N}\}$, from [9], Proposition 2.3 we see that A is an L-Dunford-Pettis set, and hence by our hypothesis $(TS)'(A)$ is a relatively compact set in c_0. It follows from [1], Section 3.2, Exercise 14 that

$$|f_n(T(x_n))| = |T'(f_n)(x_n)| \leqslant \sup_{g\in T'(A)} |g(x_n)| \to 0.$$

This proves that T is a weak Dunford-Pettis operator. □

As a consequence, we obtain:

Corollary 2.2. *Let X be a Banach space. The following statements are equivalent:*

(1) *X has the Dunford-Pettis property;*
(2) *L-Dunford-Pettis subsets of X' are L-sets;*
(3) *DPcc operators from X into an arbitrary Banach space Z are Dunford-Pettis;*
(4) *weakly compact operators from X into an arbitrary Banach space Z are Dunford-Pettis;*
(5) *the adjoint of each weakly compact operator from an arbitrary Banach space Z into X carries L-Dunford-Pettis sets in X' to relatively compact sets in Z'.*

Corollary 2.3. *Let T be an operator from a reflexive Banach space X into a Banach space Y.*

An operator T' is a Dunford-Pettis operator if and only if T' carries L-Dunford-Pettis sets in Y' to relatively compact sets in X'.

P r o o f. For "only if" part since T' is Dunford-Pettis operator then T' is weak Dunford-Pettis and by [3], Theorem 3.1 we see that T is weak Dunford-Pettis. As X is reflexive, the identity operator $I\colon X \to X$ is weakly compact. Since $T' = (TI)'$, it follows from Theorem 2.1 that T' carries L-Dunford-Pettis sets in Y' to relatively compact sets in X'.

For "if" part let A be a relatively weakly compact set in Y', then by [9], Proposition 2.3, A is an L-Dunford-Pettis sets in Y' and by our hypothesis $T'(A)$ is a relatively compact set in X'. This proves that T' is a Dunford-Pettis operator. □

As a simple consequence of Corollary 2.3 we obtain:

Corollary 2.4. *Let X be a reflexive Banach space. The following statements are equivalent:*

(1) *X' has the Schur property;*
(2) *X' has finite dimension;*
(3) *every L-Dunford-Pettis set in X' is relatively compact.*

The following Proposition gives a characterization of L-Dunford-Pettis sets in terms of sequences.

Proposition 2.5. *Let X be a Banach space and let A be a norm bounded subset of X'. The following statements are equivalent:*
(1) *A is an L-Dunford-Pettis set in X'.*
(2) *For every sequence (f_n) in A and every weakly null sequence (x_n) which is a Dunford-Pettis set in X, we have $f_n(x_n) \to 0$ as $n \to \infty$.*

Proof. (2) $\Rightarrow$ (1) Assume by way of contradiction that A is not an L-Dunford-Pettis set in X'. Then there exists a weakly null sequence (x_n) which is a Dunford-Pettis subset of X such that $\sup_{f \in A} |f(x_n)| > \varepsilon > 0$ for some $\varepsilon > 0$ and each n. Hence, for every n there exists some f_n in A such that $|f_n(x_n)| > \varepsilon$, which is impossible due to our hypothesis (2). This proves that A is an L-Dunford-Pettis set in X'.

(1) $\Rightarrow$ (2) Let (f_n) be a sequence in A and (x_n) a weakly null sequence which is a Dunford-Pettis set in X. Since

$$|f_n(x_n)| \leqslant \sup_{f \in A} |f(x_n)|$$

for every n, and A is an L-Dunford-Pettis set in X', hence $f_n(x_n) \to 0$ as $n \to \infty$. This completes the proof. □

Now, we give a characterization of DPcc operators from a Banach space into a Banach lattice.

Theorem 2.6. *Let $T\colon X \to F$ be an operator from a Banach space into a Banach lattice. The following statements are equivalent:*
(1) *T is a DPcc operator;*
(2) *$T'(B_{F'})$ is an L-Dunford-Pettis set;*
(3) *$T'([-f, f])$ and $\{T'(f_n)\colon n \in \mathbb{N}\}$ are L-Dunford-Pettis sets for each $f \in B_{F'}^{+}$ and for each disjoint sequence $(f_n) \subset B_{F'}^{+}$;*
(4) *$|T(x_n)| \to 0$ weakly in F and $f_n(T(x_n)) \to 0$ for every weak null and Dunford-Pettis sequence (x_n) in X and for each disjoint sequence $(f_n) \subset B_{F'}^{+}$.*

Proof. (1) $\Rightarrow$ (2) Follows from the equality $\sup_{f \in T'(B_{F'})} |f(x_n)| = \|T(x_n)\|$ for every weak null and Dunford-Pettis sequence (x_n) in X.

(2) ⇒ (3) Obvious.

(3) ⇒ (4) Let (x_n) be a weakly null and Dunford-Pettis sequence in X and let (f_n) be a disjoint sequence in $B^+_{F'}$. As $\{T'(f_n)\colon n \in \mathbb{N}\}$ is an L-Dunford-Pettis set in X', hence from Proposition 2.5 we see that $f_n(T(x_n)) = T'(f_n)(x_n) \to 0$ as $n \to \infty$.

On the other hand, let $f \in B^+_{F'}$ then it follows from [1], Theorem 1.23 that

$$\begin{aligned} f(|T(x_n)|) &= \sup\{g(T(x_n))\colon g \in [-f, f]\} = \sup\{T'g(x_n)\colon g \in [-f, f]\} \\ &= \sup\{h(x_n)\colon h \in T'([-f, f])\}. \end{aligned}$$

Since $T'([-f, f])$ is an L-Dunford-Pettis set in X', we conclude that $|T(x_n)| \to 0$ weakly in F.

(4) ⇒ (1) It follows from Dodds and Fremlin, see [6], Corollary 2. □

In particular, we obtain the following result:

Corollary 2.7. *Let E be a Banach lattice. The following statements are equivalent:*

(1) *E has the DPrc property;*
(2) *$B_{E'}$ is an L-Dunford-Pettis set;*
(3) *$[-f, f]$ and $\{f_n\colon n \in \mathbb{N}\}$ are L-Dunford-Pettis sets for each $f \in B^+_{E'}$ and for each disjoint sequence $(f_n) \subset B^+_{E'}$;*
(4) *$|x_n| \to 0$ weakly in E and $f_n(x_n) \to 0$ for every weak null and Dunford-Pettis sequence (x_n) in E and for each disjoint sequence $(f_n) \subset B^+_{E'}$.*

In the next result we give an operator characterization of the class of L-Dunford-Pettis sets to coincide with that of Dunford-Pettis (respectively, relatively compact) sets in a dual Banach space.

Theorem 2.8. *Let X be a Banach space.*

(1) *Every L-Dunford-Pettis set in X' is Dunford-Pettis if, and only if, T'' is Dunford-Pettis whenever Y is an arbitrary Banach space and $T\colon X \to Y$ is a DPcc operator.*
(2) *Every L-Dunford-Pettis set in X' is relatively compact if, and only if, T is compact whenever Y is an arbitrary Banach space and $T\colon X \to Y$ is a DPcc operator.*

P r o o f. (1) For the "only if" part, let Y be a Banach space and $T\colon X \to Y$ a DPcc operator. Then $T'(B_{Y'})$ is an L-Dunford-Pettis set, hence $T'(B_{Y'})$ is a Dunford-Pettis set. This proves that T'' is a Dunford-Pettis operator.

For the "if" part, assume by way of contradiction that there exists an L-Dunford-Pettis set A of X' that is not Dunford-Pettis. Then there exist a weakly null sequence

$(f_n) \subset X''$, a sequence $(g_n) \subset A$ and $\varepsilon > 0$ such that $|f_n(g_n)| > \varepsilon$. Consider the operator $T\colon X \to \ell^\infty$ defined by

$$T(x) = (g_n(x))_{n\geqslant 1}$$

for all $x \in X$. We show that T is DPcc. As $(g_n) \subseteq A$ is an L-Dunford-Pettis set for every weakly null sequence (x_m) which is a DP set in X we have

$$\|T(x_m)\| = \sup_n |g_n(x_m)| \to 0 \quad \text{as } m \to \infty,$$

so T is a Dunford-Pettis completely continuous operator, and we have

$$T'((\lambda_n)_{n=1}^\infty) = \sum_{n=1}^\infty \lambda_n g_n$$

for every $(\lambda_n)_{n=1}^\infty \in \ell^1 \subset (\ell^\infty)'$. If e'_n is the usual basis element in ℓ^1 then $T'(e'_n) = g_n$ for all $n \in \mathbb{N}$. By our hypothesis T'' is Dunford-Pettis. Hence, $T'(B_{(\ell^\infty)'})$ is a Dunford-Pettis set in X'. Now, we have

$$\varepsilon < |f_n(g_n)| = |f_n(T'(e'_n))| \leqslant \sup_{x\in B_{(\ell^\infty)'}} |f_n(T'(x))| \to 0,$$

as $n \to \infty$. We obtain a contradiction.

(2) For the "only if" part, let Y be a Banach space and $T\colon X \to Y$ a DPcc operator. Then $T'(B_{Y'})$ is an L-Dunford-Pettis set, hence $T'(B_{Y'})$ is a relatively compact set. This proves that T' is a compact operator, and hence T is also compact.

For the "if" part, assume by way of contradiction that there exists an L-Dunford-Pettis subset A of X' that is not relatively compact. So there is a sequence $(f_n) \subseteq A$ with no convergent subsequence. It is clear that the operator $T\colon X \to \ell^\infty$ defined by $T(x) = (f_n(x))$ for all $x \in X$ is DPcc. Now, we prove that T is not compact. We have $T'((\lambda_n)_{n=1}^\infty) = \sum_{n=1}^\infty \lambda_n f_n$ for every $(\lambda_n)_{n=1}^\infty \in \ell^1 \subset (\ell^\infty)'$. If e'_n is the usual basis element in ℓ^1 then $T'(e'_n) = f_n$ for all $n \in \mathbb{N}$. Thus, T' is not a compact operator and neither is T. We obtain a contradiction, and we are done. □

For proof of the next proposition, we need the following lemma which is just Lemma 1.3 of [10].

Lemma 2.9. *Let X be a Banach space.*

A sequence (x_n) in X is DP if and only if $f_n(x_n) \to 0$ as $n \to \infty$ for every weakly null sequence (f_n) in X'.

Proposition 2.10. *Let $T\colon E\to F$ be a positive operator between two Banach lattices.*

If T is a weak almost Dunford-Pettis operator then T carries each disjoint weakly null sequence (x_n) in E to a Dunford-Pettis one in F.

Proof. Assume by way of contradiction that there exists a disjoint weakly null sequence (x_n) in E such that $(T(x_n))$ is not a Dunford-Pettis sequence in F. By Lemma 2.9 there exists a weakly null sequence (f_n) in F' such that $f_n(T(x_n))$ does not converge to 0. Then there exist some $\varepsilon>0$ and a subsequence (which we denote by $f_n(T(x_n))$ again) satisfying $|f_n(T(x_n))|>\varepsilon$ for all $n\in\mathbb{N}$. By the inequality $|f_n(T(x_n))|\leqslant|f_n|(T(|x_n|))$ for all $n\in\mathbb{N}$, we get that $|f_n|(T(|x_n|))>\varepsilon$ for all $n\in\mathbb{N}$. As (x_n) is a disjoint weakly null sequence in E, it follows from [11], Remark 1 that $(|x_n|)$ is a weakly null sequence in E, and hence $(T(|x_n|))$ is a weakly null sequence in F. Now, an easy inductive argument shows that there exist a subsequence (z_n) of $(|x_n|)$ and a subsequence (g_n) of (f_n) such that

$$|g_n|(T(z_n))>\varepsilon$$

and

$$4^n\sum_{i=1}^{n}|g_i|(T(z_{n+1}))<\frac{1}{n}$$

for all $n\geqslant1$. Put

$$h=\sum_{n=1}^{\infty}2^{-n}|g_n|$$

and

$$h_n=\left(|g_{n+1}|-4^n\sum_{i=1}^{n}|g_i|-2^{-n}h\right)^+.$$

By [1], Lemma 4.35 the sequence (h_n) is disjoint. Since $0\leqslant h_n\leqslant|g_{n+1}|$ for all $n\geqslant1$ and (g_n) is weakly null in F', it follows from [1], Theorem 4.34 that (h_n) is a weakly null in F'. As T is a weak almost Dunford-Pettis operator, we see that $T(z_{n+1})$ is an almost Dunford-Pettis sequence in F, therefore $h_n(T(z_{n+1}))\to0$ as $n\to\infty$.

On the other hand, we have

$$h_n(T(z_{n+1}))\geqslant\left(|g_{n+1}|-4^n\sum_{i=1}^{n}|g_i|-2^{-n}h\right)(T(z_{n+1}))\geqslant\varepsilon-\frac{1}{n}-2^{-n}h(T(z_{n+1}))$$

and we see that $h_n(T(z_{n+1}))\geqslant\varepsilon/2$ must hold for all n sufficiently large (because $2^{-n}h(T(z_{n+1}))\to0$). This leads to a contradiction, and we are done. □

The next result characterizes positive weak almost Dunford-Pettis operators between two Banach lattices.

Theorem 2.11. *Let $T\colon E \to F$ be a positive operator between two Banach lattices. The following statements are equivalent:*

(1) *T is weak almost Dunford-Pettis;*

(2) *T' carries each L-Dunford-Pettis subset of F' to an almost L-set in E';*

(3) *for an arbitrary Banach space Z and every DPcc operator $S\colon F \to Z$ the product ST is almost Dunford-Pettis;*

(4) *for an arbitrary Banach space Z and every weakly compact operator $S\colon F \to Z$ the product ST is almost Dunford-Pettis;*

(5) *for every weakly compact operator $S\colon F \to c_0$ the product ST is almost Dunford-Pettis.*

Proof. (1) $\Rightarrow$ (2) Let A be an L-Dunford-Pettis set in F', we prove that $T'(A)$ is an almost L-set in E'. Let (x_n) be a disjoint weakly null sequence in E, by our hypothesis and Proposition 2.10 we see that $(T(x_n))$ is a weakly null and Dunford-Pettis set in F. This implies that

$$\sup_{f\in T'(A)} |f(x_n)| = \sup_{g\in A} |g(T(x_n))| \to 0$$

as $n \to \infty$, and we conclude that $T'(A)$ is an almost L-set in E'.

(2) $\Rightarrow$ (3) Let Z be a Banach space and let $S\colon F \to Z$ be a DPcc operator. Then $S'(B_{Z'})$ is an L-Dunford-Pettis subset of F', and by our hypothesis we see that $T'(S'(B_{Z'}))$ is an almost L-set in E'. Thus ST is an almost Dunford-Pettis operator.

(3) $\Rightarrow$ (4) It follows from [10], Corollary 1.1.

(4) $\Rightarrow$ (5) Obvious.

(5) $\Rightarrow$ (1) Let (x_n) be a disjoint weakly null sequence in E and let (f_n) be a disjoint weakly null sequence in F', we prove that $f_n(T(x_n)) \to 0$ as $n \to \infty$. Consider the operator $S\colon F \to c_0$ defined by

$$S(x) = (f_n(x))_{n=1}^{\infty}.$$

Theorem 5.26 of [1] proves that S is a weakly compact operator, and by our hypothesis ST is almost Dunford-Pettis. Since

$$|f_n(T(x_n))| \leqslant \|S(T(x_n))\|_{\infty} \to 0$$

as $n \to \infty$, it follows from [4], Theorem 2.5, assertion (6) that T is weak almost Dunford-Pettis, as desired. □

As a consequence we derive the following characterizations of the weak Dunford-Pettis property.

Corollary 2.12. *Let E be a Banach lattice. The following statements are equivalent:*

(1) *E has the weak Dunford-Pettis property;*
(2) *L-Dunford-Pettis subsets of E' are almost L-sets;*
(3) *every DPcc operator from E into an arbitrary Banach space Z is almost Dunford-Pettis;*
(4) *every weakly compact operator from E into an arbitrary Banach space Z is almost Dunford-Pettis;*
(5) *every weakly compact operator from E into c_0 is almost Dunford-Pettis.*

References

[1] *C. D. Aliprantis, O. Burkinshaw*: Positive Operators. Springer, Berlin, 2006. zbl MR doi
[2] *B. Aqzzouz, K. Bouras*: (L) sets and almost (L) sets in Banach lattices. Quaest. Math. *36* (2013), 107–118. zbl MR doi
[3] *B. Aqzzouz, K. Bouras, M. Moussa*: Duality property for positive weak Dunford-Pettis operators. Int. J. Math. Math. Sci. *2011* (2011), Article ID 609287, 12 pages. zbl MR doi
[4] *K. Bouras, M. Moussa*: Banach lattice with weak Dunford-Pettis property. Int. J. Math. Comput. Phys. Electr. Comput. Eng. *5* (2011), 174–178.
[5] *J. X. Chen, Z. L. Chen, G. X. Ji*: Almost limited sets in Banach lattices. J. Math. Anal. Appl. *412* (2014), 547–553. zbl MR doi
[6] *P. G. Dodds, D. H. Fremlin*: Compact operators in Banach lattices. Isr. J. Math. *34* (1979), 287–320. zbl MR doi
[7] *I. Ghenciu, P. Lewis*: The Dunford-Pettis property, the Gelfand-Phillips property, and *L*-sets. Colloq. Math. *106* (2006), 311–324. zbl MR doi
[8] *P. Meyer-Nieberg*: Banach Lattices. Universitext, Springer, Berlin, 1991. zbl MR doi
[9] *A. Retbi, B. El Wahbi*: L-Dunford-Pettis property in Banach spaces. Methods Func. Anal. Topol. *22* (2016), 387–392. zbl MR
[10] *Y. Wen, J. Chen*: Characterizations of Banach spaces with relatively compact Dunford-Pettis sets. Adv. Math., Beijing *45* (2014), 122–132. zbl MR doi
[11] *W. Wnuk*: Banach lattices with the weak Dunford-Pettis property. Atti Semin. Mat. Fis. Univ. Modena *42* (1994), 227–236. zbl MR

Author's address: Abderrahman Retbi, Université Ibn Tofail, Faculté des Sciences, Département de Mathématiques, Avenue de L'Université, B.P.133, Kénitra, Morocco, e-mail: abderrahmanretbi@hotmail.com.