Applications of Mathematics

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THE COLLATZ-WIELANDT QUOTIENT FOR PAIRS
OF NONNEGATIVE OPERATORS

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Abstract. In this paper we consider two versions of the Collatz-Wielandt quotient for a pair of nonnegative operators A, B that map a given pointed generating cone in the first space into a given pointed generating cone in the second space. If the two spaces and two cones are identical, and B is the identity operator, then one version of this quotient is the spectral radius of A. In some applications, as commodity pricing, power control in wireless networks and quantum information theory, one needs to deal with the Collatz-Wielandt quotient for two nonnegative operators. In this paper we treat the two important cases: a pair of rectangular nonnegative matrices and a pair of completely positive operators. We give a characterization of minimal optimal solutions and polynomially computable bounds on the Collatz-Wielandt quotient.

Keywords: Perron-Frobenius theory; Collatz-Wielandt quotient; completely positive operator; commodity pricing; wireless network; quantum information theory

MSC 2020: 15A22, 15A45, 15B48, 15B57, 94A40

1. Introduction

The celebrated Perron-Frobenius theorem describes important spectral properties of a square matrix A with nonnegative entries [32], [14], [15], [16]. In particular, the spectral radius $\varrho(A)$ (the maximum of absolute values of all eigenvalues of A) is an eigenvalue of A. Furthermore, to $\varrho(A)$ corresponds a nonnegative eigenvector $\mathbf{y}$:

$$(1) \qquad A\mathbf{y} = \varrho(A)\mathbf{y}, \quad \mathbf{y} \gneqq \mathbf{0},$$

which is called the Perron-Frobenius eigenvector, abbreviated as PF-eigenvector. If A is irreducible, then $\mathbf{y} > \mathbf{0}$ and is unique up to multiplication by a positive

scalar. There are many classical and recent books giving a full account of the Perron-Frobenius theory of nonnegative matrices, for example [3], [12], [17], [18], [24], [30], [31], [36]. It is well known that PF-theory found innumerous applications in all sciences. See for example [1], [2], [34], [39] and references therein.

We denote by $\mathbb{R}^{m \times n} \supset \mathbb{R}_+^{m \times n}$ the sets of real valued and nonnegative valued $m \times n$ matrices, respectively. ($\mathbb{R}^m = \mathbb{R}^{m \times 1} \supset \mathbb{R}_+^m = \mathbb{R}_+^{m \times 1}$ the set of column vectors and the subset of nonnegative column vectors with m coordinates, respectively.) Let $[n] = \{1, \ldots, n\}$, $0/0 = 0$, $+/0 = \infty$.

One of the most applicable feature of PF-theory is the Collatz-Wielandt characterization of $\varrho(A)$ for $A \in \mathbb{R}_+^{m \times m}$, see [8], [40]:

$$(2) \qquad \inf_{\mathbf{x} = (x_1, \ldots, x_m)^\top > \mathbf{0}} \max_{i \in [m]} \frac{(A\mathbf{x})_i}{x_i} = \varrho(A).$$

Theorem 6.4.5 in [12] gives a necessary and sufficient conditions on A that the above infimum is achieved for a positive $\mathbf{x}$. In a simple noiseless model in wireless networks $1/\varrho(A)$ is the reception threshold, [39], [34] and [12], §6.9.

Given a pair of nonsquare matrices $A, B \in \mathbb{R}^{m \times n}$ one can consider the generalized eigenvalue problem

$$(3) \qquad A\mathbf{x} = \lambda B\mathbf{x}, \quad A, B \in \mathbb{R}^{m \times n}, \ \lambda \in \mathbb{C}.$$

In order to assure that one has a finite number of eigenvalues, one needs to assume that $\max(\operatorname{rank} A, \operatorname{rank} B) = n$, which implies that $m \geqslant n$. There is an extensive literature on this problem, see for example [9], [4], [7] and references therein. A first attempt to generalize Perron-Frobenius theory to (3), to the best knowledge of the author, is by Mangasarian [28]. He showed the assumption that $B^\top \mathbf{y} \geqslant \mathbf{0}$ implies that $A^\top \mathbf{y} \geqslant \mathbf{0}$ yields that (3) has a discrete and finite spectrum, and the eigenvalue with the largest absolute value is real, nonnegative and the corresponding eigenvector is nonnegative.

The Perron-Frobenius theory was generalized to nonnegative operators A with respect to a closed pointed generating cone $\mathbf{K}$ in finite and infinite dimensional Banach spaces [26], [25], [35], [3]. There is also a natural generalization of the Collatz-Wielandt characterizations to the spectral radius of $\varrho(A)$, see [10], [11].

The aim of this paper is to consider the Collatz-Wielandt type infmax problem for a pair of nonnegative operators $A, B \colon \mathbb{R}^{N_1} \to \mathbb{R}^{N_2}$, with respect to closed pointed generating cones $\mathbf{K}_i \subset \mathbb{R}^{N_i}$ for $i = 1, 2$: $A\mathbf{K}_1, B\mathbf{K}_1 \subseteq \mathbf{K}_2$. Denote by $\mathbf{K}_i^o$ the interior of $\mathbf{K}_i$. Let

$$(4) \qquad r(A, B, \mathbf{x}) = \inf\{t, \ t \in [0, \infty], \ tB\mathbf{x} - A\mathbf{x} \in \mathbf{K}_2\} \quad \text{for } \mathbf{x} \in \mathbf{K}_1 \setminus \{\mathbf{0}\}.$$

We set $r(A, B, \mathbf{x}) = \infty$ if $tB\mathbf{x} - A\mathbf{x} \notin \mathbf{K}_2$ for each $t \geqslant 0$. Define

(5) $$\varrho(A, B) = \inf\{r(A, B, \mathbf{x}), \mathbf{x} \in \mathbf{K}_1^o\}.$$

In general, $\varrho(A, B)$ can have any value in $[0, \infty]$. We call $\varrho(A, B)$ the *Collatz-Wielandt quotient*.

We also consider the following variation of $\varrho(A, B)$:

(6) $$\hat{\varrho}(A, B) = \inf\{r(A, B, \mathbf{x}), \mathbf{x} \in \mathbf{K}_1 \setminus \{\mathbf{0}\}\}.$$

We call $\hat{\varrho}(A, B)$ the *weak Collatz-Wielandt quotient*. Clearly, $\hat{\varrho}(A, B) \leqslant \varrho(A, B)$. Even in the classical case, where $A \in \mathbb{R}_+^{m \times m}$ and B is the identity matrix, one may have the strict inequality $\hat{\varrho}(A, I) < \varrho(A, I) = \varrho(A)$. A simple example is the following one. Assume that A is a direct sum of k irreducible matrices $A_1, \ldots, A_k$, where $\varrho(A_1) > \varrho(A_2) > \ldots > \varrho(A_k)$. (So A is block diagonal $\mathrm{diag}(A_1, \ldots, A_k)$.) Then $\varrho(A, I) = \varrho(A_1)$ and $\hat{\varrho}(A, I) = \varrho(A_k)$. See Section 3. We show that if either A or B are positive, then the Collatz-Wielandt quotient and the weak Collatz-Wielandt quotient are equal. Furthermore, we have the following stability results. Suppose that we have two sequences of positive matrices A_l and B_l that converge to A and B, respectively. Then $\lim_{l \to \infty} \varrho(A, B_l) = \hat{\varrho}(A, B)$, and $\lim_{l \to \infty} \varrho(A_l, B) = \varrho(A, B)$ provided that B does not have a zero row. Thus, the Collatz-Wielandt quotient and the weak Collatz-Wielandt quotient seem to be equally important quantities.

In the first part of this paper we consider the Collatz-Wielandt quotient for a pair of rectangular nonnegative matrices $A, B \in \mathbb{R}_+^{m \times n}$, i.e.:

(7) $$\varrho(A, B) := \inf_{\mathbf{x} = (x_1, \ldots, x_n)^\top > \mathbf{0}} \max_{i \in [m]} \frac{(A\mathbf{x})_i}{(B\mathbf{x})_i}.$$

(So $\mathbf{K}_i = \mathbb{R}_+^{N_i}$ for $i = 1, 2$ and $N_1 = n$, $N_2 = m$.)

We now give a simple model of commodity pricing, where the above Collatz-Wielandt ratio arises. (Another example in wireless networks discussed in [1], [2] is discussed in Section 6.) Assume that we have m producers of commodities which produce n commodities. Each producer i produces a subset of commodities $C(i) \subset [n]$. (We do not exclude the possibility that two producers produce the same commodity j.) Assume that the price of commodity j is $x_j > 0$. Then $\mathbf{x} = (x_1, \ldots, x_n)^\top > \mathbf{0}$ is the pricing vector. The expected value of the cost of producer i for one unit of his products is $\sum_{j=1}^n a_{ij} x_j$. The expected value of the profit of producer i for one unit is $\sum_{j=1}^n b_{ij} x_j$. One can impose the obvious conditions that $a_{ij} = 0$ if $j \in C(i)$ (producer i does not buy the commodity it produces), and $b_{ij} = 0$ if $j \notin C(i)$ (the producer sells

only the items it produces). Then the ratio of the profit to the expense for producer i is $(B\mathbf{x})_i/(A\mathbf{x})_i$. We call this ratio *profit factor*. In order that each producer will stay in business for the pricing vector $\mathbf{x}$ one needs to satisfy the minimum profit factor requirement: $\min_{i \in [m]} (B\mathbf{x})_i/(A\mathbf{x})_i \geqslant \beta$. Then the optimal pricing choice is the solution to the supremum problem

$$\sup_{\mathbf{x}>\mathbf{0}} \min_{i\in[m]} \frac{(B\mathbf{x})_i}{(A\mathbf{x})_i} = \frac{1}{\varrho(A,B)}.$$

We are interested in a nontrivial case, where $\varrho(A,B) < \infty$. It is easy to show that this inequality holds if and only if the following condition is satisfied: For each zero row i of B row i of A is zero.

We now summarize our results for the extremal problem (7). Assume that $\varrho(A,B) \in (0,\infty)$. (It is easy to characterize the case $\varrho(A,B) = 0$.) Then there exists $\mathbf{y} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ such that $\varrho(A,B) = r(A,B,\mathbf{y})$ with the following property: There exists a sequence $\mathbf{y}_k > \mathbf{0}$ for $k \in \mathbb{N}$ such that $\lim_{k\to\infty} \mathbf{y}_k = \mathbf{y}$ and $\lim_{k\to\infty} r(A,B,\mathbf{y}_k) = \varrho(A,B) = r(A,B,\mathbf{y})$. Such $\mathbf{y}$ is called an *optimal* $\mathbf{y}$. An optimal vector $\mathbf{y}$ is called *minimal optimal* if $\mathbf{y}$ is an optimal vector, and there is no optimal vector $\mathbf{z}$ whose support is strictly contained in the support of $\mathbf{y}$. We call $\mathbf{y}$ a *generalized Perron-Frobenius vector*, abbreviated as GPF-eigenvector, if

$$(8) \qquad\qquad A\mathbf{y} = \varrho(A,B)B\mathbf{y}, \quad \mathbf{y} \gneq \mathbf{0}.$$

Note that if (3) has an eigenvector $\mathbf{x} > \mathbf{0}$ with a corresponding $\lambda > 0$, then $\varrho(A,B) \leqslant \lambda$ and it is easy to give examples where $\varrho(A,B) < \lambda$ and each optimal $\mathbf{y}$ is not a GPF-eigenvector. (See the example in the end of Section 5.) We next show, as briefly pointed in [2], that for any $\varepsilon \in (0,1)$ we can find one of the following: Either $\varrho(A,B) < \varepsilon$ or we can find an approximation of $\varrho(A,B,\varepsilon)$ such that $|\varrho(A,B,\varepsilon) - \varrho(A,B)| \leqslant \varepsilon\varrho(A,B)$, in polynomial time. This follows from the well known fact that a solvability of linear system of equations is polynomial in the data, see [20], [27].

We show that each minimal optimal $\mathbf{y}$ has at most m positive coordinates. The existence of an optimal $\mathbf{y}$ with at most $m+1$ positive coordinates in a general setting is shown in [2]. Furthermore, if there exists an optimal vector with $l \geqslant m$ positive coordinates, then the rank of the matrix $A' - \varrho(A,B)B'$ is less than m. (Here $A', B' \in \mathbb{R}^{m\times l}_+$ are the submatrices induced by l positive entries of $\mathbf{y}$.) This result implies that for each minimal optimal $\mathbf{y}'$ with l positive coordinates there exists a minimal optimal $\mathbf{y}$ with the same support as $\mathbf{y}$ such that $(A\mathbf{y} - \varrho(A,B)B\mathbf{y})_i = 0$ for at least l indices $i \in [m]$. That is, there exists a minimal optimal solution that is

a GPF-eigenvector of the system $\widetilde{A}\mathbf{y} = \varrho(A, B)\widetilde{B}\mathbf{y}$, where $\widetilde{A}$, $\widetilde{B}$ are the submatrices of A, B obtained from A, B by erasing a set of rows $\mathcal{I}$ in A, B, respectively. For the optimal commodity pricing model that we introduced above, the above results have the following meaning: First each zero coordinate j of $\mathbf{y}$ implies that the commodity j is not produced. The producers corresponding to the set $\mathcal{I}$ have their profit ratio above $1/\varrho(A, B)$. For all other producers the profit ratio is $1/\varrho(A, B)$. Similar results are shown for $\hat{\varrho}(A, B)$.

We also give the following generalization of the main result in [2]. Namely, if B has no zero row and each column has one positive element, then there is an optimal solution which is a GPF-eigenvector. That is, in the wireless model of transmitters-receivers, where each receiver i can obtain a signal from several transmitters, which can only transmit to the receiver i, there is a choice to pick exactly one transmitter $j(i)$. (However, if the system is not irreducible, as defined in [2], this choice would imply that some other transmitters to receiver i' should be shut off.)

The second part of this paper is a generalization of the above results to pairs of completely positive operators, which frequently appear in quantum information theory as quantum channels. Denote by $H_n \supset H_{+,n} \supset H_{+,1,n}$ the real space of $n \times n$ Hermitian matrices, the cone of positive semidefinite matrices and the convex set of positive semidefinite matrices of trace one, respectively. Note that $H_{+,n}$ is a pointed generating cone in $H_n \equiv \mathbb{R}^{n^2}$. In quantum information theory (QIT), $H_{+,1,n}$ is the set of density matrices (mixed states). Recall that $\mathcal{C} \colon H_n \to H_m$ is called a *completely positive operator*, abbreviated as CP-operator, if

$$(9) \qquad \mathcal{C}(X) = \sum_{j=1}^{k} T_j X T_j^*, \quad T_j \in \mathbb{C}^{m \times n}, \; j \in [k].$$

(Here $\mathbb{C}^{m \times n}$ is the space of $m \times n$ complex valued matrices and $T^* = \overline{T}^{\top}$ for $T \in \mathbb{C}^{m \times n}$.) Then $\mathcal{C}(H_{+,n}) \subseteq H_{+,m}$, that is, $\mathcal{C}$ is a nonnegative operator with respect to the pair of cones $H_{+,n}, H_{+,m}$. In QIT, $\mathcal{C}$ is called quantum channel if

$$(10) \qquad \sum_{j=1}^{k} T_j^* T_j = I_n.$$

That is, $\mathcal{C}$ is a quantum channel if and only if $\mathcal{C}$ is a CP trace preserving operator. In particular, $\mathcal{C}$ maps a density matrix to a density matrix. Quantum channel is one of the most significant notions in QIT, see [38], [22], [37], [21], [29], [23], [33]. The second main problem we discuss are $\varrho(\mathcal{A}, \mathcal{B})$ and $\hat{\varrho}(\mathcal{A}, \mathcal{B})$ for two CP-operators $\mathcal{A}, \mathcal{B} \colon H_{+,n} \to H_{+,m}$. The quantities $\varrho(\mathcal{A}, \mathcal{B})$ and $\hat{\varrho}(\mathcal{A}, \mathcal{B})$ could be viewed as the quantum analog of the optimal commodity pricing assignment discussed above. We show that most of

our results on $\hat{\varrho}(A, B)$ generalize to $\hat{\varrho}(\mathcal{A}, \mathcal{B})$, and some results on $\varrho(A, B)$ generalize to $\varrho(\mathcal{A}, \mathcal{B})$. We show that there exists weak optimal $Y \in \mathrm{H}_{+,n} \setminus \{0\}$ such that $r(\mathcal{A}, \mathcal{B}, Y) = \hat{\varrho}(\mathcal{A}, \mathcal{B})$. A weak optimal Y is called *minimal* if there is no optimal Z such that range Z is strictly contained in range Y. We show that a minimal weak optimal Y has rank at most m. Assume that Y' is a minimal weak optimal with rank l. Then there exists a minimal weak optimal Y satisfying range $Y' =$ range Y such that rank $(\hat{\varrho}(\mathcal{A}, \mathcal{B})\mathcal{B} - \mathcal{A})(Y) \leqslant m - l$. In particular, if $l = m$, then Y is a weak GPF-eigenvector. Assume that $\mathcal{B}$ is δ-positive for a given rational $\delta > 0$. (This assumption can be verified in polynomial time.) Then $\varrho(\mathcal{A}, \mathcal{B}) = \hat{\varrho}(\mathcal{A}, \mathcal{B})$. Furthermore, $\varrho(\mathcal{A}, \mathcal{B})$ has an $\varepsilon \in (0, 1)$ approximation in polynomial time in $\langle \mathcal{A} \rangle + \langle \mathcal{B} \rangle + \langle \delta \rangle + \langle \varepsilon \rangle$. (We need the assumption that $\mathcal{B}$ is δ-positive because verifying the existence of a nonzero positive semidefinite matrix satisfying $(\mathcal{A} - t\mathcal{B})(X) \leqslant 0$ is a feasibility problem in semidefinite programming, which may not be polynomially solvable.)

We now survey briefly the content of the paper. Section 2 discusses basic properties of $\varrho(A, B)$ and $\hat{\varrho}(A, B)$. Section 3 discusses that classical case of the pair $A, B \in \mathbb{R}_+^{m \times n}$, where $m = n$ and B is the idenity matrix I. We show that $\varrho(A, I) = \varrho(A)$. If A is not irreducible, then one may have the strict inequality $\hat{\varrho}(A, I) < \varrho(A, I)$. We characterize completely $\hat{\varrho}(A, I)$. In Section 4 we give a polynomial time approximation algorithm to $\varrho(A, B)$ and $\hat{\varrho}(A, B)$. In Section 5 we give various properties of minimal optimal and minimal weak optimal vectors for the pair $A, B \in \mathbb{R}_+^{m \times n}$. In Section 6 we discuss WN-pairs $A, B \in \mathbb{R}_+^{m \times n}$ arising in wireless network. That is, B has no zero row and one positive element in each column. Such pairs were introduced and studied in [1], [2]. We give generalizations of the results in [1], [2], since we do not restrict ourselves to S-irreducible systems. Sections 7, 8 and 9 are devoted to a study of the Collatz-Wielandt quotients for pairs of completely positive operators.

2. PRELIMINARY RESULTS

For a positive integer n let $\mathbf{1}_n = (1, \ldots, 1)^\top \in \mathbb{R}^n$. Let $S \subseteq [m], T \subseteq [n]$. Denote by $\mathbf{1}_S = (x_1, \ldots, x_m)^\top \in \mathbb{R}_+^m$ the characteristic vector of S, i.e. $x_i = 1$ if $i \in S$ and $x_i = 0$ otherwise. So $\mathbf{1}_\emptyset = \mathbf{0}, \mathbf{1}_{[m]} = \mathbf{1}_m$. Assume that $F \in \mathbb{R}^{m \times n}$. Denote by $F(S, T)$ the matrix obtained from F be deleting the rows of F in the set S and the columns in the set T. So $F(S, T) \in \mathbb{R}^{(m-|S|) \times (n-|T|)}$. If either $S = [m]$ or $T = [n]$, we denote $F(S, T)$ by $\emptyset$. Denote by $F[S, T]$ the matrix $F([m] \setminus S, [n] \setminus T)$.

Assume that $A, B \in \mathbb{R}_+^{m \times n}$. For $\mathbf{x} = (x_1, \ldots, x_n)^\top \gvertneqq \mathbf{0}$ we define

$$r(A, B, \mathbf{x}) = \inf\{t, \ t \geqslant 0, \ A\mathbf{x} \leqslant tB\mathbf{x}\}.$$

Note that $r(A, B, \mathbf{x}) \in [0, \infty]$. That is, $r(A, B, \mathbf{x}) = \infty$ if and only if there exists $i \in [m]$ such that $(A\mathbf{x})_i > 0$ and $(B\mathbf{x})_i = 0$. Equivalently

$$r(A, B, \mathbf{x}) = \max\left\{\frac{(A\mathbf{x})_i}{(B\mathbf{x})_i}, \ i \in [m]\right\}.$$

Hence,

(11) $\quad \varrho(A, B) = \inf\{r(A, B, \mathbf{x}), \ \mathbf{x} > \mathbf{0}\}, \ \hat{\varrho}(A, B) = \inf\{r(A, B, \mathbf{x}), \ \mathbf{x} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}\}.$

The following lemma is deduced straightforwardly.

Lemma 2.1. *Assume that* $A, B \in \mathbb{R}^{m \times n}_+$.

(1) $\varrho(0, B) = \hat{\varrho}(0, B) = 0.$

(2) $\varrho(A, B) = \infty$ *if and only if* B *has a zero row* i, *while the* i*th row of* A *is not zero.*

(3) *Let* S *be a strict subset of* $[m]$. *Then for each* $\mathbf{x} \gneqq \mathbf{0}$, $r(A(S, \emptyset), B(S, \emptyset), \mathbf{x}) \leqslant r(A, B, \mathbf{x})$. *In particular,* $\varrho(A, B) \geqslant \varrho(A(S, \emptyset), B(S, \emptyset))$ *and* $\hat{\varrho}(A, B) \geqslant \hat{\varrho}(A(S, \emptyset), B(S, \emptyset))$. *Suppose furthermore that for each* $i \in S$ *the row* i *of* A *and* B *are zero. Then for each* $\mathbf{x} \gneqq \mathbf{0}$, $r(A(S, \emptyset), B(S, \emptyset), \mathbf{x}) = r(A, B, \mathbf{x})$. *In particular,* $\varrho(A, B) = \varrho(A(S, \emptyset), B(S, \emptyset))$ *and* $\hat{\varrho}(A, B) = \hat{\varrho}(A(S, \emptyset), B(S, \emptyset))$.

(4) *Let* T *be a strict subset of* $[n]$ *such that* $A[[m], T] = B[[m], T] = 0$. *Then* $\varrho(A, B) = \varrho(A(\emptyset, T), B(\emptyset, T))$ *and* $\hat{\varrho}(A, B) = \hat{\varrho}(A(\emptyset, T), B(\emptyset, T))$.

(5) *Suppose that for each zero row* i *of* B *the row* i *of* A *is zero. Then* $r(A, B, \mathbf{1}_n) < \infty$.

(6) *There exists* $\mathbf{x} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ *such that* $r(A, B, \mathbf{x}) < \infty$ *if and only if there exists a nonempty subset* $T \subseteq [n]$ *such that* $r(A, B, \mathbf{1}_T) < \infty$.

(7) *The weak Collatz-Wielandt quotient is positive if and only if the union of the supports of the rows of* A *is* $[n]$, *i.e.* $A^\top \mathbf{1}_m > \mathbf{0}$.

(8) *Assume that* $A_1, B_1 \in \mathbb{R}^{m \times n}_+$ *and* $A_1 \leqslant A, B \leqslant B_1$. *Then* $\varrho(A_1, B_1) \leqslant \varrho(A, B)$ *and* $\hat{\varrho}(A_1, B_1) \leqslant \hat{\varrho}(A, B)$.

The following lemma gives a lower bound on $\hat{\varrho}(A, B)$:

Lemma 2.2. *Let* $A = [a_{ij}], B = [b_{ij}] \in \mathbb{R}^{m \times n}_+$. *Then*

(12) $$\hat{\varrho}(A, B) \geqslant \min_{j \in [n]} \frac{\sum_{i=1}^m a_{ij}}{\sum_{i=1}^m b_{ij}}.$$

P r o o f. Clearly, it is enough to assume that $\hat{\varrho}(A, B) < \infty$. Assume that $\mathbf{x} \gneqq \mathbf{0}$ and $r(A, B, \mathbf{x}) < \infty$. Observe that

$$r(A, B, \mathbf{x}) \geqslant \frac{\sum_{i=1}^m (A\mathbf{x})_i}{\sum_{i=1}^m (B\mathbf{x})_i} = \frac{\mathbf{1}_m A\mathbf{x}}{\mathbf{1}_m B\mathbf{x}} = \frac{\sum_{j=1}^n (\mathbf{1}_m A)_j x_j}{\sum_{j=1}^n (\mathbf{1}_m B)_j x_j} \geqslant \min_{j \in [n]} \frac{(\mathbf{1}_m A)_j}{(\mathbf{1}_m B)_j}.$$

$\square$

Lemma 2.3. *Assume that $A, B \in \mathbb{R}_+^{m \times n}$. Let $\mathcal{I} \subseteq [m]$ be the set of the zero rows of B. Denote by T the union of the supports of the rows $i \in \mathcal{I}$ of A. Then one of the following conditions holds:*

(1) *If $\mathcal{I} = \emptyset$, then $\varrho(A, B) \leqslant r(A, B, \mathbf{1}_n) < \infty$.*
(2) *If $T = [n]$, then $\hat\varrho(A, B) = \infty$.*
(3) *Assume that $\mathcal{I} \neq \emptyset$ and T is a strict subset of $[n]$. Then $\hat\varrho(A, B) = \hat\varrho(A(\mathcal{I}, T), B(\mathcal{I}, T))$.*

P r o o f.   (1) If $\mathcal{I} = \emptyset$, then $(B\mathbf{1}_n)_i > 0$ for each i and $r(A, B, \mathbf{1}_n) = \max\limits_{i \in [m]} (A\mathbf{1}_n)_i / (B\mathbf{1}_n)_i < \infty$.

(2) Suppose that $T = [n]$. Let $\mathbf{x} \gneq \mathbf{0}$. Then there exists $i \in \mathcal{I}$ such that $(A\mathbf{x})_i > 0$ and $(B\mathbf{x})_i = 0$. Hence, $r(A, B, \mathbf{x}) = \infty$, which yields that $\hat\varrho(A, B) = \infty$.

(3) Suppose that $\mathcal{I} \neq \emptyset$, $T \subset [n]$. Let $\mathbf{x} \gneq \mathbf{0}$ and assume that $(\text{supp } \mathbf{x}) \cap T \neq \emptyset$. Then there exists $i \in \mathcal{I}$ such that $(A\mathbf{x})_i > 0$. As $(B\mathbf{x})_i = 0$, it follows that $r(A, B, \mathbf{x}) = \infty$. Hence, to determine $\hat\varrho(A, B)$ it is enough to consider $\inf\{r(A, B, \mathbf{x}), \ \mathbf{x} \gneq \mathbf{0}, \ \text{supp } \mathbf{x} \subseteq [n] \setminus T\}$. Note that if $\text{supp } \mathbf{x} \subseteq [n] \setminus T$, then $(A\mathbf{x})_i = (B\mathbf{x})_i = 0$ for $i \in \mathcal{I}$. Therefore, $\hat\varrho(A, B) = \hat\varrho(A(\mathcal{I}, T), B(\mathcal{I}, T))$.   □

The above lemma gives rise to a polynomial time algorithm to check if $\hat\varrho(A, B)$ is finite or infinite:

A l g o r i t h m 2.4.   Given $A = [a_{ij}]$, $B = [b_{ij}] \in \mathbb{R}_+^{m \times n}$, set $S = [n]$, $\mathcal{I} = \mathcal{J} \subseteq [m]$ the set of zero rows of B and T the union of the supports of the rows $i \in \mathcal{I}$ of A;
While $\mathcal{I} \neq \emptyset$ and $T \neq S$
  Replace A, B, S by $A(\mathcal{I}, T), B(\mathcal{I}, T), S \setminus T$;
  Replace $\mathcal{I}$ by the set of zero rows of B;
  Replace T by the union of the supports of the rows $i \in \mathcal{I}$ of A;
  Replace $\mathcal{J}$ by $\mathcal{I} \cup \mathcal{J}$;
Else
  If $\mathcal{I} = \emptyset$, then $\hat\varrho(A, B) \leqslant r(A, B, 1_S) < \infty$ and stop;
  If $T = S$, then $\hat\varrho(A, B) = \infty$ and stop;

Denote by $\Pi_n \subset \mathbb{R}_+^n$ the set of probability vectors on $\mathbb{R}_+^n$. Let Π_n^o be the interior of Π_n, i.e. all probability vectors with positive coordinates.

We now discuss some properties of $\varrho(A, B)$ and $\hat\varrho(A, B)$.

Lemma 2.5. *Let $A, B \in \mathbb{R}_+^{m \times n}$.*

(1) *The function $r(A, B, \mathbf{x})$ is lower semicontinuous on $\mathbb{R}_+^{m \times n} \times \mathbb{R}_+^{m \times n} \times (\mathbb{R}_+^n \setminus \{\mathbf{0}\})$.*
(2) *Let $T \subset [n]$ be a nonempty subset such that $B\mathbf{1}_T > \mathbf{0}$. Assume that $\mathbf{x} \in \mathbb{R}_+^n$, $\sup \mathbf{y} = T$. Suppose that the sequence $\mathbf{0} < \mathbf{x}_k \in \mathbb{R}^n$, $k \in \mathbb{N}$, converges to $\mathbf{x}$. Then $\lim\limits_{k \to \infty} r(A, B, \mathbf{x}_k) = r(A, B, \mathbf{x})$. In particular, $\varrho(A[[m], T], B[[m], T]) \geqslant \varrho(A, B)$.*

(3) *Assume that*

$$A = [A_1 \, A_2], \ A_1 = \begin{bmatrix} A_{11} \\ 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}, \ B = [B_1 \, B_2],$$

$$B_1 = \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}, \ B_2 = \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix},$$

where $A_{11}, B_{11} \in \mathbb{R}^{m' \times l}$, $1 \leqslant m' < m$, $1 \leqslant l < n$, and $B_{11}\mathbf{1}_l > \mathbf{0}$. Suppose that $\varrho(A_{11}, B_{11}) < \varrho(A, B)$. Then $\varrho(A_{22}, B_{22}) = \varrho(A, B)$.

P r o o f. (1) Suppose that $A_k, B_k \in \mathbb{R}_+^{m \times n}$, $\mathbf{x}_k \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$, $k \in \mathbb{N}$ and assume that

$$\lim_{k \to \infty} A_k = A, \quad \lim_{k \to \infty} B_k = B, \quad \lim_{k \to \infty} \mathbf{x}_k = \mathbf{x} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}.$$

Suppose that $t = \liminf_{k \to \infty} r(A_k, B_k, \mathbf{x}_k) \in [0, \infty]$. If $t = \infty$, then $r(A, B, \mathbf{x}) \leqslant t$. Assume that $t \in [0, \infty)$. By passing to subsequences we can assume without loss of generality that $\lim_{k \to \infty} r(A_k, B_k, \mathbf{x}_k) = t$. As $A_k \mathbf{x}_k \leqslant r(A_k, B_k, \mathbf{x}_k) \mathbf{x}_k$, we deduce that $A\mathbf{x} \leqslant tB\mathbf{x}$. Hence, $r(A, B, \mathbf{x}) \leqslant t$.

(2) As $B\mathbf{1}_T > \mathbf{0}$ and $\sup \mathbf{x} = T$, we deduce that $B\mathbf{x} > \mathbf{0}$. Therefore,

$$\varrho(A, B) \leqslant \lim_{k \to \infty} r(A, B, \mathbf{x}_k) = \lim_{k \to \infty} \max_{i \in [m]} \frac{(A\mathbf{x}_k)_i}{(B\mathbf{x}_k)_i} = \max_{i \in [m]} \frac{(A\mathbf{x})_i}{(B\mathbf{x})_i} = r(A, B, \mathbf{x}).$$

Let $\mathbf{z} \in \mathbb{R}^{|T|}$ be the projection of $\mathbf{x}$ on its support. Then $\mathbf{z} > \mathbf{0}$ and $r(A, B, \mathbf{x}) = r(A[[m], T], B[[m], T], \mathbf{z})$. Hence $\varrho(A[[m], T], B[[m], T]) \geqslant \varrho(A, B)$.

(3) Assume that $\varrho(A_{11}, B_{11}) < t_0 = \varrho(A, B)$. Suppose to the contrary that $\varrho(A_{22}, B_{22}) < t_0$. Let $\mathbf{0} < \mathbf{z} \in \mathbb{R}^l$, $\mathbf{0} < \mathbf{u} \in \mathbb{R}^{n-l}$ such that $r(A_{11}, B_{11}, \mathbf{z}) < t_0, r(A_{22}, B_{22}, \mathbf{u}) < t_0$. For $s > 0$ let $\mathbf{x}(s) = (\mathbf{z}^\top, s\mathbf{u}^\top)^\top$. As $B_{11}\mathbf{z} > \mathbf{0}$, there exists $K > 0$ such that $B_{11}\mathbf{z} \geqslant KA_{12}\mathbf{u}$. Then

$$A_{11}\mathbf{z} + sA_{12}\mathbf{u} \leqslant r(A_{11}, B_{11}, \mathbf{z})B_{11}\mathbf{z} + sKB_{11}\mathbf{z} = (r(A_{11}, B_{11} + sK)B_{11}\mathbf{z},$$

$$A_{22}s\mathbf{u} \leqslant r(A_{22}, B_{22})s\mathbf{u}.$$

Thus, for s small enough we have the inequality $r(A, B, \mathbf{x}(s)) < t_0$, which is a contradiction. $\square$

Lemma 2.6. *Let $m, n \geqslant 1$ be integers. Assume that $A, B \in \mathbb{R}_+^{m \times n}$. Then:*

(1) *There exists $\mathbf{y} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ such that $\hat{\varrho}(A, B) = r(A, B, \mathbf{y})$. (Such $\mathbf{y}$ is called a weak optimal $\mathbf{y}$.)*

(2) *There exists $\mathbf{y} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ such that $\varrho(A, B) = r(A, B, \mathbf{y})$ with the following property: There exists a sequence $\mathbf{y}_k > \mathbf{0}$ for $k \in \mathbb{N}$ such that $\lim_{k \to \infty} \mathbf{y}_k = \mathbf{y}$ and $\lim_{k \to \infty} r(A, B, \mathbf{y}_k) = \varrho(A, B) = r(A, B, \mathbf{y})$. (Such $\mathbf{y}$ is called an optimal $\mathbf{y}$.)*

P r o o f.   (1) If $\hat{\varrho}(A, B) = \infty$, then each $\mathbf{x} \in \Pi_n$ is weak optimal. Assume that $\hat{\varrho}(A, B) < \infty$. Choose a sequence of $\mathbf{y}_k \in \Pi_n$ such that $t_k := r(A, B, \mathbf{y}_k) \in (0, \infty)$, $k \in \mathbb{N}$, $t_k \geqslant t_{k+1}$ for $k \in \mathbb{N}$, such that $\lim_{k \to \infty} = \hat{\varrho}(A, B)$. Pick up a subsequence of $\mathbf{y}_k$ which converges to $\mathbf{y} \in \Pi_n$. Part (1) of Lemma 2.5 yields that $r(A, B, \mathbf{y}) \leqslant \hat{\varrho}(A, B)$. Thus $r(A, B, \mathbf{y}) = \hat{\varrho}(A, B)$ and $\mathbf{y}$ is weak optimal.

(2) If $\varrho(A, B) = \infty$, then each $\mathbf{x} \in \Pi_n^\circ$ is optimal. Assume that $\varrho(A, B) < \infty$. Without loss of generality we may assume that B does not have a zero row. We show by induction on n that there exists an optimal $\mathbf{y}$. For $n = 1$ this claim is trivial. Assume that the claim holds for $n \leqslant N$. Suppose that $n = N + 1$. There exists a sequence of $\mathbf{y}_k \in \Pi_n^\circ$ such that $t_k := r(A, B, \mathbf{y}_k) \in (0, \infty)$, $k \in \mathbb{N}$, $t_k \geqslant t_{k+1}$ for $k \in \mathbb{N}$, such that $\lim_{k \to \infty} = \varrho(A, B)$. Pick up a subsequence of $\mathbf{y}_k$ which converges to $\mathbf{w} \in \Pi_n$. Part (1) of Lemma 2.5 yields that $r(A, B, \mathbf{w}) \leqslant \varrho(A, B)$. If $\varrho(A, B) = 0$, we deduce that $\mathbf{w}$ is optimal. Assume that $\varrho(A, B) > 0$. If $r(A, B, \mathbf{w}) = \varrho(A, B)$, then $\mathbf{w}$ is minimal. Assume that $r(A, B, \mathbf{w}) < \varrho(A, B)$. Hence, $T = \mathrm{supp}\, \mathbf{w}$ is a strict subset of $[n]$. Furthermore, part (2) of Lemma 2.5 yields that $B\mathbf{1}_T$ is not positive.

By relabeling the elements of $[n]$ we can assume that $T = [l]$ for some $l \in [n-1]$. Let $\mathcal{K} = \{l+1, \ldots, n\}$ and denote

$$A_1 = A[[m], [l]], \;\; B_1 = B[[m], [l]], \;\; A_2 = A[[m], \mathcal{K}], \;\; B_2 = B[[m], \mathcal{K}].$$

Let $\mathcal{I}$ be the set of zero rows of B_1. As $B_1\mathbf{1}_l$ is not positive, $|\mathcal{I}| \geqslant 1$. As $r(A, B, \mathbf{w}) < \infty$, we deduce that $\mathcal{I}$ is a subset of zero rows of A_1. Let $\mathbf{0} < \mathbf{z} \in \mathbb{R}^l$ be the projection of $\mathbf{w}$ on its support. Relabel the rows of A and B so that $\mathcal{I} = \{m'+1, \ldots, m\}$.

Assume first that $m' = 0$, i.e. $\mathcal{I} = [m]$. So $A_1 = B_1 = 0$. Part 4 of Lemma 2.1 yields that $\varrho(A, B) = \varrho(A_2, B_2)$. As $|\mathcal{K}| = n - l < n$, we can apply the induction hypothesis to (A_2, B_2) to deduce the existence of an optimal $\mathbf{u} \in \Pi_{n-l}$, $\mathrm{supp}\, \mathbf{y} \subseteq \mathcal{K}$. That is, there exists a sequence $\mathbf{0} < \mathbf{u}_k \in \mathbb{R}^{n-l}$, $k \in \mathbb{N}$ such that $\lim_{k \to \infty} \mathbf{u}_k = \mathbf{u}$ and $\lim_{k \to \infty} r(A_2, B_2, \mathbf{u}_k) = \varrho(A_2, B_2)$. Choose a sequence $\mathbf{0} < \mathbf{z}_k \in \mathbb{R}^l$ such that $\lim_{k \to \infty} \mathbf{z}_k = \mathbf{0}$. Let $\mathbf{v}_k = (\mathbf{z}_k^\top, \mathbf{u}_k^\top) > \mathbf{0}$ for $k \in \mathbb{N}$. Clearly $r(A, B, \mathbf{v}_k) = r(A_2, B_2, \mathbf{u}_k)$. Hence $\mathbf{y} = (\mathbf{0}^\top, \mathbf{u}^\top)^\top)$ is optimal.

Assume that $m' = m - |\mathcal{I}| \in [m-1]$. It now follows that the conditions of part (3) of Lemma 2.5 hold. The induction hypothesis yields that there exists $\mathbf{u} \in \mathbb{R}_+^{n-l} \setminus \{\mathbf{0}\}$ such that $r(A_{22}, B_{22}, \mathbf{u}) = \varrho(A_{22}, B_{22}) = \varrho(A, B)$. Furthermore, there exists a sequence $\mathbf{0} < \mathbf{v}_k \in \mathbb{R}^{n-l}$, $k \in \mathbb{N}$ such that $\lim_{k \to \infty} \mathbf{v}_k = \mathbf{v}$ and $\lim_{k \to \infty} r(A_2, B_2, \mathbf{v}_k) = \varrho(A_2, B_2)$. For $s > 0$ let $\mathbf{v}(s) = (\mathbf{z}^\top, s\mathbf{v})^\top$. As in the proof of part (3) of Lemma 2.5 we deduce $r(A, B, \mathbf{v}(s)) \leqslant \max(r(A_{11}, B_{11}, \mathbf{z}) + Ks, \varrho(A, B))$. Choose $s_0 > 0$ such

that $r(A_{11}, B_{11}, \mathbf{z}) + Ks_0 < \varrho(A, B)$. Then $r(A, B, \mathbf{v}(s_0)) = \varrho(A_2, B_2) = \varrho(A, B)$.
Set $\mathbf{v}_k = (\mathbf{z}^\top, s_0 \mathbf{u}_k^\top)^\top > \mathbf{0}, k \in \mathbb{N}$. So $\lim_{k \to \infty} \mathbf{v}_k = \mathbf{v}(t_0)$. Choose $K' > K$ such that
$r(A_{11}, B_{11}, \mathbf{z}) + K's_0 < \varrho(A, B)$. As $\lim_{k \to \infty} \mathbf{u}_k \to \mathbf{u}$, it follows that there exists N such
that $A_{12}\mathbf{u}_k \leqslant K'B\mathbf{z}$ for $k > N$. Therefore, $r(A, B, \mathbf{v}_k) = \varrho(A_2, B_2, \mathbf{u}_k)$ for $k > N$.
This implies that $\mathbf{v}(s_0)$ is optimal. $\qquad\square$

We call $\mathbf{y}$ a *minimal (weak) optimal* if $\mathbf{y}$ is (weak) optimal and there is no (weak) optimal $\mathbf{w}$ such that the support of $\mathbf{w}$ is strictly contained in the support of $\mathbf{y}$. A vector $\mathbf{y} \gneqq \mathbf{0}$ is called a *weak GPF-eigenvector* if $A\mathbf{y} = \hat{\varrho}(A, B)\mathbf{y}$.

The next lemma discusses connections between $\varrho(A, B)$ and $\hat{\varrho}(A, B)$.

Lemma 2.7. *Let $A, B \in \mathbb{R}_+^{m,n}$. Then:*

(1) $\hat{\varrho}(A, B) \leqslant \varrho(A, B)$.

(2) *Let $\mathbf{y}$ be weak optimal. If either $A\mathbf{y} > \mathbf{0}$ or $B\mathbf{y} > \mathbf{0}$, then $\varrho(A, B) = \hat{\varrho}(A, B)$ and $\mathbf{y}$ is optimal.*

(3) *Assume that either $A > 0$ or $B > 0$. Then $\varrho(A, B) = \hat{\varrho}(A, B)$ and each weak optimal $\mathbf{y}$ is optimal.*

(4) *Assume that $0 < B_l \in \mathbb{R}^{m \times n}$ for $l \in \mathbb{N}$ and $\lim_{l \to \infty} B_l = B$. Then $\lim_{l \to \infty} \varrho(A, B_l) = \hat{\varrho}(A, B)$.*

(5) *Assume that $0 < A_l \in \mathbb{R}^{m \times n}$ for $l \in \mathbb{N}$ and $\lim_{l \to \infty} A_l = A$. If B does not have a zero row, then $\lim_{l \to \infty} \varrho(A_l, B) = \varrho(A, B)$.*

P r o o f.   (1) The inequality $\hat{\varrho}(A, B) \leqslant \varrho(A, B)$ is clear.

(2) Let $\mathbf{y} \in \Pi_n$ be weak optimal. So $r(A, B, \mathbf{y}) = \hat{\varrho}(A, B)$. Assume first that $B\mathbf{y} > \mathbf{0}$. For $t > 0$ define $\mathbf{y}(t) = \mathbf{y} + t\mathbf{1}_n$. Note that $\lim_{t \searrow 0} \mathbf{y}(t) = \mathbf{y}$. As $(B\mathbf{y})_i > 0$ for $i \in [n]$, it follows that

$$\lim_{t \searrow 0} \frac{(A\mathbf{y}(t))_i}{(B\mathbf{y}(t))_i} = \frac{(A\mathbf{y})_i}{(B\mathbf{y})_i}.$$

Hence

$$\varrho(A, B) \leqslant \lim_{t \searrow 0} r(A, B, \mathbf{y}(t)) = r(A, B, \mathbf{y}) = \hat{\varrho}(A, B) \Rightarrow \varrho(A, B) = \hat{\varrho}(A, B),$$

and $\mathbf{y}$ is optimal.

Assume second that $A\mathbf{y} > \mathbf{0}$. Suppose that B has a zero row. Then $\varrho(A, B) = \hat{\varrho}(A, B) = \infty$ and $\mathbf{y}$ is optimal. Assume now that B does not have a zero row. Then $\varrho(A, B) < \infty$. As $A\mathbf{y} \leqslant \hat{\varrho}(A, B)B\mathbf{y}$, it follows that $\hat{\varrho}(A, B) > 0$ and $B\mathbf{y} > \mathbf{0}$. Therefore the first case yields $\varrho(A, B) = \hat{\varrho}(A, B)$ and $\mathbf{y}$ is optimal.

(3) This claim follows from part 2.

(4) Assume that we have a sequence $\mathbf{x}_l \in \Pi_n$ such that $r(A, B_l) = r(A, B_l, \mathbf{x}_l)$ for each $l \in \mathbb{N}$. Denote $t = \limsup r(A, B_l)$, $s = \liminf r(A, B_l)$. Let $B_l = C_l + D_l, C_l, D_l \geqslant 0$, where $\operatorname{supp} C_l = \operatorname{supp} B$ and $\operatorname{supp} C_l \cap \operatorname{supp} D_l = \emptyset$. Clearly, $\lim_{l \to \infty} C_l = B$ and $\lim_{l \to \infty} D_l = 0$. Fix $\varepsilon > 0$. Then there exists $M(\varepsilon)$ such that $B \leqslant (1 + \varepsilon) C_l$ for $l > M(\varepsilon)$. In particular, $B \leqslant (1 + \varepsilon) B_l$ for $l > M(\varepsilon)$. Hence,

$$\frac{1}{1 + \varepsilon} r(A, B_l, \mathbf{x}) = r(A, (1 + \varepsilon) B_l, \mathbf{x}) \leqslant r(A, B, \mathbf{x}).$$

Therefore,

$$\frac{1}{1 + \varepsilon} \varrho(A, B_l) = \frac{1}{1 + \varepsilon} \hat{\varrho}(A, B_l) = \hat{\varrho}(A, (1 + \varepsilon) B_l) \leqslant \hat{\varrho}(A, B) \quad \text{for } l > M(\varepsilon).$$

As $\varepsilon > 0$ was chosen arbitrarily, it follows that $t \leqslant \hat{\varrho}(A, B)$.

There exists a subsequence $1 \leqslant l_1 < l_2 < \ldots$ such that $\lim_{k \to \infty} \varrho(A, B_{l_k}) = s$ and $\lim_{k \to \infty} \mathbf{x}_{l_k} = \mathbf{x} \in \Pi_n$. The inequality $A\mathbf{x}_{l_k} \leqslant r(A, B_{l_k}) B_{l_k} \mathbf{x}_{l_k}$ yields $A\mathbf{x} \leqslant sB\mathbf{x}$. (We may have that $s = \infty$.) Suppose first that $s < \infty$. Hence $s \geqslant \hat{\varrho}(A, B)$. Combine that with the inequality $t \leqslant \hat{\varrho}(A, B)$ to deduce that $s = t = \hat{\varrho}(A, B)$. Assume second that $s = \infty$. Then $t = \infty$ and the inequality $t \leqslant \hat{\varrho}(A, B)$ yields that $\hat{\varrho}(A, B) = \infty$.

(5) Denote $t = \limsup r(A_l, B)$, $s = \liminf r(A_l, B)$. Let $A_l = C_l + D_l, C_l, D_l \geqslant 0$, where $\operatorname{supp} C_l = \operatorname{supp} A$ and $\operatorname{supp} C_l \cap \operatorname{supp} D_l = \emptyset$. Clearly, $\lim_{l \to \infty} C_l = A$ and $\lim_{l \to \infty} D_l = 0$. Fix $\varepsilon \in (0, 1)$ in the rest of the proof. Then there exists $M(\varepsilon)$ such that $(1 - \varepsilon) C_l \leqslant A \leqslant (1 + \varepsilon) C_l$ for $l > M(\varepsilon)$. In particular, $A \leqslant (1 + \varepsilon) A_l$ for $l > M(\varepsilon)$. Hence, for $\mathbf{x} \geqslant \mathbf{0}$

$$(1 + \varepsilon) r(A_l, B, \mathbf{x}) = r((1 + \varepsilon) A_l, B, \mathbf{x}) \geqslant r(A, B, \mathbf{x}).$$

Therefore,

$$(1 + \varepsilon) \varrho(A_l, B) = \varrho((1 + \varepsilon) A_l, B) \geqslant \varrho(A, B) \quad \text{for } l > M(\varepsilon).$$

As $\varepsilon > 0$ was chosen arbitrarily, it follows that $s \geqslant \varrho(A, B)$. As B does not have zero row, it follows that $\varrho(A, B) \leqslant r(A, B, \mathbf{1}_n) < \infty$. Let $\mathbf{x}(\varepsilon) \in \Pi_n^\circ$ satisfy $A\mathbf{x}(\varepsilon) \leqslant (\varrho(A, B) + \varepsilon) B\mathbf{x}(\varepsilon)$. As B does not have a zero row, it follows that $B\mathbf{x}(\varepsilon) > \mathbf{0}$. Since $\lim_{l \to \infty} D_l = 0$, there exists $L(\varepsilon) \geqslant M(\varepsilon)$ such that $D_l \mathbf{x}(\varepsilon) \leqslant \varepsilon B\mathbf{x}(\varepsilon)$ for $l > L(\varepsilon)$. Then

$$A_l \mathbf{x}(\varepsilon) = C_l \mathbf{x}(\varepsilon) + D_l \mathbf{x}(\varepsilon) \leqslant (1 - \varepsilon)^{-1} A\mathbf{x}(\varepsilon) + \varepsilon B\mathbf{x}(\varepsilon)$$
$$\leqslant ((1 - \varepsilon)^{-1} (\varrho(A, B) + \varepsilon) + \varepsilon) B\mathbf{x}(\varepsilon) \quad \text{for } l > L(\varepsilon).$$

That is, $\varrho(A_l, B) \leqslant ((1 - \varepsilon)^{-1}(\varrho(A, B) + \varepsilon)$ for $l > L(\varepsilon)$. Therefore

$$t \leqslant ((1 - \varepsilon)^{-1}(\varrho(A, B) + \varepsilon) + \varepsilon).$$

As ε was an arbitrary number in the open interval $(0, 1)$, it follows that $t \leqslant \varrho(A, B)$. Combine that with the inequality $s \geqslant \varrho(A, B)$ to deduce that $s = t = \varrho(A, B)$. $\square$

3. The classical case

In this section we discuss the case where $m = n$ and B is the identity matrix I. We first recall some basic results on directed graphs $\vec{G} = (V, \vec{E})$. Here V is a finite set of vertices and $\vec{E} \subseteq V \times V$ is the set of diedges of $\vec{G}$. An ordered tuple $(v, w) \in \vec{E}$ is a diedge from v to w. The diedge (v, v) is called a *loop*. A dipath $\vec{P}$ in $\vec{G}$ is an ordered set of diedges $\{(v_1, v_2), (v_2, v_3), \ldots, (v_p, v_{p+1})\}$ for a positive integer p. A dipath $\vec{P}$ is closed if $v_{p+1} = v_1$. The digraph $\vec{G}$ is called *acyclic* or *diforest* if $\vec{G}$ does not have a closed path. The digraph $\vec{G}$ is called *strongly connected* if for any two distinct vertices $v, w \in V$ there is a dipath in $\vec{G}$ from v to w. (A digraph on one vertex with no diedge is strongly connected.)

For $W \subseteq V$ we define the *induced subdigraph* $\vec{G}(W) = (W, \vec{E}(W))$, where $\vec{E}(W)$ is the set of diedges in $\vec{E}$ that connects two vertices in W. Assume that $\vec{G}$ is not strongly connected. Then the subgraph $\vec{G}(W)$ is called *strongly connected component* of $\vec{G}$ if the subgraph $\vec{G}(W)$ is strongly connected but $\vec{G}(U)$ is not strongly connected for each U that strictly contains W. Let $V = \bigcup_{i=1}^{k} V_i$ be the partition of V corresponding to the strongly connected components of $\vec{G}$. That is, $\vec{G}(V_i)$ for $i \in [k]$ are all strongly connected components of $\vec{G}$. The *reduced digraph* $\vec{G}_{\mathrm{red}} = (V_{\mathrm{red}}, \vec{E}_{\mathrm{red}})$ of $\vec{G}$ is defined as follows: First, $V_{\mathrm{red}} = \{\{V_1\}, \ldots, \{V_k\}\}$. Second, a diedge $(\{V_i\}, \{V_j\})$ is in $\vec{E}_{\mathrm{red}}$ if $i \neq j$ and there is a diedge in $\vec{E}$ from V_i to V_j. For a strongly connected digraph $\vec{G}$ we let $V_{\mathrm{red}} = \{\{V\}\}$ and $\vec{E}_{\mathrm{red}} = \emptyset$. It is straightforward to show that $\vec{G}_{\mathrm{red}}$ is acyclic. With each digraph we associated an undirected graph $G = (V, E)$, where undirected edge $\{i, j\}$ is in E if either (i, j) or (j, i) in $\vec{E}$. Then G is a union of its connected components $G(W_j)$, $j \in [c]$, where each V_i is a subset of some W_j. Clearly, each W_j is a union of some subsets $V_1, \ldots, V_k$. The subset W_j induces a subdigraph $\vec{G}(W_j)$. Observe that $\vec{G}(W_j)$ induces a reduced digraph $\vec{G}(W_j)_{\mathrm{red}}$, which is a subdigraph of the reduced graph of $\vec{G}_{\mathrm{red}}$. The subdigraph $\vec{G}(W_j)_{\mathrm{red}}$ is called a *ditree* of $\vec{G}_{\mathrm{red}}$. A vertex $\{V_p\}$ is called a *source of* $\vec{G}_{\mathrm{red}}$ if there is no $\{V_q\}$ such that there is a diedge in $\vec{G}_{\mathrm{red}}$ from $\{V_q\}$ to $\{V_p\}$. A vertex $\{V_p\}$ is called a *sink of* $\vec{G}_{\mathrm{red}}$ if there is no $\{V_q\}$ such that there is a diedge in $\vec{G}_{\mathrm{red}}$ from $\{V_p\}$ to $\{V_q\}$. Clearly, each ditree $\vec{G}(W_j)_{\mathrm{red}}$

contains at least one source $\{V_p\}$ and one sink $\{V_q\}$. Denote by $\mathcal{R}(\vec{G}) \subseteq [k]$ all indices $p \in [k]$ such that $\{V_p\}$ is a source.

Let $A = [a_{ij}] \in \mathbb{R}_+^{m \times m}$. One associates with A the digraph $\vec{G}(A) = ([m], \vec{E}(A))$. A diedge (i, j) is in $\vec{E}(A)$ if and only if $a_{ij} > 0$. A is called *irreducible* if $\vec{G}(A)$ is strongly connected. Let $G(A)$ be the induced undirected graph. Then after renaming the indices, A is permutation similar to a block diagonal matrix $\mathrm{diag}(A_1, \ldots, A_c)$ if and only if $G(A)$ has c-connected components. Each A_j corresponds to $\vec{G}(W_j)$. For each nonzero subset $U \subseteq [m]$ denote by $A[U]$ the submatrix $[a_{ij}]_{i,j \in U}$. Each A_j can be assumed to be in the Frobenius normal form [12], Theorem 6.4.4. It is a block upper triangular form, where each diagonal block is an irreducible matrix $A[V_i]$, where $V_i \subseteq W_j$. (The order of the diagonal blocks depends on the labeling of the vertices of the ditree $\vec{G}(W_j)_{\mathrm{red}}$. One such labeling is given in [12], Theorem 6.4.4.) We assume that the top diagonal block of $A[W_j]$ is $A[V_p]$, where $\{V_p\}$ is a source in the ditree $\vec{G}(W_j)_{\mathrm{red}}$ and the bottom diagonal block is $A[V_q]$, where $\{V_q\}$ is a sink in $\vec{G}(W_j)_{\mathrm{red}}$. Furthermore, every source $\{V_p\}$ can be chosen to be the top diagonal block in $A[W_j]$. In particular

$$\varrho(A) = \max_{i \in [k]} \varrho(A[V_i]), \quad \varrho(A[W_j]) = \max_{V_i \subseteq W_j} \varrho(A[V_i]).$$

We first bring the well-known result due to Wielandt [40].

Lemma 3.1. *Let $A \in \mathbb{R}_+^{m \times m}$ be an irreducible matrix. Then $\varrho(A, I) = \hat{\varrho}(A, I) = \varrho(A)$. Furthermore, in characterization (2) in Theorem 3.2 equality holds if and only if $\mathbf{x}$ is the PF-eigenvector of A.*

P r o o f. The equality $\varrho(A, I) = \varrho(A)$, i.e. the characterization (2) was proved by Wielandt [40]. Wielandt also showed that equality in (2) holds if and only if $\mathbf{x}$ is the PF-eigenvector of A. The equality $\varrho(A, I) = \hat{\varrho}(A, I)$ follows from the following observation: Assume that $\mathbf{x} = (x_1, \ldots, x_m)^\top \gneqq 0$ and $1 \leqslant |\mathrm{supp}\, \mathbf{x}| < m$. Then $r(A, I, \mathbf{x}) = \infty$. Indeed, let $\mathcal{I} = [m] \setminus \mathrm{supp}\, \mathbf{x}$. As $\vec{G}(A)$ is irreducible, there exists $(i, j) \in \vec{E}(A)$ such that $i \in \mathcal{I}$ and $j \in \mathrm{supp}\, \mathbf{x}$. Hence $a_{ij} > 0$. Therefore, $x_i = 0$ and $(A\mathbf{x})_i > 0$, which yield that $(A\mathbf{x})_i / x_i = \infty$. $\quad\square$

Theorem 3.2. *Assume that $A \in \mathbb{R}_+^{m \times m}$. Then:*

(1) *$\varrho(A, I) = \varrho(A)$, and $A\mathbf{y} = \varrho(A)\mathbf{y}$ for some $\mathbf{y} \gneqq \mathbf{0}$.*

(2) *There exists $\mathbf{x} = (x_1, \ldots, x_m)^\top > \mathbf{0}$ such that $\varrho(A) = \max_{i \in [m]} (A\mathbf{x})_i / x_i$ if and only*

if the following condition holds: Let $\vec{G}_{\mathrm{red}}$ be the induced reduced graph by A with the set of vertices $\{\{V_1\}, \ldots, \{V_k\}\}$. The equality $\varrho(A[V_j]) = \varrho(A)$ implies that $\{V_j\}$ is a sink of $\vec{G}_{\mathrm{red}}$.

(3) *The vector* $\mathbf{x} \in \Pi_m^\circ$ *such that* $\varrho(A) = \max\limits_{i \in [m]} (A\mathbf{x})_i / x_i$ *is unique if and only if* A *is irreducible.*

(4) $\hat{\varrho}(A, I) = \min\limits_{i \in \mathcal{R}(\vec{G}(A))} \varrho(A[V_i])$ *and* $A\mathbf{w} = \hat{\varrho}(A, I)\mathbf{w}$ *for some* $\mathbf{w} \gneq \mathbf{0}$.

P r o o f. (1) Recall Perron's result that claims that a positive square matrix C has a positive eigenvector $\mathbf{u}$ corresponding to $\varrho(C)$. Assume that $B \in \mathbb{R}_+^{m \times m}$. Let $J \in \mathbb{R}_+^{m \times m}$ be a matrix whose all entries are 1. Set $B_l = B + J/l$ for each positive integer l. Perron's theorem implies the existence of $\mathbf{u}_l \in \Pi_m^o$ such that $B_l \mathbf{u}_l = \varrho(B_l)\mathbf{u}_l$. Since Π_m is compact, there is a subsequence of $\{\mathbf{u}_l\}$ which converges to $\mathbf{u} \in \Pi_m$. Clearly, $\lim\limits_{l \to \infty} \varrho(B_l) = \varrho(B)$. Hence, $B\mathbf{u} = \varrho(B)\mathbf{u}$. Choose $B = A$. Then $r(A, I, \mathbf{u}_l) < \varrho(A_l, I.\mathbf{u}_l) = \varrho(A_l)$. Therefore $\varrho(A, I) \leqslant \varrho(A)$. Furthermore A has a nonnegative eigenvector $\mathbf{y}$ corresponding to A. Let $\mathbf{x} > \mathbf{0}$. We show that $r(A, I, \mathbf{x}) \geqslant \varrho(A)$. Assume that $A^\top \mathbf{u} = \varrho(A)\mathbf{u}$, where $\mathbf{u} \gneq \mathbf{0}$. Then $\mathbf{u}^\top (r(A, I, \mathbf{x})\mathbf{x}) \geqslant \mathbf{u}^\top A\mathbf{x} = \varrho(A)\mathbf{u}^\top \mathbf{x}$. As $\mathbf{x} > \mathbf{0}$, we deduce that $\mathbf{u}^\top \mathbf{x} > 0$. Hence, $r(A, I) \geqslant \varrho(A)$. (This argument is in [8].) Hence $\varrho(A, I) \geqslant \varrho(A)$ and $\varrho(A, I) = \varrho(A)$.

(2) Assume that there exists $\mathbf{x} = (x_1, \ldots, x_m)^\top > \mathbf{0}$ such that $\varrho(A) = \max\limits_{i \in [m]} (A\mathbf{x})_i / x_i$. Let $\{\{V_1\}, \ldots, \{V_k\}\}$ be the vertices of the reduced graph induced by A. Suppose that $\varrho(A) = \varrho(A[V_j])$. Let $\mathbf{z} = (z_1, \ldots, z_l)^\top \in \mathbb{R}_+^{|V_j|}$ be the subvector of $\mathbf{x}$ restricted to the set V_j. Hence

$$\varrho(A) = \varrho(A[V_j]) \leqslant \max_{i \in [|V_j|]} \frac{(A[V_j]\mathbf{z})_i}{z_i} \leqslant \max_{i \in V_j} \frac{(A\mathbf{x})_i}{x_i} \leqslant \varrho(A).$$

Therefore, all the inequalities are equalities. The first equality and Lemma 3.1 yield that $A[V_j]\mathbf{z} = \varrho(A[V_j])\mathbf{z}$. The second equality yields that V_j is a sink.

Vice versa assume that $\varrho(A[V_j]) = \varrho(A)$ if and only if $\{V_j\}$ is a sink of $\vec{G}_{\mathrm{red}}$. We now show that there exists $\mathbf{x} = (x_1, \ldots, x_m)^\top > \mathbf{0}$ such that equality holds in (2). Without loss of generality we may assume that $\varrho(A) > 0$. (Otherwise $A = 0$.) Let $\mathcal{C}_0(\vec{G}_{\mathrm{red}}) \subseteq \{\{V_1\}, \ldots, \{V_k\}\}$ be the set of the sink vertices in $\vec{G}_{\mathrm{red}}$. Assume that $\{V_j\}$ is a sink. Then the restriction of $\mathbf{x}$ to V_j is the PF-eigenvector $\mathbf{x}_j > \mathbf{0}$ of $A[V_j]$. If $\mathcal{C}_0(\vec{G}_{\mathrm{red}}) = \{\{V_1\}, \ldots, \{V_k\}\}$, we easily deduce that equality holds in (2) for this $\mathbf{x}$. If not, let us consider the subsets $\mathcal{C}_l(\vec{G}_{\mathrm{red}}) \subset \{\{V_1\}, \ldots, \{V_k\}\}$ for $l = 0, 1, \ldots, p$, which is a partition of $\{\{V_1\}, \ldots, \{V_k\}\}$, defined as follows. For $l \geqslant 1$ the set $\mathcal{C}_l(\vec{G}_{\mathrm{red}}) \subset \{\{V_1\}, \ldots, \{V_k\}\}$ consists of vertices in $\vec{G}_{\mathrm{red}}$ with the maximal path length l in the digraph $\vec{G}_{\mathrm{red}}$ starting at these vertices. Thus, the diedges in $\vec{G}_{\mathrm{red}}$ from $\mathcal{C}_l(\vec{G}_{\mathrm{red}})$ go only to $\mathcal{C}_r(\vec{G}_{\mathrm{red}})$ for $r = 0, 1, \ldots, l - 1$, and each $\{V_j\} \in \mathcal{C}_l(\vec{G}_{\mathrm{red}})$ has at least one diedge. (This corresponds to the Frobenius normal form given in [12], Theorem 6.4.4.) Suppose that we already determined the restriction of $\mathbf{x}$ to $\{V_q\} \in \mathcal{C}_r(\vec{G}_{\mathrm{red}})$ for $r \leqslant l - 1$, which is denoted by $\mathbf{x}_q$, where $\mathbf{x}_q > 0$. We now

show how to determine $\mathbf{x}_j$ for each $\{V_j\} \in \mathcal{C}_l(\vec{G}_{\mathrm{red}})$. Recall that $\varrho(A[V_j]) < \varrho(A)$. Choose $t_j \in (\varrho(A[V_j]), \varrho(A)]$. We now determine $\mathbf{x}_j$ by the condition $(A\mathbf{x})[V_j] = t_j\mathbf{x}_j$. Denote by $A[V_j, V_q]$ the restriction of A to rows in the set V_j and columns in the set in V_q. Then the above condition is equivalent to

$$(t_j I_{V_j} - A[V_j])\mathbf{x}_j = \sum_{V_q \in \bigcup\limits_{r=0}^{l-1} \mathcal{C}_r(\vec{G}_{\mathrm{red}})} A[V_j, V_q]\mathbf{x}_q.$$

Since all $\mathbf{x}_q > \mathbf{0}$ for $V_q \in \bigcup\limits_{r=0}^{l-1} \mathcal{C}_r(\vec{G}_{\mathrm{red}})$ and at least one of $A[V_j, V_q]$ is a nonzero nonnegative matrix, it follows that the right hand side in the above equality is a nonzero nonnegative vector. As $t_j > \varrho(A(V_j))$ and $A[V_j]$ is irreducible, it follows that $(t_j I_{V_j} - A[V_j])^{-1}$ is a positive matrix [12], Lemma 6.4.3. Therefore,

$$\mathbf{x}_j = (t_j I_{V_j} - A(V_j))^{-1} \sum_{V_q \in \bigcup\limits_{r=0}^{l-1} \mathcal{C}_r(\vec{G}_{\mathrm{red}})} A[V_j, V_q]\mathbf{x}_q > \mathbf{0}.$$

This shows that the constructed $\mathbf{x}$ is a positive vector. It is left to show that $\varrho(A) = \max\limits_{i \in [m]} (A\mathbf{x})_i/x_i$. Assume that $i \in V_j$. Then our construction gives that $(A\mathbf{x})_i/x_i = t_j \leqslant \varrho(A)$. For a sink $\{V_j\}$ such that $\varrho(A(V_j)) = \varrho(A)$ we have that $t_j = \varrho(A)$. Hence, $\varrho(A) = \max\limits_{i \in [m]} (A\mathbf{x})_i/x_i$.

(3) The proof of 2 shows that $\mathbf{x} \in \Pi_m^o$ is unique if and only if A is an irreducible matrix.

(4) Clearly, it is enough to show the equality $\hat{\varrho}(A, I) = \min\limits_{i \in \mathcal{R}(\vec{G}(A))} \varrho(A[V_i])$ in the case where $\vec{G}(A)_{\mathrm{red}}$ is a ditree, that is, $G(A)$ is a connected graph. Let $\{V_i\}$ be a source of $\vec{G}(A)_{\mathrm{red}}$. So we can choose a Frobenius normal form so that the irreducible matrix $A[V_i]$ appears in the first diagonal block of the Frobenius normal form F of A. Let $A[V_i]\mathbf{z} = \varrho(A[V_i])\mathbf{z}$, where $\mathbf{z} > 0$. Extend $\mathbf{z}$ to $\mathbb{R}_+^m$ by addind zero entries for indices $\{|V_i| + 1, \ldots, m\}$ to obtain the vector $\mathbf{v} \in \mathbb{R}_+^m$. Then $F\mathbf{v} = \varrho(A[U_i])\mathbf{v}$. Rename the indices to deduce that $A\mathbf{w} = \varrho(A[U_i])\mathbf{w}$. Hence, $r(A, I, \mathbf{w}) = \varrho(A[U_i])$ and $\hat{\varrho}(A, I) \leqslant \varrho(A[U_i])$. This shows that $\hat{\varrho}(A, I) \leqslant \alpha = \min\limits_{i \in \mathcal{R}(\vec{G}(A))} \varrho(A[V_i])$.

It is left to show the reverse inequality $r(A, I, \mathbf{x}) \geqslant \alpha$ for each $\mathbf{x} \gneqq \mathbf{0}$ such that $r(A, I, \mathbf{x}) < \infty$. Suppose first that $\mathbf{x} > \mathbf{0}$. The above arguments yield that $r(A, I, \mathbf{x}) \geqslant \varrho(A) \geqslant \alpha$. Assume now that supp $\mathbf{x}$ is a strict subset of $[m]$. Let $\mathcal{I} \subseteq [k]$ be the set of all $i \in [k]$ such that supp $\mathbf{x} \cap V_i \neq \emptyset$. We claim that for each $i \in \mathcal{I}$ we have the equality supp $\mathbf{x} \cap V_i = V_i$. Indeed, assume that supp $\mathbf{x} \cap V_i$ is a strict subset of V_i. Let $\mathbf{x}_i = \mathbf{x}[V_i] \in \mathbb{R}_+^{|V_i|}$ be the restriction of $\mathbf{x}$ to V_i. Thus $\mathbf{x}_i \gneqq 0$.

Since $A[V_i]$ is an irreducible matrix, the proof of Lemma 3.1 yields that there exists $p \in V_i \setminus \text{supp } \mathbf{x}$ such that $(A[V_i]\mathbf{x}_i)_p > 0$. As $(A\mathbf{x})_p \geqslant (A[V_i]\mathbf{x}_i)_p$, it follows that $(A\mathbf{x})_p/x_p = +/0 = \infty$. So $r(A, I, \mathbf{x}) = \infty$, contrary to our assumption. Assume that $\{V_i\}$ is a source in $\vec{G}(A)_{\text{red}}$. Clearly $r(A, I, \mathbf{x}) \geqslant r(A[V_i], I, \mathbf{x}_i)$. Lemma 3.1 yields that $r(A[V_i], I, \mathbf{x}_i) \geqslant \varrho(A[V_i])$. Hence, $r(A[V_i], I, \mathbf{x}_i) \geqslant \alpha$. Assume that $\{V_i\}$ is not a source. Hence, there is $j \in [k]$ such that $(j, i) \in \vec{E}(\vec{G}(A)_{\text{red}})$. We claim that $\text{supp } \mathbf{x} \cap V_j = V_j$. Suppose not. So $\text{supp } \mathbf{x} \cap V_j = \emptyset$. From the definition of the reduced graph $\vec{G}(A)_{\text{red}}$ it follows there is $p \in V_j$ and $q \in V_i$ such that $a_{pq} > 0$. Hence, $(A\mathbf{x})_p > 0$ and $x_p = 0$. Thus $r(A, I, \mathbf{x}) = \infty$, which contradicts our assumption. Thus $\text{supp } \mathbf{x} \cap V_j = V_j$. Repeating this argument a number of steps we deduce that there is a source vertex $\{V_l\}$ in $\vec{G}(A)_{\text{red}}$ such that $\text{supp } \mathbf{x} \cap V_l = V_l$. The previous arguments yield that $r(A, I, \mathbf{x}) \geqslant \varrho(A[V_l]) \geqslant \alpha$. $\qquad \square$

We remark that part 1 of this theorem is well known, part 2 is close to [12], Theorem 6.4.5, part 3 is perhaps known, and part 4 seems to be new.

E x a m p l e 3.3.   Let
$$A = \begin{bmatrix} a & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1+a \end{bmatrix},$$
where $a \in (0, 1]$, and $B = I_3$. Then

(1) $\varrho(A, I_3) = \varrho(A) = 1 + a$, $\mathbf{x} = (1 + 2a^{-1}, 2a^{-1}, 1)^\top$ is optimal and $\mathbf{y} = (1, 1, 0)^\top$ is minimal optimal;

(2) $\hat{\varrho}(A, I) = a$ and $\mathbf{y} = (1, 0, 0)^\top$ is the unique weak minimal vector in Π_3.

P r o o f.   (1) It is straightforward to show that $r(A, I, \mathbf{x}) = (1 + a)$, hence $\mathbf{x}$ is optimal. Take $\mathbf{y}(t) = (1, 1, t)^\top$ for $t > 0$. Then $r(A, B, \mathbf{y}(t)) = 1 + a + t$. Let $t \to 0$ to deduce that $\mathbf{y}$ is optimal. Suppose to the contrary that $\mathbf{y}$ is not minimal optimal. Thus either $\mathbf{u} = (1, 0, 0)^\top$ or $\mathbf{v} = (0, 1, 0)^\top$ is minimal optimal. But $r(A, I, \mathbf{u}) = a$ and $r(A, I, \mathbf{v}) = \infty$. Contradiction.

(2) Part 4 of Theorem 3.2 yields that $\hat{\varrho}(A, I) = a$. It is straightforward to show that $\mathbf{y} = (1, 0, 0)^\top$ is the unique weak minimal vector in Π_3. $\qquad \square$

4. Polynomial approximation of $\varrho(A, B)$ and $\hat{\varrho}(A, B)$

We first recall the fundamental result that a feasibility of a system of linear inequalities and minimization of a linear function are polynomial. For simplicity we state a variant of this fact in the following setting that we need.

Lemma 4.1. *Let $C \in \mathbb{Q}^{m \times n}$, $\mathbf{b} = (b_1, \ldots, b_m)^\top \in \mathbb{Q}^m$, $\mathbf{c} = (c_1, \ldots, c_n)^\top \in \mathbb{Q}^n$. Then one can find in polynomial time in $\langle C \rangle + \langle \mathbf{b} \rangle$ if the following polytope is empty or not:*

$$(13) \qquad C\mathbf{x} \geqslant \mathbf{b}, \quad \mathbf{x} = (x_1, \ldots, x_n)^\top \geqslant \mathbf{0}, \quad \sum_{i=1}^{n} x_i = 1.$$

Furthermore, if the above polytope is nonempty, then the minimum of the linear function $\mathbf{c}^\top \mathbf{x}$ over this polytope can be found in polynomial time.

This result is well known, see for example [27], [20]. To be precise, [27], Corollary 2.3.7 assumes for simplicity that system (13) is solvable, and then one can find the maximum or minimum of $\mathbf{c}^\top \mathbf{x}$, where $\mathbf{c} \in \mathbb{Q}^n$ is given and $\mathbf{x}$ satisfies (13). To apply [27], Corollary 2.3.7 for solvable system we consider the following linear programing problems LP_j for $j \in [m]$. Let $\mathbf{c}_j^\top$ be the jth row of C. Let $C_j \in \mathbb{Q}^{(j-1) \times n}$, $\mathbf{b}_j \in \mathbb{Q}^{j-1}$ be the matrix and the column obtained from C and $\mathbf{b}$ by deleting the $m - j + 1$ rows $j, \ldots, m$, respectively. Assume that the system

$$(14) \qquad C_j \mathbf{x} \geqslant \mathbf{b}_j, \quad \mathbf{x} = (x_1, \ldots, x_n)^\top \geqslant \mathbf{0}, \quad \sum_{i=1}^{n} x_i = 1$$

is solvable for $j + 1 = l \in [n]$. (This is trivially true for $l = 1$.) Now consider the maximum problem $\max \mathbf{c}_l^\top \mathbf{x}$ over the set given by (14) for $j = l$. Assume that the maximum is achieved at $\mathbf{x}_l$. Then the polytope given by (14) for $j = l + 1$ is nonempty if and only if $\mathbf{c}^\top \mathbf{x}_l \geqslant b_l$. Hence, by running at most m linear programming problems we can determine in polynomial time if system (13) is feasible or not.

Theorem 4.2. *Let $A, B \in \mathbb{Q}_+^{m \times n} \setminus \{0\}$. Assume that $\varrho(A, B) < \infty$. Then for any $\varepsilon \in (0, 1) \cap \mathbb{Q}$ one of the following statements can be verified in polynomial time in $\langle A \rangle + \langle B \rangle + \langle \varepsilon \rangle$:*

(1) *The Collatz-Wielandt quotient satisfies $\varrho(A, B) < \varepsilon$.*
(2) *The Collatz-Wielandt quotient is positive and one can find $\tilde{\varrho}(A, B) \in \mathbb{Q}_+ \setminus \{0\}$ such that*

$$(15) \qquad \tilde{\varrho}(A, B) \leqslant \varrho(A, B) \leqslant (1 + \varepsilon)\tilde{\varrho}(A, B).$$

P r o o f. Let $t_0 = r(A, B, \mathbf{1}_n) < \infty$ and $N = \lceil \log_2 \varepsilon^{-1} \rceil + 1$. Set $k = 1$, $t_k = \frac{1}{2}t_{k-1}$ and $C = t_k B - A$. Consider system (13) with $\mathbf{b} = \mathbf{0}$. Assume first that this system is solvable. Let $\mu_{i,k}$ be the maximum of x_i for system (13) with $\mathbf{b} = \mathbf{0}$ for $i \in [n]$. Assume first that $\mu_{i,k} > 0$ for each $i \in [n]$. We claim that $\varrho(A, B) \leqslant t_k$. Indeed, assume that $\mu_{i,k} = x_{i,i,k}$, where $\mathbf{x}_{i,k} = (x_{1,i,k}, \ldots, x_{n,i,k})^\top \in \mathbb{Q}^n$ satisfies system (13) with $\mathbf{b} = \mathbf{0}$. Set $\mathbf{x} = \frac{1}{n} \sum_{i=1}^n \mathbf{x}_{i,k}$ to deduce that $r(A, B, \mathbf{x}) \leqslant t_k$. Assume that for $k = 2, \ldots, N$ we have the inequality $\varrho(A, B) \leqslant t_k$. Then $\varrho(A, B) \leqslant t_N < \varepsilon$ and we showed condition (1).

Suppose now that for the smallest value $k \in [N]$ one of the following conditions hold: Either system (13) with $C = t_k B - A$ and $\mathbf{b} = \mathbf{0}$ is not solvable or $\mu_{i,k} = 0$ for some $i \in [n]$. Then $0 < t_k \leqslant \varrho(A, B)$. Set $l = 0$, $f_l = t_k$, $g_l = t_{k-1}$. Then $f_l \leqslant \varrho(A, B) \leqslant g_l$. If $(g_l - f_l)/f_l \leqslant \varepsilon$, then $\tilde{\varrho}(A, B) = f_l$. If not, set $M = \lceil \log_{4/3} (g_l - f_0)/f_0\varepsilon \rceil$ and apply now the bisection algorithm: Let $h_l = (f_l + g_l)/2$ and $C = h_l B - A$. Consider system (13) with $\mathbf{b} = \mathbf{0}$. Assume first that this system is solvable. Let $\mu_{i,l}$ be the minimum of x_i for system (13) with $\mathbf{b} = \mathbf{0}$ for $i \in [n]$. Suppose that $\mu_{i,l} > 0$ for each i. Then $f_{l+1} = f_l$, $g_{l+1} = (f_l + g_l)/2$. If $\mu_{i,l} = 0$ for some i or system (13) with $\mathbf{b} = \mathbf{0}$ is not solvable, set $f_{l+1} = (f_l + g_l)/2$, $g_{l+1} = g_l$. Clearly $f_{l+1} \leqslant \varrho(A, B) \leqslant g_{l+1}$. Continue this bisection to $l = M$. Then $\tilde{\varrho}(A, B) = f_M$ and condition (2) holds. $\qquad\square$

We now discuss briefly an approximation of $\hat{\varrho}(A, B)$. First use Algorithm 2.4 to determine if $\hat{\varrho}(A, B) < \infty$. Assume that $\hat{\varrho}(A, B) < \infty$. Next assume that $\hat{\varrho}(A, B) > 0$, that is, condition 7 of Lemma 2.1 holds. Then the right hand side of (12) is a positive lower bound for $\hat{\varrho}(A, B)$. Use Algorithm 2.4 to find a nonempty subset $S \subseteq [n]$ such that $r(A, B, \mathbf{1}_S) < \infty$. Thus $r(A, B, \mathbf{1}_S)$ is an upper bound on $\hat{\varrho}(A, B)$. Next apply a simplified version of the bisection algorithm used in the proof of Theorem 4.2 (without considering the minimum problem) to deduce:

Proposition 4.3. *Let* $A, B \in \mathbb{Q}_+^{m \times n} \setminus \{0\}$. *Assume that* $0 < \hat{\varrho}(A, B) < \infty$. *Then for any* $\varepsilon \in (0, 1) \cap \mathbb{Q}$ *one can find* $\bar{\varrho}(A, B) \in \mathbb{Q}_+ \setminus \{0\}$ *in polynomial time in* $\langle A \rangle + \langle B \rangle + \langle \varepsilon \rangle^*$ *such that*

$$(16) \qquad\qquad \bar{\varrho}(A, B) \leqslant \hat{\varrho}(A, B) \leqslant (1 + \varepsilon)\bar{\varrho}(A, B).$$

We first discuss weak optimal solutions which are easier to characterize.

Theorem 5.1. *Let m, n be positive integers. Assume that $A = [a_{ij}]$, $B = [b_{ij}] \in \mathbb{R}_+^{m \times n}$. Suppose that $\hat{\varrho}(A, B) < \infty$.*

(1) *Assume that $\mathbf{y} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ is weak optimal. Then at least one coordinate of $(A - \hat{\varrho}(A, B)B)\mathbf{y}$ is zero.*

(2) *Assume that there exists a weak optimal vector $\mathbf{y} \in \mathbb{R}_+^n$ with l positive coordinates. Let $A', B' \in \mathbb{R}^{m \times l}$ be the submatrices of A and B, respectively, induced by the positive coordinates of $\mathbf{y}$. If $l \geqslant m$, then $\operatorname{rank}(A' - \hat{\varrho}(A, B)B') < m$.*

(3) *A minimal weak optimal $\mathbf{y}$ has at most m positive coordinates.*

(4) *Let $\mathbf{y}$ be a minimal weak optimal with m positive coordinates. Then $\mathbf{y}$ is a weak GPF-vector. Furthermore, $\operatorname{rank}(A' - \hat{\varrho}(A, B)B') = m - 1$.*

(5) *Let $\mathbf{y}'$ be a minimal weak optimal with $l < m$ positive coordinates. Then there exists a minimal weak optimal $\mathbf{y}$ satisfying $\operatorname{supp} \mathbf{y} = \operatorname{supp} \mathbf{y}'$ with the following property: Let $\mathcal{K} = \{k \in [m], (A\mathbf{y})_k = \hat{\varrho}(A, B)(B\mathbf{y})_k\}$. Then $|\mathcal{K}| \geqslant l$.*

P r o o f. (1) Let $D(t) = A - tB$ and $t_0 = \hat{\varrho}(A, B)$. As $r(A, B, \mathbf{y}) = t_0$, it follows that $C(t_0)\mathbf{y} \leqslant \mathbf{0}$. Suppose to the contrary that $C(t_0)\mathbf{y} < \mathbf{0}$. Then $t_0 > 0$. Furthermore, there exists $0 \leqslant t_1 < t_0$ such that $C(t_1)\mathbf{y} \leqslant \mathbf{0}$. Hence, $r(A, B, \mathbf{y}) \leqslant t_1 < t_0$, contrary to our assumption.

(2) Assume that $\mathbf{y}$ is weak optimal vector which has l positive coordinates. Let $A', B' \in \mathbb{R}^{m \times l}$ be defined as in the theorem. Assume that $\mathbf{0} < \mathbf{z} \in \mathbb{R}^l$ is the subvector of $\mathbf{y}$ induced by its positive coordinates. Let $C(t) = A' - tB'$ and $t_0 = \hat{\varrho}(A, B)$. Then $C(t_0)\mathbf{z} = -\mathbf{w}$, $\mathbf{w} \in \mathbb{R}_+^m$. Assume $l \geqslant m$ and $\operatorname{rank} C(t_0) = m$. Then there exists an $m \times m$ submatrix of $C(t_0)$ which is nonsingular. By permuting the columns of $C(t_0)$ we can assume the following. Let $A' = [A_1 \, A_2]$, $B' = [B_1 \, B_2]$, where $A_1, B_1 \in \mathbb{R}^{m \times m}$ and $\det(A_1 - t_0 B_1) \neq 0$. Denote $C_1(t) = A_1 - tB_1$, $C_2(t) = A_2 - tC_2$. Assume that $\mathbf{z}^\top = (\mathbf{u}^\top, \mathbf{v}^\top)$, $\mathbf{0} < \mathbf{u} \in \mathbb{R}^m$, $\mathbf{0} < \mathbf{v} \in \mathbb{R}^{l-m}$. Thus $C_1(t_0)\mathbf{u} = -(C_2(t_0)\mathbf{v} + \mathbf{w})$. Since $\det C_1(t_0) \neq 0$, there exists $\varepsilon > 0$ such that $\det C_1(t) \neq 0$ for $|t - t_0| < \varepsilon$. Observe that $\mathbf{u} = -C_1(t_0)^{-1}(C_2(t_0)\mathbf{v} + \mathbf{w}) > \mathbf{0}$. Let $\mathbf{u}(t) = -C_1(t)^{-1}(C_2(t)\mathbf{v} + \mathbf{w})$ for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$. Then $\mathbf{u}(t)$ is continuous in the interval $(t_0 - \varepsilon, t_0 + \varepsilon)$. Hence, there exists $\varepsilon_1 \in (0, \varepsilon)$ such that $\mathbf{u}(t) > \mathbf{0}$ for $|t - t_0| \leqslant \varepsilon_1$. Set $t_1 = t_0 - \varepsilon_1$ and $\mathbf{z}' = (\mathbf{u}(t_1)^\top, \mathbf{v}^\top)^\top$. Thus $C(t_1)\mathbf{z}' = -\mathbf{w}$, which implies that $r(A', B', \mathbf{z}') \leqslant t_1 < \hat{\varrho}(A', B') = \hat{\varrho}(A, B)$. This contradicts the definition of $\hat{\varrho}(A', B')$. Hence, $\operatorname{rank} C(t_0) < m$.

(3) Assume to the contrary that $\mathbf{y}$ is a minimal weak optimal solution with $l > m$ positive coordinates. Let $A', B', C(t), \mathbf{z}, \mathbf{w}, t_0$ be defined as in part 2 of the proof. We showed that $\operatorname{rank} C(t_0) < m$. Hence, $\dim \ker C(t_0) \geqslant 2$. Choose $\mathbf{x} \in \ker C(t_0) \setminus \{\mathbf{0}\}$

such that $\mathbf{x}$ has at least one negative coordinate and one positive coordinate. So $\mathbf{y}$ is not proportional to $\mathbf{x}$. Let $\mathbf{z}(s) = \mathbf{z} + s\mathbf{x}$ for $s \geqslant 0$. Note that $C(t_0)\mathbf{z}(s) = -\mathbf{w}$. Let $s_0 > 0$ be the biggest s such that $\mathbf{z}(s) \geqslant \mathbf{0}$. Then $\mathbf{z}(s_0) \gneq \mathbf{0}$, $\mathbf{z}(s_0)$ has at least one zero component and $C(t_0)\mathbf{z}(s_0) = -\mathbf{w}$. Thus $r(A', B', \mathbf{z}(s_0)) \leqslant t_0 = \hat{\varrho}(A', B')$. Hence $r(A', B', \mathbf{z}(s_0)) = t_0$, which contradicts the minimality of $\mathbf{y}$.

(4) Assume that $\mathbf{y}$ is a minimal weak optimal with m positive coordinates. Use the notations of parts 2 and 3 of the proof. We claim that $C(t_0)\mathbf{z} = 0$. Suppose not. By part 2 rank $C(t_0) < m$. Let $\mathbf{x} \in \ker C(t_0) \setminus \{\mathbf{0}\}$. So $\mathbf{y}$ is not proportional to $\mathbf{x}$. By considering $\pm\mathbf{x}$ we may assume that $\mathbf{x}$ has at least one negative coordinate. Define $\mathbf{z}(s_0)$ as in part (3) to deduce that $\mathbf{y}$ is not minimal. So $C(t_0)\mathbf{z} = 0$. Assume to the contrary that rank $C(t_0) < m - 1$. Choose $\mathbf{x} \in \ker C(t_0)$ to have positive and negative coordinates. Then we conclude as above that $\mathbf{z}$ is not minimal.

(5) Let $\mathbf{y}'$ be a minimal weak optimal with $l < m$ positive coordinates. Part 1 of the theorem yields that $C(t_0)\mathbf{y}'$ has at least one zero coordinate. Thus if $l = 1$, part 5 of the theorem is trivial.

Assume that $l > 1$. Consider all minimal weak optimal $\tilde{\mathbf{y}}$ such that supp $\tilde{\mathbf{y}} =$ supp $\mathbf{y}'$. Let $\mathcal{K}(\tilde{\mathbf{y}}) = \{k \in [m], (A\tilde{\mathbf{y}})_k = \hat{\varrho}(A, B)(B\tilde{\mathbf{y}})_k\}$. Choose a minimal weak optimal $\mathbf{y}$ such that $|\mathcal{K}(\mathbf{y})| = p$ is maximal. We claim that $p \geqslant l$. Suppose not. Let $\mathcal{K} = \mathcal{K}(\mathbf{y})$. Assume the notations of part 2. Let $\tilde{A} = A'[\mathcal{K}, [n]]$, $\tilde{B} = B'[\mathcal{K}, [n]]$, $\tilde{C}(t) = C(t)[\mathcal{K}, [n]]$. Hence $\tilde{C}(t_0))\mathbf{z} = \mathbf{0}$. Suppose first that rank $\tilde{C}(t_0) \leqslant l - 2$. Hence, there exists $\mathbf{u} \in \mathbb{R}^l$ satisfying $\tilde{C}(t_0)\mathbf{u} = 0$ such that $\mathbf{u}$ has positive and negative coordinates. Thus, $\mathbf{z}$ and $\mathbf{u}$ are linearly independent. Let $\mathbf{v} \in \mathbb{R}^m$ be the extension of $\mathbf{u}$ by adding zero coordinates. In particular, $\mathbf{v}$ has a zero coordinate where $\mathbf{y}$ has zero coordinate. Let $s \geqslant 0$ and consider $\mathbf{z}(s) = \mathbf{z} + s\mathbf{u}$ and $\mathbf{y}(s) = \mathbf{y} + s\mathbf{v}$. Let $s_1 > 0$ be the smallest value such that $\mathbf{z}(s_1) \geqslant \mathbf{0}$ and $z(s_1)$ has at least one zero coordinate. Since $\mathbf{y}$ was minimal, we deduce that $\mathbf{y}(s_1)$ is not weak optimal. That is, there exists $s_2 \in (0, s_1)$ with the following property: There exists $j \in [m] \setminus \mathcal{K}$ so that $(A\mathbf{y}(s_2))_j = t_0(B\mathbf{y}(s_2))_j$ and $(A\mathbf{y}(s_2))_k \leqslant t_0(B\mathbf{y}(s_2))_k$ for $k \in [m] \setminus \{\mathcal{K} \cup \{j\}\}$. Clearly $(A\mathbf{y}(s_2))_k = t_0(B\mathbf{y}(s_2))_k$ for $k \in \mathcal{K} \cup \{j\}$. So $\mathbf{y}(s_2)$ is optimal and $|\mathcal{K}(\mathbf{y}(s_2))| > |\mathcal{K}| = |\mathcal{K}(\mathbf{y})|$. This contradicts the choice of $\mathbf{y}$.

It is left to consider the case where $|\mathcal{K}| = l - 1$ and rank $\tilde{C}(t_0) = l - 1$. We proceed similarly as in the proof of (2). Permute the columns of $C(t)$ so that $\tilde{C}(t) = [\tilde{C}_1(t) \, \tilde{C}_2(t)] \in \mathbb{R}^{(l-1)\times l}$ and $\tilde{C}_1(t_0) \in \mathbb{R}^{(l-1)\times(l-1)}$ is a nonsingular matrix. Therefore, $\tilde{C}_1(t)$ is nonsingular for $|t - t_0| < \varepsilon$ for some $\varepsilon > 0$. Assume that $\mathbf{z}^\top = (\mathbf{u}^\top, \mathbf{v}^\top)$. Thus $\mathbf{u} = -(\tilde{C}_1(t_0))^{-1}\tilde{C}_1(t_0)\mathbf{v}$. As $\mathbf{u} > 0$, it follows that $\mathbf{u}(t) = -(\tilde{C}(t))^{-1}\tilde{C}_1(t)\mathbf{v} > 0$ for $|t - t_0| < \varepsilon'$ for some $\varepsilon' \in (0, \varepsilon)$. Let $\mathbf{w}(t)^\top = (\mathbf{u}(t)^\top, \mathbf{v}^\top)$. As $(A'\mathbf{w}(t_0))_j < (t_0 - \varepsilon_1)(B'\mathbf{w}(t_0))_j$ for $j \in [m] \setminus \mathcal{K}$ and some $\varepsilon_1 > 0$, there exists $t' \in (t_0 - \varepsilon', t_0)$ such that $\tilde{C}(t')\mathbf{w}(t) = \mathbf{0}$ and $(A'\mathbf{w}(t'))_j < t'(B'\mathbf{w}(t'))_j$ for $j \in [m] \setminus \mathcal{K}$. That is,

$\hat{\varrho}(A', B') \leqslant t'$. This contradicts our assumption that $\hat{\varrho}(A', B') = t_0$. Therefore, $|\mathcal{K}| \geqslant l$. □

Parts (4)–(5) and their proofs yield:

Corollary 5.2. *Let m, n be positive integers. Assume that $A, B \in \mathbb{R}_+^{m \times n}$. Suppose that $\hat{\varrho}(A, B) < \infty$. Then there exists a minimal weak optimal $\mathbf{y} \in \mathbb{R}_+^n$ with at most m positive coordinates, and a minimal $\mathcal{I} \subset [m]$, possibly an empty set, with the following properties:*

$$(17) \qquad A(\mathcal{I}, \emptyset)\mathbf{y} = \hat{\varrho}(A, B)B(\mathcal{I}, \emptyset)\mathbf{y}, \quad m - |\mathcal{I}| \geqslant |\operatorname{supp} \mathbf{y}|.$$

Furthermore, if $m - |\mathcal{I}| = |\operatorname{supp} \mathbf{y}|$, then the only nonzero solution of $(A(\mathcal{I}), \emptyset) - \hat{\varrho}(A, B)B(\mathcal{I}, \emptyset))\mathbf{x} = \mathbf{0}$, whose nonzero coordinates lie in $\operatorname{supp} \mathbf{y}$, are multiples of $\mathbf{y}$.

We now show that some similar results apply to optimal vectors.

Theorem 5.3. *Let m, n be positive integers. Assume that $A = [a_{ij}]$, $B = [b_{ij}] \in \mathbb{R}_+^{m \times n}$. Suppose that $\varrho(A, B) < \infty$.*

(1) *Assume that $\mathbf{y} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ is optimal. Then at least one coordinate of $(A - \varrho(A, B)B)\mathbf{y}$ is zero.*

(2) *Assume that there exists an optimal vector $\mathbf{y} \in \mathbb{R}_+^n$ with l positive coordinates. Let $A', B' \in \mathbb{R}^{m \times l}$ be the submatrices of A and B, respectively, induced by the positive coordinates of $\mathbf{y}$. If $l \geqslant m$, then $\operatorname{rank}(A' - \varrho(A, B)B') < m$.*

(3) *There exists an optimal $\mathbf{y}$ which has at most m positive coordinates.*

P r o o f.   (1) Part 2 of Lemma 2.6 yields the existence of a sequence $\mathbf{y}_l \in \Pi_n^\circ$ such that $\lim_{l \to \infty} \mathbf{y}_l = \mathbf{y}$, and $\lim_{l \to \infty} r(A, B, \mathbf{y}_l) = r(A, B, \mathbf{y}) = \varrho(A, B)$. Clearly, at least one coordinate of $(A - r(A, B, \mathbf{y}_l)B)\mathbf{y}_l$ is zero. Hence, there exists an infinite subsequence $\{l_p\}$, $p \in \mathbb{N}$ such that a fixed coordinate of $(A - r(A, B, \mathbf{y}_l)B)\mathbf{y}_{l_p}$ is zero. Letting $p \to \infty$ we deduce the claim.

(2) Let $t_0 = \varrho(A, B)$. Assume first that $B\mathbf{y} > \mathbf{0}$. Then part (3) of Lemma 2.5 yields that $r(A, B, \mathbf{x}) \geqslant \varrho(A, B)$ for each $\mathbf{x} \in \mathbb{R}_+^n$, $\sup \mathbf{x} = \sup \mathbf{y}$. Then we proceed as in the proof of part 2 of Theorem 5.1, using the notations and the results in this proof. Rename the columns of A and B so that the first l coordinates of $\mathbf{y}$ are positive. Let $A' = A[[m], [l]]$, $B' = B[[m], [l]]$, $C(t) = A' - tB'$, and $\mathbf{0} < \mathbf{z} \in \mathbb{R}^l$ be the projection of $\mathbf{y}$ on its first l coordinates. As $B\mathbf{y} = B'\mathbf{z} > \mathbf{0}$, it follows that we can choose $t_1 < t_0$ such that $B'\mathbf{z} > \mathbf{0}$. Hence, $r(A', B', \mathbf{z}) \leqslant t_1 < t_0$, which contradicts the claim that $r(A', B', \mathbf{z}) \geqslant t_0$. Hence, $\operatorname{rank} C(t_0) < m$.

578

Assume that $\mathcal{I} \neq \emptyset$ is the set of zero coordinates of $B\mathbf{y}$. That is, $\mathcal{I}$ is the set of the zero rows of B'. As $\varrho(A, B) < \infty$, we deduce that the set of zero rows of A' contains $\mathcal{I}$. Clearly, rank $C(t_0) < m$.

(3) We prove the claim by induction on n. For $n = 1$ the vector $\mathbf{y} = (1)$ is optimal, hence $m \geqslant l = 1$. Assume that the claim holds for $n \leqslant N$ and suppose that $n = N + 1$. Let $\mathbf{y}$ be a minimal optimal with vector with minimal number of positive coordinates l. Assume the conventions of parts (1)–(2) of the proof. Suppose to the contrary that $l > m$. Then we proceed to the proof of part (3) of Theorem 5.1. Let t_0 and $\mathbf{0} < \mathbf{z} \in \mathbb{R}^l$ be as in part 2. Let $\mathcal{J} = \{l + 1, \ldots, n\}$. (If $l = n$, then $\mathcal{J} = \emptyset$.) Assume that $\mathbf{x} \in \ker C(t_0) \setminus \{\mathbf{0}\}$ such that $\mathbf{x}$ has at least one negative coordinate and one positive coordinate. Set $\mathbf{z}(s) = \mathbf{z} + s\mathbf{x}$ and choose $s_0 > 0$ the biggest s such that $\mathbf{z}(s_0) \geqslant \mathbf{0}$. Then $\mathbf{z}' = \mathbf{z}(s_0) \gneqq \mathbf{0}$ and $\mathbf{z}'$ has at least one zero coordinate. Clearly, $r(A', B', \mathbf{z}') \leqslant r(A', B', \mathbf{z}) = t_0$.

Suppose first that $r(A', B', \mathbf{z}') < t_0$. Part (2) of Lemma 2.5 yields that $B'\mathbf{z}'$ is not a positive vector. Hence B' has at least one zero row. Let $\mathcal{I}$ be the set of zero rows of B. As $r(A', B', \mathbf{z}') < t_0$, it follows that the set of zero rows of A contains $\mathcal{I}$. Hence $\hat{\varrho}(A', B') \leqslant r(A', B', \mathbf{z}') < t_0$.

Assume first that $|\mathcal{I}| = m$. Then $A' = B' = 0$. If $\mathcal{J} = \emptyset$, then $A = A'$, $B = B'$, and vector $(1, 0, \ldots, 0)^\top$ is a minimal optimal vector, contrary to our assumption. Hence $|\mathcal{J}| \geqslant 1$. Clearly, $\varrho(A, B) = \varrho(A[[m], [\mathcal{J}]], B[[m], [\mathcal{J}]])$ and an optimal vector of $(A[[m], [\mathcal{J}]], B[[m], [\mathcal{J}]])$ can be extended trivially, by adding zero coordinates, to an optimal vector of (A, B). The induction hypothesis on the pair $(A[[m], [\mathcal{J}]], B[[m], [\mathcal{J}]])$ yields the existence of an optimal $\mathbf{w} \in \mathbb{R}_+^{n-l}$ satisfying $|\sup \mathbf{w}| \leqslant m$. Extend trivially $\mathbf{w}$ to $\mathbb{R}_+^n$ to obtain an optimal vector $\mathbf{y}$ with $\sup \mathbf{y} \leqslant m$, contrary to our assumption.

Assume that $|\mathcal{I}| = m - m' \in [m - 1]$. Rename the rows of A and B so that $\mathcal{I} = \{m' + 1, \ldots, m\}$. Let $C = A[[m'], [l]]$, $D = B[[m'], [l]]$. Clearly $\hat{\varrho}(C, D) = \hat{\varrho}(A', B') < t_0$. Let us consider a weak minimal vector $\mathbf{w} \in \mathbb{R}_+^l$ corresponding to (C, D) with minimal number of positive coordinates. Theorem 5.1 yields that $p = |\sup \mathbf{w}| \leqslant m'$. Rename the first l columns of A and B so that the first p coordinates of $\mathbf{w}$ are positive. Let $\mathbf{0} < \mathbf{u} \in \mathbb{R}^p$ be the projection of $\mathbf{w}$ on its first p-coordinates. Let $C' = C[[m'], [p]]$, $D' = D[[m'], [p]]$. Observe that $\hat{\varrho}(C, D) = r(C, D, \mathbf{w}) = r(C', D', \mathbf{u}) = \hat{\varrho}(C', D') < t_0$. Let $\mathcal{I}'$ be the set of zero rows of D'. We claim that $m' \geqslant p + |\mathcal{I}'|$. Recall the set of zero rows in C' contains $\mathcal{I}'$. Let $A_{11} = C'(\mathcal{I}', \emptyset)$, $B_{11} = D'(\mathcal{I}', \emptyset)$. Clearly, $\hat{\varrho}(C', D') = \hat{\varrho}(A_{11}, B_{11})$. Furthermore, $\mathbf{x} \in \mathbb{R}_+^p$ is a weak optimal vector of (A_{11}, B_{11}) if and only if it is a weak optimal vector of (C', D'). Note that each weak optimal vector of (C', D') can be trivially extended to a weak optimal vector of (C, D). Since a weak optimal vector of (C, D) has at least p positive co-

ordinates, it follows that each optimal vector of (A_{11}, B_{11}) is positive. Hence, $\varrho(A_{11}, B_{11}) = \hat{\varrho}(A_{11}, B_{11}) < t_0$.

Apply Theorem 5.1 to minimal optimal vectors of (A_{11}, B_{11}) to deduce the inequality $m' - |\mathcal{I}'| \geqslant p$. Let $m'' = m' - |\mathcal{I}'|$ rearrange the first m' rows of A and B so that $\mathcal{I} = \{m'' + 1, \ldots, m'\}$. Let $\mathcal{K} = \{p + 1, \ldots, n\}$. Observe

$$A = [A_1 \, A_2], \quad A_1 = A[[m], [p]] = \begin{bmatrix} A_{11} \\ 0 \end{bmatrix}, \quad A_2 = A[[m], [\mathcal{K}]] = \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix},$$

$$B = [B_1 \, B_2], \quad B_1 = B[[m], [p]] = \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}, \quad B_2 = B[[m], [\mathcal{K}]] = \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix},$$

$$A_{11}, B_{11} \in \mathbb{R}_+^{m'' \times p}, \quad A_{22}, B_{22} \in \mathbb{R}^{(m-m'') \times (n-p)}.$$

As $\mathcal{I}'$ was the set of zero rows of D', we deduce that B_{11} has no zero rows. Hence $B_{11}\mathbf{1}_p > \mathbf{0}$. Thus the conditions of part (3) of Lemma 2.5 hold. Thus $\varrho(A_{22}, B_{22}) = t_0$. We now can apply the induction hypothesis on the optimal vector of the pair (A_{22}, B_{22}). That is, there exists an optimal $\mathbf{v} \in \mathbb{R}_+^{n-p}$ satisfying $|\sup \mathbf{u}| \leqslant m - m''$. We now use the arguments of the proof of part (2) of Lemma 2.6 to deduce that $\mathbf{y} = (\mathbf{u}^\top, s_1 \mathbf{v}^\top)^\top$ is an optimal for the pair (A, B) for some $s_1 > 0$. Note that $|\sup \mathbf{y}| = |\sup \mathbf{u}| + |\sup \mathbf{v}| \leqslant p + m - m'' \leqslant m'' + (m - m'') = m$, which contradicts our assumption.

It is left to discuss the case where $r(A', B', \mathbf{z}(s_0)) = \varrho(A, B)$. Let $\hat{\mathbf{x}} \in \mathbb{R}^n$ be the trivial extensions of $\mathbf{x}$ to $\mathbb{R}^n$. Then $A\hat{\mathbf{x}} = t_0\hat{\mathbf{x}}$. Let $\mathbf{y}(s) = \mathbf{y} + s\hat{\mathbf{x}} \in \mathbb{R}^n$. Then $(A - t_0 B)\mathbf{y}(s) = (A - t_0 B)\mathbf{y} \leqslant \mathbf{0}$. Hence $r(A, B, \mathbf{y}(s)) \leqslant t_0$ for $\mathbf{y}(s) \geqslant \mathbf{0}$. This holds for $s \in [0, s_0]$. Note that our assumption is that $r(A, B, \mathbf{y}(s_0)) = t_0$. We claim that $\mathbf{y}(s_0)$ is optimal. Recall that there exists a sequence $\mathbf{0} < \mathbf{y}_k \in \mathbb{R}^n$ such that $\lim_{k \to \infty} \mathbf{y}_k = \mathbf{y}$, $\lim_{k \to \infty} r(A, B, \mathbf{y}_k) = r(A, B, \mathbf{y}) = t_0 > 0$. Let $\mathbf{y}_l(s) = \mathbf{y}_l + s\hat{\mathbf{x}}$. Then

$$(18) \qquad (A - t_0 B)\mathbf{y}_k(s) = (A - t_0 B)\mathbf{y}_k \leqslant (r(A, B, \mathbf{y}_k) - t_0)\mathbf{y}_k, \quad k \in \mathbb{N}.$$

Let $\mathbf{y}_k = \tilde{\mathbf{y}}_k + \mathbf{v}_k$, where $\tilde{\mathbf{y}}_k$ is obtained by replacing the positive coordinates of $\mathbf{y}_k$ with zero coordinates in the places $\mathbf{y}$ has zero coordinates. Then $\mathbf{v}_k = \mathbf{y}_k - \tilde{\mathbf{y}}_k \geqslant \mathbf{0}$. The coordinates of $\mathbf{v}_k$ are positive where the coordinates of $\tilde{\mathbf{y}}_k$ are zero, and the coordinates of $\mathbf{v}_k$ are zero where the coordinates of $\tilde{\mathbf{y}}_k$ are positive. Thus $\mathbf{y}_k(s) = \mathbf{y}(s) + (\tilde{\mathbf{y}}_k - \mathbf{y}) + \mathbf{v}_k$. Fix $s \in (0, s_0)$. We claim that there exists $K(s) > 1$ and $N(s) > 0$ such that for $k > N(s)$ we have the inequality $\mathbf{y}_k \leqslant K(s)\mathbf{y}_k(s)$. Indeed, as $\mathbf{y}(s)$ has positive coordinates where $\mathbf{y}$ has positive coordinates, it follows that there exists $K(s) > 1$ such that $\mathbf{y} \leqslant K(s)\mathbf{y}(s)/2$. Clearly $\mathbf{v}_k \leqslant K(s)\mathbf{v}_k$ for all $k \in \mathbb{N}$. As $\lim_{k \to \infty}(\tilde{\mathbf{y}}_k - \mathbf{y}) = \mathbf{0}$, it follows that there exists $N(s)$ such that $(\tilde{\mathbf{y}}_k - \mathbf{y}) \leqslant K(s)\mathbf{y}(s)/2$ for $k > N(s)$. Hence, $\mathbf{y}_k \leqslant K(s)\mathbf{y}_k(s)$ and $\mathbf{y}_k(s) > \mathbf{0}$ for $k > N(s)$.

Therefore, $r(A, B, \mathbf{y}_k(s)) \geqslant t_0$ for $k > N(s)$. Use (18) to deduce that $r(A, B, \mathbf{y}_k(s)) \leqslant r(A, B, \mathbf{y}_k)$ for $k > N(s)$.

Choose an increasing sequence $0 < s_1 < s_2 < \ldots$ which converges to s_0. Choose an increasing subsequence l_j, $j \in \mathbb{N}$ such that $l_j > N(s_j)$, $|\mathbf{y}_{l_j}(s_j) - \mathbf{y}(s_j)| < 1/j$ and $r(A, B, \mathbf{y}_j(s_j)) \in [t_0, t_0 + 1/j]$. So $\lim_{j \to \infty} \mathbf{y}_{lj}(s_j) = \mathbf{y}(s_0)$ and $\mathbf{y}(s_0)$ is optimal. This contradicts our assumption that $\mathbf{y}$ is optimal vector with minimum number of positive coordinates. $\qquad\square$

We now give a simple example of two positive invertible stochastic matrices $A, B \in \mathbb{R}_+^{2 \times 2}$ for which there is a unique optimal $\mathbf{y} \in \Pi_2$ with one positive coordinate.

Proposition 5.4. *Let*

$$A = \begin{bmatrix} a & 1-a \\ b & 1-b \end{bmatrix}, \quad B = \begin{bmatrix} 1-a & a \\ 1-b & b \end{bmatrix}, \quad 0 < b < a < \frac{1}{2}.$$

Then $\varrho(A, B) = \hat{\varrho}(A, B) = a/(1-a) < 1$ and $\mathbf{z} = (1, 0)^\top$ is the unique optimal vector in Π_2, which is not a GPF-eigenvector.

P r o o f. As $a, b \in (0, \frac{1}{2})$, it follows that

$$\min_{\mathbf{x} \in \Pi_2} \frac{(A\mathbf{x})_1}{(B\mathbf{x})_1} = \frac{(A\mathbf{z})_1}{(B\mathbf{z})_1} = \frac{a}{1-a}, \quad \min_{\mathbf{x} \in \Pi_2} \frac{(A\mathbf{x})_2}{(B\mathbf{x})_2} = \frac{(A\mathbf{z})_2}{(B\mathbf{z})_2} = \frac{b}{1-b},$$

where $\mathbf{z} = (1, 0)^\top \in \Pi_2$ is the unique vector that minimizes both ratios. As $b/(1-b) < a/(1-a)$, we deduce that $\varrho(A, B) = a/(1-a)$ and $\mathbf{z}$ is a unique optimal in Π_2. Clearly, $\varrho(A, B) < 1$. $\qquad\square$

Note that $\operatorname{rank}(A - \varrho(A, B)B) = 2$, which does not contradict part 2 of Theorem 5.1 as $l = 1 < m = 2$. Observe that $A\mathbf{1}_2 = B\mathbf{1}_2$, that is, 1 is the eigenvalue of the generalized eigenvalue problem (3) with corresponding positive eigenvector $\mathbf{1}_2$. Note that the second eigenvalue of (3) is $\lambda = -1$ with corresponding eigenvector $(1, -1)^\top$.

Recall that for $A \in \mathbb{R}_+^{n \times n}$ we have that $\varrho(A^\top) = \varrho(A)$. For a pair of $A, B \in \mathbb{R}_+^{m \times n}$ such equality does not always hold. For a pair of matrices given in Proposition 5.4 we have $\varrho(A^\top, B^\top) = (1-a)/a > \varrho(A, B)$. Indeed, observe that $(A^\top\mathbf{x})_1/(B\mathbf{x})_1 < (A^\top\mathbf{x})_2/(B\mathbf{x})_2$ for each $\mathbf{x} \in \Pi_2$. The minimum of the bigger ratio is achieved for $\mathbf{z} = (1, 0)^\top$, which yields the equality $\varrho(A^\top, B^\top) = (1-a)/a$.

Definition 6.1. A pair $A, B \in \mathbb{R}_+^{m \times n}$ is called a *WN-pair* (a wireless network pair) if $n \geqslant m$, B has no zero row and each column of B has exactly one positive entry.

A WN-pair was considered in [2]. It has the following interpretation in a wireless network [2], Introduction. Each row i in $A = [a_{ij}]$ corresponds to the *entity* i (receiver), and each nonzero element in the row i of $B = [b_{ij}]$ corresponds to an *affector* (transmitter) of the entity i. In the classical case, $m = n$ and B is a diagonal matrix with positive diagonal. That is, each entity i has one affector located at the entry (i, i) of B. In more general case the entity i may have several affectors corresponding to the positive entries in the row i of B. The assumption that each column B has one positive entry means that two different entities do not share a common affector. In view of the wireless network interpretation of the entries of A and B, it is assumed in [2] that $a_{ij}b_{ij} = 0$ for each pair (i, j). In our treatment we drop this assumption.

Note that if $m = n$, then B is called a *monomial matrix*. So $B = PD$, where P is an $m \times m$ permutation matrix and D is an $m \times m$ diagonal matrix with positive diagonal entires. Hence, $B^{-1} = D^{-1}P^\top$.

The following theorem gives an explicit formula for $\hat{\varrho}(A, B)$ of a WN-pair.

Theorem 6.2. *Assume that $A, B \in \mathbb{R}_+^{m \times n}$ is a WN-pair. Let $\mathcal{E}(A, B) \subset \Pi_n$ be a finite set of vectors $\mathbf{w}$ that satisfy the following five conditions:*

(1) *The vector $\mathbf{w} \in \Pi_n$ has $l \leqslant m$ nonzero coordinates.*
(2) *Let $\mathcal{I}$ be the set of zero rows of $B[[m], \mathrm{supp}\, \mathbf{w}]$. Then $|\mathcal{I}| = m - l$. (Hence, $B[[m] \setminus \mathcal{I}, \mathrm{supp}\, \mathbf{w}]$ is monomial.)*
(3) *$A[\mathcal{I}, \mathrm{supp}\, \mathbf{w}] = 0$.*
(4) *The matrix $C(\mathbf{w}) = B[[m] \setminus \mathcal{I}, \mathrm{supp}\, \mathbf{w}]^{-1}A[[m] \setminus \mathcal{I}, \mathrm{supp}\, \mathbf{w}]$ is irreducible.*
(5) *Let $0 < \mathbf{z} \in \mathbb{R}^l$ be the projection of $\mathbf{w}$ on $\mathrm{supp}\, \mathbf{w}$. Then the vector $\mathbf{z}$ is the PF-eigenvector of $C(\mathbf{w})$.*

The above conditions imply that $\varrho(C(\mathbf{w})) = r(A, B, \mathbf{w})$ and $A\mathbf{w} = \varrho(C(\mathbf{w}))B\mathbf{w}$. Furthermore,

$$\text{(19)} \qquad \hat{\varrho}(A, B) = \min\{r(A, B, \mathbf{w}),\ \mathbf{w} \in \mathcal{E}(A, B)\}.$$

In particular, $\mathbf{y}$ is a minimal weak optimal if and only if $\mathbf{y} \in \mathcal{E}(A, B)$ and $\mathbf{y}$ minimizes the right hand side of (19). Furthermore, each minimal weak optimal is a weak GPF-eigenvector.

P r o o f. We first justify that the assumption $|\mathcal{I}| = m - l$ in part (2) implies that $B_{11} = B[[m] \setminus \mathcal{I}, \mathrm{supp}\ \mathbf{w}]$ is a monomial matrix. Since each column of B has exactly one nonzero entry, it follows that $B_1 = B[[m], \mathrm{supp}\ \mathbf{w}]$ has l nonzero entries. Since $|\mathcal{I}| = m - l$, it follows that B_{11} has l nonzero rows. That is, each row and column of B_{11} has exactly one nonzero element. We next show that conditions (1)–(5) imply that $A\mathbf{w} = \varrho(C)B\mathbf{w}$. Let $A_1 = A[[m], \mathrm{supp}\ \mathbf{w}]$ and $A_{11} = A[[m] \setminus \mathcal{I}, \mathrm{supp}\ w]$. Since $\mathbf{z}$ is a PF-eigenvector of $C(\mathbf{w})$, it follows that $A_{11}\mathbf{z} = \varrho(C(\mathbf{w}))B_{11}\mathbf{z}$. Let $A_{21} = A[\mathcal{I}, \mathrm{supp}\ \mathbf{w}], B_{21} = B[\mathcal{I}, \mathrm{supp}\ \mathbf{w}]$. As $A_{21} = B_{21} = 0$, we deduce that $A_1\mathbf{w} = \varrho(C(\mathbf{w}))B_1\mathbf{w}$, which is equivalent to $A\mathbf{w} = \varrho(C)B\mathbf{w}$. Hence, $r(A, B, \mathbf{w}) = \varrho(C(\mathbf{w}))$.

In particular, $\hat{\varrho}(A, B) \leqslant r(A, B, \mathbf{w})$. Denote by $\varrho_1(A, B)$ the minimum in (19). Then $\hat{\varrho}(A, B) \leqslant \varrho_1(A, B)$. To show equality (19) it is enough to show that a minimal weak optimal $\mathbf{y}$ is in $\mathcal{E}(A, B)$. We show this claim by induction on m.

For $m = 1$ equality (19) trivially holds. Assume that each minimal weak optimal $\mathbf{y}$ is in $\mathcal{E}(A, B)$ for $m \leqslant M$. Suppose that $m = M + 1$. Assume that $\mathbf{y} \in \Pi_n$ is a minimal weak optimal vector with $l = |\mathrm{supp}\ \mathbf{y}|$. Part 3 of Theorem 5.1 yields that $l \leqslant m$. Let $\mathbf{0} < \mathbf{z}$ be the projection of $\mathbf{y}$ on its support. Let $\mathcal{J} = [n] \setminus \mathrm{supp}\ \mathbf{y}$, $A' = A(\emptyset, \mathcal{J})$ and $B' = B(\emptyset, \mathcal{J})$. Then $\hat{\varrho}(A, B) = \hat{\varrho}(A', B') = r(A', B', \mathbf{z})$. Denote by $\mathcal{I}$ the set of zero rows of B'. As $\hat{\varrho}(A, B) < \infty$, it follows that $\mathcal{I}$ is a set of zero rows of A'. Let $\widetilde{A} = A(\mathcal{I}, \mathcal{J})$, $\widetilde{B} = B(\mathcal{I}, \mathcal{J}) \in \mathbb{R}_+^{m' \times l}$. Thus $\hat{\varrho}(A, B) = \hat{\varrho}(\widetilde{A}, \widetilde{B}) = r(\widetilde{A}, \widetilde{B}, \mathbf{z})$. As B does not have a zero column, it follows that B' does not have a zero column. As $\mathcal{I}$ is the set of zero rows of B', it follows that $\widetilde{B}$ does not have zero columns or zero rows. As each column of $\widetilde{B}$ has one positive element, it follows that $\widetilde{B}$ has exactly l nonzero entries. Hence, $m' \leqslant l$.

The equality $\hat{\varrho}(A, B) = \hat{\varrho}(\widetilde{A}, \widetilde{B}) = r(\widetilde{A}, \widetilde{B}, \mathbf{z})$ yields that $\mathbf{z}$ is a weak optimal solution for $\hat{\varrho}(\widetilde{A}, \widetilde{B}) = \hat{\varrho}(A, B)$. The assumption that $\mathbf{y}$ is a minimal weak optimal yields that $\mathbf{z}$ is a minimal weak optimal for $(\widetilde{A}, \widetilde{B})$.

Assume first that $m' < m$. We apply the induction hypothesis to $\widetilde{A}, \widetilde{B}$ and a minimal $\mathbf{z}$ to deduce that $m' = l$. So $\widetilde{B}$ is a monomial matrix, $\hat{\varrho}(\widetilde{A}, \widetilde{B}) = \varrho(C)$ and $\widetilde{A}\mathbf{z} = \varrho(C)\widetilde{B}\mathbf{z}$. The induction hypothesis yields that C is irreducible. Hence, $\mathbf{y} \in \mathcal{E}(A, B)$, as we claimed.

It is left to discuss the case where $m' = m = l$. In this case we use part 4 of Theorem 5.1. So $\mathbf{y}$ is a weak GPF-eigenvector. In particular, $\widetilde{A}\mathbf{z} = \hat{\varrho}(A, B)\widetilde{B}\mathbf{z}$. Since $m' = m$, it follows that $\widetilde{B}$ is a monomial matrix. Observe next that $\varrho(C) = \hat{\varrho}(\widetilde{A}, \widetilde{B}) = \hat{\varrho}(C, I)$. Furthermore, $\mathbf{z} > \mathbf{0}$ is a minimal weak optimal vector of (C, I). Apply now part (4) of Theorem 3.2. A minimal weak optimal vector of (C, I) is supported on $V_i \subseteq [m]$ which corresponds to a source in the reduced graph $\vec{G}_{\mathrm{red}}$. Furthermore, $A[V_i]$ is irreducible. Since $\mathbf{z} > \mathbf{0}$, it follows that $V_i = [m]$ and C is irreducible. Hence, $\mathbf{y} \in \mathcal{E}(A, B)$. $\qquad\square$

The following notion of S-irreducibility was introduced in [2]:

Definition 6.3. A WN-pair $A, B \in \mathbb{R}_+^{m \times n}$ is called S-*irreducible* if the following condition holds: For each subset $\mathcal{K} \subseteq [n]$ of cardinality m such that $B[[m], \mathcal{K}]$ is a monomial matrix, the matrix $B[[m], \mathcal{K}]^{-1} A[[m], \mathcal{K}]$ is irreducible.

Note that if $n = m$, then S-irreducibility is equivalent to the irreducibility of $B^{-1} A$. The following proposition gives a sufficient condition for an S-irreducible pair:

Proposition 6.4. Let $A = [a_{ij}], B = [b_{ij}] \in \mathbb{R}_+^{m \times n}$ be a WN-pair. Assume that $a_{ij} > 0$ if $b_{ij} = 0$. Then the pair A, B is irreducible.

P r o o f. Let $\mathcal{K} \subseteq [n]$ of cardinality m such that $B_1 = B[[m], \mathcal{K}]$ is a monomial matrix. Let $A_1 = A[[m], \mathcal{K}]$. So A_1 has positive elements where the elements of B_1 are zero. Therefore all off-diagonal entries of $C = B_1^{-1} A_1$ are positive, and C is irreducible. $\qquad \square$

The following theorem gives an explicit formula for $\varrho(A, B)$ of WN-pair.

Theorem 6.5. Let $A, B \in \mathbb{R}_+^{m \times n}$ be a WN-pair. Denote by $\mathcal{M}(B)$ the subset of all $\mathcal{K} \subseteq [n]$ of cardinality m such that the matrix $B[[m], \mathcal{K}]$ is monomial. Then

$$(20) \qquad \varrho(A, B) = \min\{\varrho(B[[m], \mathcal{K}]^{-1} A([m], \mathcal{K})), \ \mathcal{K} \in \mathcal{M}(B)\}.$$

For each $\mathcal{K} \in \mathcal{M}(B)$ such that $\varrho(A, B) = \varrho(B[[m], \mathcal{K}]^{-1} A([m], \mathcal{K}))$ there exists an optimal $\mathbf{y} \in \Pi_n$ with the following property: The support of $\mathbf{y}$ is contained in $\mathcal{K}$ and $\mathbf{y}$ is a GPF-vector. Assume that WN-pair is S-irreducible. Then each such $\mathbf{y}$ is minimal optimal.

P r o o f. Assume first that the pair A, B is S-irreducible. Let $\mathcal{E}(A, B)$ be defined as in Theorem 6.2. We claim that $|\text{supp } \mathbf{w}| = m$ for each $\mathbf{w} \in \mathcal{E}(A, B)$. Assume to the contrary that $l = |\text{supp } \mathbf{w}| < m$. After relabeling the elements of the set $[n]$ we can assume that supp $\mathbf{w} = [l]$. Let $\mathcal{I}$ be a zero set of $B_1 = B[[m], [l]]$. Recall that $B_{11} = B[[m] \setminus \mathcal{I}, [l]]$ is monomial and $A_{21} = A[\mathcal{I}, [l]] = 0$. Relabel the elements of $[m]$ so that $\mathcal{I} = \{l+1, \ldots, m\}$. Since B does not have zero rows, there is a subset $\mathcal{J}$ of $[n]$ of cardinality $m - l$ such that $B[\mathcal{I}, \mathcal{J}]$ is a monomial matrix. Clearly, $[l] \cap \mathcal{J} = \emptyset$. Let $\mathcal{K} = [l] \cup \mathcal{J}$. Then $B_1 = B[[m], \mathcal{K}]$ is a monomial matrix, which is a direct sum of B_{11} and $B[\mathcal{I}, \mathcal{J}]$. Let $A_1 = A[[m], \mathcal{K}]$. Since A, B is S-irreducible, it follows that $C = B_1^{-1} A_1$ is irreducible. This contradicts the fact that $C[\mathcal{I}, [l]] = 0$.

Thus, for each $\mathbf{w} \in \mathcal{E}(A, B)$ we have that $|\text{supp } \mathbf{w}| = m$. Let $\mathbf{z} \in \Pi_m^\circ$ be the projection of $\mathbf{w}$ on supp $\mathbf{w}$. Then $C(\mathbf{w}) = B[[m], \text{supp } \mathbf{w}]^{-1} A[[m], \text{supp } \mathbf{w}]$ is irreducible,

and $\mathbf{z}$ is the PF-vector of $C(\mathbf{w})$. Hence, $B\mathbf{w} = B[[m], \text{supp } \mathbf{w}]\mathbf{z} > \mathbf{0}$. Theorem 6.2 yields that $\varrho(C(\mathbf{w})) = r(A, B, \mathbf{w})$.

Vice versa, each $\mathcal{K} \in \mathcal{M}(B)$ induces $\mathbf{w} \in \mathcal{E}(A, B)$ as follows. Let $\mathbf{z} \in \Pi_m^\circ$ be the PF-eigenvector of $C = B[[m], \mathcal{K}]^{-1}A([m], \mathcal{K})$. Then $\mathbf{w} \in \Pi_n$ is obtained from $\mathbf{z}$ by adding zero coordinates. As C is irreducible, we deduce that $\mathbf{w} \in \mathcal{E}(A, B)$.

Recall that $\hat{\varrho}(A, B)$ is given by (19). Assume that $\mathbf{y}$ is a minimal weak optimal. So $\mathbf{y} \in \mathcal{E}(A, B)$ and $B\mathbf{y} > \mathbf{0}$. Lemma 2.7 yields that $\mathbf{y}$ is minimal optimal. Hence, $\varrho(A, B) = \hat{\varrho}(A, B)$. Characterization (20) follows from (19). Furthermore, $\mathbf{y}$ is GPF-vector. This proves the theorem in the case where A, B is S-irreducible.

Assume now that A, B is not S-irreducible. Let $J \in \mathbb{R}^{m \times n}$ be a matrix whose all entries are 1. For $l \in \mathbb{N}$ denote $A_l = A + J/l$. So $A_l > 0$, and Proposition 6.4 implies that the pair A_l, B is S-irreducible. Fix $\mathcal{K} \in \mathcal{M}(B)$. Let $C = B[[m], \mathcal{K}]^{-1}A([m], \mathcal{K})$ and $C_l = B[[m], \mathcal{K}]^{-1}A_l([m], \mathcal{K})$. Observe that $A_{l+1} \leqslant A_l$ for $l \in \mathbb{N}$. Hence, $C_{l+1} \leqslant C_l$ for $l \in \mathbb{N}$, and $\lim_{l \to \infty} C_l = C$. Therefore $\varrho(C_l)$, $l \in \mathbb{N}$ is a decreasing sequence which converges to $\varrho(C)$. Apply characterization (20) to $\varrho(A_l, B)$. Let $\varrho_1(A, B)$ be the right hand side of (20). It now follows that $\lim_{l \to \infty} \varrho(A_l, B) = \varrho_1(A, B)$. Part 5 of Lemma 2.7 yields that $\varrho(A, B) = \varrho_1(A, B)$. Hence characterization (20) holds.

Assume that $\mathcal{K} \in \mathcal{M}(B)$ and $\varrho(A, B) = \varrho(B[[m], \mathcal{K}]^{-1}A([m], \mathcal{K}))$. For each $l \in \mathbb{N}$ let $\mathbf{w}_l \in \Pi_n$ be be the vector induced by the PF-eigenvector $\mathbf{0} < \mathbf{z}_l \in \Pi_m^\circ$ of $B[[m], \mathcal{K}]^{-1}A_l([m], \mathcal{K})$. Let $r_l = \varrho(B[[m], \mathcal{K}]^{-1}A_l([m], \mathcal{K}))$. Then $A_l\mathbf{w}_l = r_l B\mathbf{w}_l$. Pick a convergent subsequence $\mathbf{w}_{l_k} \to \mathbf{y}$, $k \to \infty$. Then $A\mathbf{y} = \varrho(A, B)B\mathbf{y}$.

We claim that $\mathbf{y}$ is optimal. Choose $\mathbf{x}_l \in \Pi_n^\circ$ such that $r(A_l, B, \mathbf{x}_l) \leqslant (\varrho(A_l, B) + 1/l)\mathbf{x}_l$. Clearly, $\varrho(A, B) \leqslant \varrho(A_l, B) \leqslant r_l$. Set $\mathbf{v}_l = (1 - 1/l)\mathbf{y} + \mathbf{x}_l/l > \mathbf{0}$. Then $\lim_{k \to \infty} \mathbf{v}_{l_k} = \mathbf{y}$ and $\lim_{k \to \infty} r(A_l, B, \mathbf{x}_l) = \varrho(A, B)$. Hence $\mathbf{y}$ is optimal. $\qquad\square$

To summarize, if a WN-pair is S-irreducible, then each minimal optimal $\mathbf{y}$, which is a GPF-vector, corresponds to an optimal choice of one transmitter for each receiver. If a WN-pair is not S-irreducible, there exists an optimal $\mathbf{y}$, which is a GPF-vector, and corresponds to an optimal choice of one transmitter for each receiver. However, for some receivers all their transmitters may shut off. This can happen in the classical case where $m = n$ and $B = I_m$. For example:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then the only optimal $\mathbf{y} \in \Pi_4$ is $\mathbf{y} = \frac{1}{2}(1, 1, 0, 0)^\top$.

7. A pair of CP-operators

Recall that $\mathbb{C}^n$ is equipped with the standard inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^* \mathbf{x}$. Given a finite dimensional inner product space over $\mathbb{C}$, with a product $\langle \cdot, \cdot \rangle$, we denote by $\mathrm{S}(\mathbf{V}) \supset \mathrm{S}_+(\mathbf{V}) \supset \mathrm{S}_{+,1}(\mathbf{V})$ the real space of self adjoint operators $A \colon \mathbf{V} \to \mathbf{V}$, the cone of positive semidefinite operators and the convex set of all positive semidefinite operators with trace 1, respectively. By fixing an orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in $\mathbf{V}$ we identify $\mathrm{S}(\mathbf{V}), \mathrm{S}_+(\mathbf{V}), \mathrm{S}_{+,1}(\mathbf{V})$ with $\mathrm{H}_n, \mathrm{H}_{+,n}, \mathrm{H}_{+,1,n}$, respectively.

On $\mathbb{C}^{n \times n}$ we have the standard inner product $\langle U, V \rangle = \mathrm{tr}\, V^* U$. For $X \in \mathrm{H}_n$ we denote by $\lambda_1(A) \geqslant \ldots \geqslant \lambda_n(X)$ the n-real eigenvalues of X counted with their multiplicities. Recall that for $X, Y \in \mathrm{H}_n$ we say that $Y \succeq X$ if $Y - X \in \mathrm{H}_{+,n}$, i.e. $Y - X$ is positive semidefinite. Denote by $\mathrm{H}_{++,n}$ the interior of the cone $\mathrm{H}_{+,n}$, i.e. the open set of positive definite $n \times n$ Hermitian matrices. Then $Y \succ X$ if $Y - X \in \mathrm{H}_{++,n}$. Let $\mathrm{H}_{++,1,n} = \mathrm{H}_{+,1,n} \cap \mathrm{H}_{++,n}$. Denote by $CP(n,m)$ the cone of completely positive operators from H_n to H_m, given by (9). In the rest of the paper we assume that $\mathcal{A}, \mathcal{B} \in CP(n,m)$. Then we can define $\varrho(\mathcal{A}, \mathcal{B})$ and $\hat{\varrho}(\mathcal{A}, \mathcal{B})$ as in (4), (5) and (6) with respect to the cones $\mathbf{K}_1 = \mathrm{H}_{+,n}, \mathbf{K}_2 = \mathrm{H}_{+,m}$. We call $Y \in \mathrm{H}_{+,n} \setminus \{0\}$ weak optimal if $\hat{\varrho}(\mathcal{A}, \mathcal{B}) = r(\mathcal{A}, \mathcal{B}, Y)$. We call $Y \in \mathrm{H}_{+,n} \setminus \{0\}$ optimal if the following conditions hold: First, $\varrho(\mathcal{A}, \mathcal{B}) = r(\mathcal{A}, \mathcal{B}, Y)$. Second, there exists a sequence $X_l \in \mathrm{H}_{++,n}$ such that $\lim_{l \to \infty} X_l = Y$ and $\lim_{l \to \infty} r(\mathcal{A}, \mathcal{B}, X_l) = r(\mathcal{A}, \mathcal{B}, Y)$. We say that Y is a generalized Perron-Frobenius vector or weak generalized Perron-Frobenius vector if

$$(21) \qquad \mathcal{A}(Y) = \varrho(A, B)\mathcal{B}(Y), \quad \text{or} \quad \mathcal{A}(Y) = \hat{\varrho}(A, B)\mathcal{B}(Y), \quad Y \in \mathrm{H}_{+,n} \setminus \{0\},$$

respectively.

Given a pair $\mathcal{C}, \mathcal{D} \in CP(n,m)$ we say that $\mathcal{C} \succeq \mathcal{D}$ or $\mathcal{C} \succ \mathcal{D}$ if for each $X \in \mathrm{H}_{+,1,n}$ we have that $\mathcal{C}(X) \succeq \mathcal{D}(X)$ or $\mathcal{C}(X) \succ \mathcal{D}(X)$, respectively. If $\mathcal{D} = 0$, $\mathcal{C} \succ 0$ is called a *positive CP-operator*. An example of positive CP-operator $\mathcal{I}(n,m)$ is the operator $\mathcal{I}(n,m)(Z) = (\mathrm{tr}\, Z)I_m$ for any $Z \in \mathbb{C}^{n \times n}$. (We will justify briefly why $\mathcal{I}(n,m)$ is completely positive in the next section.)

In this paper we will concentrate on $\hat{\varrho}(\mathcal{A}, \mathcal{B})$ since this quantity is much easier to deal with. When the proofs of our results for CP-pair are very similar for the matrix pair $A, B \in \mathbb{R}_+^{m \times n}$, we will omit the proofs.

For $\mathcal{A}, \mathcal{B} \in CP(n,m)$ and $X \in \mathrm{H}_{n,+,1}$ we give a formula to compute $r(\mathcal{A}, \mathcal{B}, X)$. To do that we need to recall the classical definition of the Rayleigh quotient for $A, B \in \mathrm{H}_{+,m}$, see [12], §4.4:

Lemma 7.1. *Let* $A, B \in \mathrm{H}_{+,m}$. *Define*

$$(22) \qquad \lambda(A, B) = \sup_{\mathbf{x} \in \mathbb{C}^m \setminus \{\mathbf{0}\}} \frac{\mathbf{x}^* A \mathbf{x}}{\mathbf{x}^* B \mathbf{x}}.$$

Then the above supremum is achieved for some $\mathbf{y} \in \mathbb{C}^m \setminus \{\mathbf{0}\}$:

(1) $\lambda(A, B) = \infty$ *if and only if* $\ker B$ *is not a subset of* $\ker A$. *Then* $\mathbf{y} \in \ker B \setminus \ker A$.

(2) *If* $A = B = 0$, *then* $\lambda(A, B) = 0$ *and* $\mathbf{y}$ *is any nonzero vector in* $\mathbb{C}^m$.

(3) *Assume that* $\ker B \subseteq \ker A$ *and* $\dim \ker B < m$. *Let* $\mathbf{V} \subseteq \mathbb{C}^m$ *be the orthogonal complement of* $\ker B$. *Then* $\mathbf{V}$ *is an invariant subspace of* A *and* B. *Denote by* A_1, B_1 *the restrictions of* A, B *to* $\mathbf{V}$. *Then* $B_1^{-1} A_1$ *is a diagonalizable operator in* $\mathbf{V}$ *with nonnegative eigenvalues. Furthermore,*

$$(23) \qquad \lambda(A, B) = \lambda(A_1, B_1) = \varrho(B_1^{-1} A_1).$$

Moreover, a maximizing $\mathbf{y}$ *of the quotient* (22) *can be chosen to be an eigenvector of* $B_1^{-1} A_1$ *corresponding to* $\varrho(B_1^{-1} A_1)$.

In particular,

$$(24) \qquad \lambda(A, B) = \inf\{t \geqslant 0, \ tB \succeq A\}.$$

P r o o f. Parts (1) and (2) are straightforward. We now prove (3). Suppose first that $\ker B = \{\mathbf{0}\}$. So $B \in \mathrm{H}_{++,m}$. Let $C = \sqrt{B}$ be the unique root of B such that $C \in \mathrm{H}_{++,m}$. Set $\mathbf{x} = C^{-1}\mathbf{z}$. Then

$$\frac{\mathbf{x}^* A \mathbf{x}}{\mathbf{x}^* B \mathbf{x}} = \frac{\mathbf{z}^* C^{-1} A C^{-1} \mathbf{z}}{\mathbf{z}^* \mathbf{z}}.$$

Thus the supremum (22) is the maximum characterization of the maximum eigenvalue of $C^{-1} A C^{-1} \in \mathrm{H}_{+,m}$. So

$$\lambda(A, B) = \lambda(C^{-1} A C^{-1}, I_m) = \varrho(C^{-1} A C^{-1}) = \varrho(C^{-2} A) = \varrho(B^{-1} A).$$

As $C^{-1} A C^{-1} \in \mathrm{H}_{+,m}$ has nonnegative eigenvalues, it follows that $B^{-1} A$ is diagonizable with nonnegative eigenvalues.

Suppose now that $1 \leqslant \dim \ker B < m$. Let $\mathbf{V} = \ker B^\perp$. Then: $A\mathbf{V} \subseteq \mathbf{V} = B\mathbb{C}^n$. Suppose that $\mathbf{x} \in \ker B \setminus \{\mathbf{0}\}$. Then

$$\frac{\mathbf{x}^* A \mathbf{x}}{\mathbf{x}^* B \mathbf{x}} = \frac{0}{0} = 0.$$

Assume that $\mathbf{x} \in \mathbb{C}^m \setminus \ker B$. Then $\mathbf{x} = \mathbf{u} + \mathbf{v}$, where $\mathbf{u} \in \mathbf{V} \setminus \{\mathbf{0}\}$ and $\mathbf{v} \in \ker B$. Clearly,

$$\frac{\mathbf{x}^* A \mathbf{x}}{\mathbf{x}^* B \mathbf{x}} = \frac{\mathbf{u}^* A \mathbf{u}}{\mathbf{u}^* B \mathbf{u}} = \frac{\mathbf{u}^* A_1 \mathbf{u}}{\mathbf{u}^* B_1 \mathbf{u}}.$$

Apply the previous arguments to $A_1, B_1 \in \mathrm{S}_+(\mathbf{V})$ to deduce that $\lambda(A, B) = \lambda(A_1, B_1) = \varrho(B_1^{-1} A_1)$.

The characterization (24) follows straightforward from (22). $\qquad\square$

Corollary 7.2. *Let $\mathcal{A}, \mathcal{B} \in CP(n, m)$ and $X \in \mathrm{H}_{+,n} \setminus \{0\}$. Then $r(\mathcal{A}, \mathcal{B}, X) = \lambda(\mathcal{A}(X), \mathcal{B}(X))$.*

Lemma 7.3. *Let $\mathcal{A}, \mathcal{B} \in CP(n, m)$. Then:*

(1) *$\hat{\varrho}(\mathcal{A}, \mathcal{B}) \leqslant \varrho(\mathcal{A}, \mathcal{B})$.*
(2) *Assume that $\mathcal{A}_1, \mathcal{B}_1 \in CP(n, m)$ and $\mathcal{A}_1 \preceq \mathcal{A}, \mathcal{B} \preceq \mathcal{B}_1$. Then $\varrho(\mathcal{A}_1, \mathcal{B}_1) \leqslant \varrho(\mathcal{A}, \mathcal{B})$ and $\hat{\varrho}(\mathcal{A}_1, \mathcal{B}_1) \leqslant \hat{\varrho}(\mathcal{A}, \mathcal{B})$.*
(3) *There exists a weak optimal $Y \in \mathrm{H}_{n,+,1}$.*
(4) *Assume that there exists a weak optimal Y such that either $\mathcal{B}(Y) \succ 0$ or $\mathcal{A}(Y) \succ 0$. If $\varrho(\mathcal{A}, \mathcal{B}) < \infty$, then $\varrho(\mathcal{A}, \mathcal{B}) = \hat{\varrho}(\mathcal{A}, \mathcal{B})$.*
(5) *Suppose that $\varrho(\mathcal{A}, \mathcal{B}) < \infty$, and either $\mathcal{A} \succ 0$ or $\mathcal{B} \succ 0$. Then $\varrho(\mathcal{A}, \mathcal{B}) = \hat{\varrho}(A, B)$.*
(6) *Assume that $0 \prec \mathcal{D}_l \in CP(n, m)$ for $l \in \mathbb{N}$ and $\lim_{l \to \infty} \mathcal{D}_l = 0$. Then $\lim_{l \to \infty} \varrho(\mathcal{A}, \mathcal{B} + \mathcal{D}_l) = \hat{\varrho}(\mathcal{A}, \mathcal{B})$.*
(7) *Assume that $0 \prec \mathcal{D}_l \in CP(n, m)$ for $l \in \mathbb{N}$ and $\lim_{l \to \infty} \mathcal{D}_l = 0$. If $\mathcal{B}(I_n) \succ 0$, then $\lim_{l \to \infty} \varrho(\mathcal{A} + \mathcal{D}_l, \mathcal{B}) = \varrho(\mathcal{A}, \mathcal{B})$.*

P r o o f. (1) Trivial.

(2) Straightforward from the definitions.

(3) As in the proof of part (1) of Lemma 2.6.

(4) As in the proof of part (2) of Lemma 2.7.

(5) As in the proof of part (3) of Lemma 2.7.

(6) We use similar arguments as in the proof of part (4) of Lemma 2.7 with the following modifications. Let $\mathcal{B}_l = \mathcal{C}_l + \mathcal{D}_l$, where $\mathcal{C}_l = \mathcal{B}$. Then the arguments of part (4) of Lemma 2.7 apply.

(7) We use similar arguments as in the proof of part (5) of Lemma 2.7 with the following modifications. Let $\mathcal{A}_l = \mathcal{C}_l + \mathcal{D}_l$, where $\mathcal{C}_l = \mathcal{A}$. Then the arguments of part (5) of Lemma 2.7 apply. $\qquad\square$

Let $\mathcal{C} \colon \mathrm{H}_n \to H_m$ be a linear operator. Then there exists a dual operator $\mathcal{C}^\vee \colon \mathrm{H}_m \to \mathrm{H}_n$ which is defined as follows. Recall that on H_n one has the inner product $\langle X, Z \rangle = \operatorname{tr} XZ$, where $\operatorname{tr} W$ is the trace of the matrix $W \in \mathbb{C}^{n \times n}$. Then

$\mathcal{C}^\vee \colon \mathrm{H}_m \to \mathrm{H}_n$ is defined uniquely by the property $\langle \mathcal{C}(X), Z \rangle = \langle X, \mathcal{C}^\vee(Z) \rangle$ for all $X \in \mathrm{H}_n$ and $Z \in \mathrm{H}_m$. Assume that $\mathcal{C}$ is CP-operator given by (9). Then $\mathcal{C}$ is called *unital* if $\mathcal{C}(I_n) = I_m$. Recall that $\mathcal{C}^\vee$ is also completely positive and given by $C^\vee(Y) = \sum_{j=1}^{k} T_j^* Y T_j$. Thus, $\mathcal{C}$ is a quantum channel if and only if $\mathcal{C}^\vee$ is unital.

The following lemma is an analog of Lemma 2.2:

Lemma 7.4. *Let $\mathcal{A}, \mathcal{B} \in CP(n, m)$. Then:*

(1) $\hat{\varrho}(\mathcal{A}, \mathcal{B}) = 0$ *if and only if $\mathcal{A}^\vee(I_m)$ is not positive definite.*

(2) *Assume that $\mathcal{A}^\vee(I_m)$ is positive definite. Then*

$$(25) \qquad \hat{\varrho}(\mathcal{A}, \mathcal{B}) \geqslant \varrho(\mathcal{A}^\vee(I_m)^{-1} \mathcal{B}^\vee(I_m))^{-1}.$$

In particular, if $\mathcal{A}$ and $\mathcal{B}$ are quantum channels, then $\varrho(\mathcal{A}, \mathcal{B}) \geqslant 1$.

P r o o f. (1) Clearly, $\hat{\varrho}(\mathcal{A}, \mathcal{B}) = 0$ if and only if there exists $X \in \mathrm{H}_{+,1,n}$ such that $\mathcal{A}(X) = 0$. Recall that $Y \in \mathrm{H}_{m,+}$ is zero if and only if $\langle Y, I_m \rangle = 0$. So $\mathcal{A}(X) = 0$ if and only if $\langle \mathcal{A}(X), I_m \rangle = \langle X, \mathcal{A}^\vee(I_m) \rangle = 0$. Hence, $\mathcal{A}^\vee(I_m) \not\succ 0$.

Vice versa, assume that $\mathcal{A}^\vee(I_m) \not\succ 0$. Therefore, there exists a vector $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x}^* \mathbf{x} = 1$, such that $\mathcal{A}^\vee(I_m)\mathbf{x} = \mathbf{0}$. In particular,

$$0 = \mathbf{x}^* \mathcal{A}^\vee(I_m)\mathbf{x} = \langle \mathbf{x}\mathbf{x}^*, \ \mathcal{A}^\vee(I_m) \rangle = \langle \mathcal{A}(\mathbf{x}\mathbf{x}^*), \ I_m \rangle \Rightarrow \mathcal{A}(\mathbf{x}\mathbf{x}^*) = 0.$$

(2) Assume that $\mathcal{A}^\vee(I_m) \succ 0$. Let $X \in \mathrm{H}_{n,+,1}$. Assume that rank $X = r \in [n]$. Then spectral decomposition of X is $X = \sum_{i=1}^{r} \lambda_i \mathbf{x}_i \mathbf{x}_i^*$, where each $\lambda_i > 0$ and $\mathbf{x}_j^* \mathbf{x}_i = \delta_{ij}$ for $i, j \in [r]$. As $\mathcal{A}(X) \leqslant r(\mathcal{A}, \mathcal{B}, X)\mathcal{B}(X)$, it follows that

$$\langle X, \mathcal{A}^\vee(I_m) \rangle = \langle \mathcal{A}(X), \ I_m \rangle \leqslant r(\mathcal{A}, \mathcal{B}, X)\langle \mathcal{B}(X), \ I_m \rangle = r(\mathcal{A}, \mathcal{B}, X)\langle X, \mathcal{B}^\vee(I_m) \rangle.$$

Hence,

$$\begin{aligned} r(\mathcal{A}, \mathcal{B}, X)^{-1} &\leqslant \frac{\langle X, \mathcal{B}^\vee(I_m) \rangle}{\langle X, \mathcal{A}^\vee(I_m) \rangle} = \frac{\sum_{i=1}^{r} \lambda_i \mathbf{x}_i^* \mathcal{B}^\vee(I_m)\mathbf{x}_i}{\sum_{i=1}^{r} \lambda_i \mathbf{x}_i^* \mathcal{A}^\vee(I_m)\mathbf{x}_i} \\ &\leqslant \max_{i \in [r]} \frac{\mathbf{x}_i^* \mathcal{B}^\vee(I_m)\mathbf{x}_i}{\mathbf{x}_i^* \mathcal{A}^\vee(I_m)\mathbf{x}_i} \leqslant \lambda(\mathcal{B}^\vee(I_m), \mathcal{A}^\vee(I_m)) = \varrho(\mathcal{A}^\vee(I_m)^{-1}\mathcal{B}^\vee(I_m)). \end{aligned}$$

This establishes (25).

Assume that $\mathcal{A}$ and $\mathcal{B}$ are quantum channels. Then $\mathcal{A}^\vee(I_m) = \mathcal{B}^\vee(I_m) = I_n$. Hence, $\varrho(\mathcal{A}^\vee(I_m)^{-1}\mathcal{B}^\vee(I_m)) = 1$ and then $\hat{\varrho}(\mathcal{A}, \mathcal{B}) \geqslant 1$. $\square$

8. Polynomial approximation of $\varrho(\mathcal{A}, \mathcal{B})$ for δ-positive $\mathcal{B}$

In this section we assume that $\mathcal{A}, \mathcal{B} \in CP(n, m)$. Suppose furthermore that $0 < \varrho(\mathcal{A}, \mathcal{B}) < \infty$. We now want to apply the bisection algorithm as in the proof of Theorem 4.2. To this end, for a given $t > 0$ we need to decide if the intersection $(t\mathcal{B} - \mathcal{A})(\mathrm{H}_{+,1,n}) \cap \mathrm{H}_{+,m}$ is empty or not.

Let $X \in \mathrm{H}_n$ and $Z \in \mathrm{H}_m$. The square of the distance between $t\mathcal{B}(X) - \mathcal{A}(X)$ and Z is given by the following quadratic convex function:

$$(26) \quad f_t(X, Z) = \langle t\mathcal{B}(X) - \mathcal{A}(X) - Z, t\mathcal{B}(X) - \mathcal{A}(X) - Z \rangle, \quad X \in \mathrm{H}_n, Z \in \mathrm{H}_m.$$

We assume that (X, Z) are in the cone $\mathrm{H}_{+,n} \times \mathrm{H}_{+,m}$ subject to the linear constrain $\operatorname{tr} X = 1$. Note that $f_t(X, Z) \geqslant 0$. Finding the distance between the two convex sets $(t\mathcal{B} - \mathcal{A})(\mathrm{H}_{+,1,n})$ and $\mathrm{H}_{+,m}$ is equivalent to the minimization problem

$$(27) \qquad \mu_0(t) = \min\{f_t(X, Z),\ X \in \mathrm{H}_{+,1,n},\ Z \in \mathrm{H}_{+,m}\}.$$

This minimization problem can be dealt with by the standard interior point methods [5]. Fix $\tau > 0$. Assume that we found an approximation $\mu(t) \in \mathbb{Q}_{++}$ of $\mu_0(t)$ by an interior method within precision τ in polynomial time in the data. If $\mu(t) \geqslant 2\tau$, then $\operatorname{dist}((t\mathcal{B} - \mathcal{A})(\mathrm{H}_{+,1,n}), \mathrm{H}_{+,m}) \geqslant \tau$. Hence $\varrho(\mathcal{A}, \mathcal{B}) > t$. Suppose that $\mu(t) < 2\tau$. How can we estimate from above $\varrho(A, B)$? Recall that $\mu(t) = f_t(X(t), Z(t))$. So

$$(28) \qquad t\mathcal{B}(X(t)) = \mathcal{A}(X(t)) + Z(t) + W,$$
$$X(t) \in \mathrm{H}_{++,1,n},\ Z(t) \in \mathrm{H}_{++,m},\ W \in \mathrm{H}_m,\ \|W\|^2 = \operatorname{tr} W^2 = \mu(t) < 2\tau.$$

To find an upper bound for $\varrho(\mathcal{A}, \mathcal{B})$ from (28) we need to assume a positivity condition on $\mathcal{B}$.

Definition 8.1. Assume that m, n are two positive integers and $\delta \geqslant 0$. Then:

(1) Denote by $\mathcal{I}(n, m)\colon \mathrm{H}_n \to \mathrm{H}_m$ the linear transformation $\mathcal{I}(n, m)(X) = (\operatorname{tr} X)I_m$ for $X \in \mathrm{H}_n$.
(2) A real linear transformation $\mathcal{L}\colon \mathrm{H}_n \to \mathrm{H}_m$ is called δ-positive if $\mathcal{L} - \delta\mathcal{I}(n, m)$ is completely positive.

Let $\mathcal{L}\colon \mathrm{H}_n \to \mathrm{H}_m$ be a real linear transformation. Since any $F \in \mathbb{C}^{n \times n}$ is of the form $F = X + iY$, where $X, Y \in \mathrm{H}_n$ and $i^2 = -1$, it follows that $\mathbb{C}^{n \times n}$ is the complexification of H_n. Hence, $\mathcal{L}$ extends to linear operators $\widehat{\mathcal{L}}\colon \mathbb{C}^{n \times n} \to \mathbb{C}^{m \times m}$ over $\mathbb{C}$ by letting $\widehat{\mathcal{L}}(iX) = i\mathcal{L}(X)$ for $X \in \mathrm{H}_n$. We will identify $\widehat{\mathcal{L}}$ with $\mathcal{L}$ and no

confusion will arise. Note that for $U \in \mathbb{C}^{n \times n}$ we have that $\mathcal{L}(U^*) = \mathcal{L}(U)^*$. Associate with $\mathcal{L}$ the following block Hermitian matrix of dimension mn:

$$(29) \quad Z(\mathcal{L}) = \begin{bmatrix} \mathcal{L}(\mathbf{e}_1\mathbf{e}_1^*) & \mathcal{L}(\mathbf{e}_1\mathbf{e}_2^*) & \dots & \mathcal{L}(\mathbf{e}_1\mathbf{e}_n^*) \\ \mathcal{L}(\mathbf{e}_2\mathbf{e}_1^*) & \mathcal{L}(\mathbf{e}_1\mathbf{e}_2^*) & \dots & \mathcal{L}(\mathbf{e}_2\mathbf{e}_n^*) \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{L}(\mathbf{e}_n\mathbf{e}_1^*) & \mathcal{L}(\mathbf{e}_n\mathbf{e}_2^*) & \dots & \mathcal{L}(\mathbf{e}_n\mathbf{e}_n^*) \end{bmatrix}, \quad \mathbf{e}_i = (\delta_{1i}, \dots, \delta_{ni})^\top, \ i \in [n].$$

Denote by $\lambda_{mn}(Z(\mathcal{L}))$ the smallest eigenvalue of $Z(\mathcal{L})$. The following lemma follows from Choi's characterization of completely positive operators [6], [13].

Lemma 8.2. *Let $\delta \geqslant 0$. A real linear transformation $\mathcal{L} \colon \mathrm{H}_n \to \mathrm{H}_m$ is δ-positive if and only if $\lambda_{mn}(Z(\mathcal{L})) \geqslant \delta$.*

P r o o f.   Recall Choi's theorem [6] that $\mathcal{L} - \delta\mathcal{I}(n,m)$ is completely positive if and only if $\lambda_{mn}(Z(\mathcal{L} - \delta\mathcal{I}(n,m))) \geqslant 0$. Clearly, for $U \in \mathbb{C}^{n \times n}$ we have that $\mathcal{I}(n,m)(U) = (\operatorname{tr} U)I_m$. Hence $Z(\mathcal{I}(n,m)) = I_{mn}$, and $Z(\mathcal{L} - \delta\mathcal{I}(n,m)) = Z(\mathcal{L}) - \delta I_{mn}$. Thus $\lambda_{mn}(Z(\mathcal{L} - \delta\mathcal{I}(n,m))) = \lambda_{mn}(Z(\mathcal{L})) - \delta \geqslant 0$ if and only if $\lambda_{mn}(Z(\mathcal{L})) \geqslant \delta$. $\square$

Corollary 8.3. *Let $\delta \geqslant 0$ and assume that $\mathcal{L} \colon \mathrm{H}_n \to \mathrm{H}_m$ is δ-positive. Then $\mathcal{L}$ is completely positive. In particular, $\mathcal{I}(n,m) \in CP(n,m)$.*

Let $\mathcal{C} \colon \mathbb{C}^{n \times n} \to \mathbb{C}^{m \times m}$ be a linear transformation over $\mathbb{C}$. Then $\mathcal{C}$ is called rationally represented if the entries of each matrix $\mathcal{L}(\mathbf{e}_i\mathbf{e}_j^*)$ are Gaussian rationals, denoted as $\mathbb{Q} + i\mathbb{Q}$. Assume that $\mathcal{C}$ is rationally represented. Denote by $\langle \mathcal{C} \rangle = \sum\limits_{i,j=1}^{n} \langle \mathcal{L}(\mathbf{e}_i\mathbf{e}_j^*) \rangle$ the complexity of $\mathcal{C}$.

Theorem 8.4. *Let $\mathcal{A}, \mathcal{B} \in CP(n,m)$ be rationally represented. Assume furthermore that $\mathcal{B}$ is δ-positive for a given rational $\delta > 0$. Then $\hat{\varrho}(\mathcal{A}, \mathcal{B}) = \varrho(\mathcal{A}, \mathcal{B})$. Suppose that $\varrho(\mathcal{A}, \mathcal{B}) > 0$. Then for any $\varepsilon \in (0,1) \cap \mathbb{Q}$ one can find $\tilde{\varrho}(\mathcal{A}, \mathcal{B}) \in \mathbb{Q}_{++}$, in polynomial time in $\langle \mathcal{A} \rangle + \langle \mathcal{B} \rangle + \langle \delta \rangle + \langle \varepsilon \rangle$, such that*

$$(30) \qquad\qquad \tilde{\varrho}(\mathcal{A}, \mathcal{B}) \leqslant \varrho(\mathcal{A}, \mathcal{B}) \leqslant (1 + \varepsilon)\tilde{\varrho}(\mathcal{A}, \mathcal{B}).$$

P r o o f.   As $\mathcal{B}(I_n) \succeq \delta(\operatorname{tr} I_n)I_m = \delta n I_m$, we deduce that $r(\mathcal{A}, \mathcal{B}, I_n) \leqslant (n\delta)^{-1}\varrho(\mathcal{A}(I_n)) < t_0 \in \mathbb{Q}_{++}$. Hence, $\varrho(\mathcal{A}, \mathcal{B}) < t_0$. As $\mathcal{B} \succ 0$, part 5 of Lemma 7.3 yields that $\hat{\varrho}(A, B) = \varrho(A, B)$. As $\hat{\varrho}(\mathcal{A}, \mathcal{B}) > 0$, Lemma 7.4 yields that $\mathcal{A}^\vee(I_m) \succ 0$. Hence, $\varrho(\mathcal{A}, \mathcal{B}) = \hat{\varrho}(\mathcal{A}, \mathcal{B}) \geqslant \varrho(\mathcal{A}^\vee(I_m)^{-1}\mathcal{B}^\vee(I_m))^{-1} > s_0 \in \mathbb{Q}_{++}$. Note that the values of s_0, t_0 are based on the eigenvalue computations of $\lambda(A, B)$, given by (22), whose approximation is polynomial [19].

We now start a bisection problem as in the proof of Theorem 4.2. Suppose that we know that $\varrho(\mathcal{A}, \mathcal{B}) \in [s_k, t_k]$, where $s_k, t_k \in \mathbb{Q}_{++}$. Let $t = (s_k + t_k)/2$. Consider the minimum problem (27). If $\mu(t) \geqslant 2\tau$, then $\varrho(\mathcal{A}, \mathcal{B}) > t$. and we let $s_{k+1} = t$, $t_{k+1} = t_k$. Assume now $\mu(t) < 2\tau$. We claim that

$$(31) \qquad \qquad \varrho(\mathcal{A}, \mathcal{B}) \leqslant t + \frac{\sqrt{2\tau}}{\delta}.$$

Indeed, let $X(t)$, $Z(t)$ and W be defined as in (28). Let $\lambda_1(W) \geqslant \ldots \geqslant \lambda_m(W)$ be the m-eigenvalues of W. Then

$$2\tau > \|W\|^2 = \langle W, W \rangle = \sum_{j=1}^{m} \lambda_j(W)^2 \geqslant \lambda_m(W)^2.$$

As $\sqrt{2\tau} I_m + W \geqslant 0$, we obtain

$$\left(t + \frac{\sqrt{2\tau}}{\delta} \right) \mathcal{B}(X(t)) - \mathcal{A}(X(t)) = Z(t) + \frac{\sqrt{2\tau}}{\delta} \mathcal{B}(X(t)) + W \geqslant Z(t) + \frac{\sqrt{2\tau}}{\delta} \delta I_m + W \geqslant 0.$$

Thus $r(\mathcal{A}, \mathcal{B}, X(t)) \leqslant t + \sqrt{2\tau}/\delta$, which implies (31). Now choose $\tau = (s_k + t_k)^2 \delta^2/32$. So in the case $\mu(t) < 2\tau$ we set $s_{k+1} = s_k$, $t_{k+1} = (s_k + 3t_k)/4$.

To conclude we showed that $\varrho(\mathcal{A}, \mathcal{B}) \in [s_{k+1}, t_{k+1}]$, where $[s_{k+1}, t_{k+1}] \subset [s_k, t_k]$ and $|t_{k+1} - s_{k+1}| \leqslant \frac{3}{4} |t_k - s_k|$. Hence, in polynomial time in $\langle A \rangle + \langle B \rangle + \langle \delta \rangle + \langle \varepsilon \rangle$ we get the approximation $\tilde{\varrho}(\mathcal{A}, \mathcal{B}) \in \mathbb{Q}_{++}$ satisfying (30). $\qquad \square$

9. Minimal weak optimal solutions for CP-operators

For $X \in \mathrm{H}_{+,n}$ we call range $X \subseteq \mathbb{C}^n$ the support of X, and denote supp $X = $ range X. So dim supp $X = $ rank X. Assume that $\mathcal{A}, \mathcal{B} \in CP(n, m)$. Suppose furthermore that $\hat{\varrho}(\mathcal{A}, \mathcal{B}) < \infty$. A weak optimal $Y \in \mathrm{H}_{+,n}$ is called minimal weak optimal if there is no weak optimal $X \in \mathrm{H}_{+,n} \setminus \{0\}$ such that supp $X \subsetneq$ supp Y.

The following result is an analog of Theorem 5.1.

Theorem 9.1. *Let m, n be positive integers. Assume that $\mathcal{A}, \mathcal{B} \in CP(n, m)$. Suppose that $\hat{\varrho}(\mathcal{A}, \mathcal{B}) \in (0, \infty)$.*

(1) *Assume that $Y \in \mathrm{H}_{+,n} \setminus \{0\}$ is weak optimal. Then at least one of the eigenvalues of $(\mathcal{A} - \hat{\varrho}(\mathcal{A}, \mathcal{B})\mathcal{B})(Y)$ is zero.*

(2) *Assume that there exists a weak optimal $Y \in \mathrm{H}_{+,n}$ whose rank is $l \geqslant 1$. Let $\mathbf{V} = $ range Y. Denote by $\mathcal{A}', \mathcal{B}' \colon \mathrm{S}(\mathbf{V}) \to \mathrm{H}_m$ the restrictions of $\mathcal{A}, \mathcal{B}$ to all*

$X \in \mathrm{H}_n$ such that range $X \subseteq \mathbf{V}$. Then $\mathcal{A}', \mathcal{B}'$ are CP-operators. If $l \geqslant m$, then rank $(\mathcal{A}' - \hat{\varrho}(\mathcal{A}, \mathcal{B}) \mathcal{B}') < m^2$.

(3) A minimal weak optimal Y has rank at most m.

(4) Assume that Y is a minimal weak optimal with rank m. Then Y satisfies (21). Furthermore rank $(\mathcal{A}' - \hat{\varrho}(\mathcal{A}, \mathcal{B}) \mathcal{B}') = m^2 - 1$.

(5) Let Y' be a minimal weak optimal with rank $l < m$. Then there exists a minimal weak optimal Y satisfying range $Y = $ range Y' with the following property: The matrix $(\mathcal{A} - \hat{\varrho}(\mathcal{A}, \mathcal{B}) \mathcal{B})(Y)$ has at least l zero eigenvalues.

P r o o f. (1) Let $\mathcal{D}(t) = \mathcal{A} - t\mathcal{B}$ and $t_0 = \hat{\varrho}(\mathcal{A}, \mathcal{B})$. As $r(\mathcal{A}, \mathcal{B}, Y) = t_0$, it follows that $\mathcal{D}(t_0)(Y) \leqslant 0$. Suppose to the contrary that $\mathcal{D}(t_0)(Y) < 0$, As $t_0 > 0$, there exists $0 \leqslant t_1 < t_0$ such that $\mathcal{D}(t_1)(Y) \leqslant 0$. Hence, $r(\mathcal{A}, \mathcal{B}, Y) \leqslant t_1 < t_0$, contrary to our assumption.

(2) Choose an orthonormal basis $\mathbf{g}_1, \ldots, \mathbf{g}_l$ of $\mathbf{V}$. Then $P_{\mathbf{V}} = \sum_{i=1}^{l} \mathbf{g}_i \mathbf{g}_i^*$ is the orthogonal projection on $\mathbf{V}$. Note that for each $X \in \mathrm{H}_n$, $P_{\mathbf{V}} X P_{\mathbf{V}} \in \mathrm{S}(\mathbf{V})$. (Observe that $P_{\mathbf{V}}^* = P_{\mathbf{V}}$.) Suppose that $\mathcal{C} \colon \mathrm{H}_n \to \mathrm{H}_m$ is a CP-operator given by (9). Define $\widehat{\mathcal{C}} \colon \mathrm{H}_n \to \mathrm{H}_m$ by $\widehat{\mathcal{C}}(X) = \mathcal{C}(P_{\mathbf{V}} X P_{\mathbf{V}})$. Then $\widehat{\mathcal{C}}(X) = \sum_{j=1}^{k} (T_j P_{\mathbf{V}}) X (T_j P_{\mathbf{V}})^*$. So $\widehat{\mathcal{C}}$ is also a CP-operator. Hence $\mathcal{A}', \mathcal{B}'$ are CP-operators. By choosing an orthonormal basis $\mathbf{g}_1, \ldots, \mathbf{g}_l$ of $\mathbf{V}$ we can identify $\mathrm{S}(\mathbf{V})$ with H_l. So $\mathcal{A}', \mathcal{B}' \colon \mathrm{H}_l \to \mathrm{H}_m$. Recall that $\dim \mathrm{H}_l = l^2$, $\dim \mathrm{H}_m = m^2$. Clearly, $\hat{\varrho}(\mathcal{A}', \mathcal{B}') = \hat{\varrho}(\mathcal{A}, \mathcal{B})$. By abusing the notation we assume that an optimal Y is in $\mathrm{H}_{+,l}$.

Assume that $l \geqslant m$. Let $\mathcal{C}(t) = \mathcal{A}' - t\mathcal{B}' \colon \mathrm{H}_l \to \mathrm{H}_m$. Thus $\mathcal{C}(t_0)(Y) = -W$ for some $W \in \mathrm{H}_{+,m}$. Clearly, rank $\mathcal{C}(t_0) \leqslant m^2$. Assume to the contrary that rank $\mathcal{C}(t_0) = m^2$. Hence, there exists an m^2-dimensional subspace $\mathbf{W} \subseteq \mathrm{H}_l$ such that $\mathcal{C}(t_0) | \mathbf{W}$ is an invertible operator. Let $\mathcal{A}_1, \mathcal{B}_1$ and $\mathcal{A}_2, \mathcal{B}_2$ be the restrictions of $\mathcal{A}', \mathcal{B}'$ to $\mathbf{W}$ and $\mathbf{W}^\perp$, respectively. Define $\mathcal{C}_1(t) = \mathcal{A}_1 - t\mathcal{B}_1$, $\mathcal{C}_2(t) = \mathcal{A}_2 - t\mathcal{B}_2$. As $\mathcal{C}_1(t_0)$ is invertible, it follows that there exists $\varepsilon > 0$ such that $C_1(t) \colon \mathbf{W} \to \mathrm{H}_m$ is invertible for $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$. Let $Y = Y_1 + Y_2$, where $Y_1 \in \mathbf{W}$, $Y_2 \in \mathbf{W}^\perp$. Assume that $Y_1(t) = -\mathcal{C}_1(t)^{-1}(\mathcal{C}_2(t)(Y_2) + W)$. So $Y_1(t_0) = Y_1$. Hence $\mathcal{C}(t)(Y_1(t) + Y_2) = -W$. As rank $(Y_1 + Y_2) = l$, it follows that there exits $t_0 - \varepsilon < t_1 < t_0$ such that $Y_1(t) + Y_2 \in \mathrm{H}_{+,l}$ and rank $(Y_1(t) + Y_2) = l$. Therefore, $r(\mathcal{A}, \mathcal{B}, Y_1(t) + Y_2) \leqslant t_1 < \hat{\varrho}(\mathcal{A}, \mathcal{B})$, which contradicts the definition of $\hat{\varrho}(\mathcal{A}, \mathcal{B})$.

(3) Assume to the contrary that Y is a minimal weak optimal with rank $l > m$. Assume as in 2 that we restricted ourselves to $\mathcal{A}', \mathcal{B}' \colon \mathrm{H}_l \to \mathrm{H}_m$. Let t_0 and $\mathcal{C}(t_0)$ be defined as above. As $l > m$, it follows that $\dim \mathrm{H}_l = l^2 \geqslant (m+1)^2 = m^2 + 2m + 1$. So $\dim \ker \mathcal{C}(t_0) \geqslant 2m + 1 \geqslant 2$. Hence, there exists an indefinite matrix $X \in \mathrm{H}_l$ with at least one positive and one negative eigenvalue such that $\mathcal{C}(t_0)(X) = 0$. Let $s \in [0, \infty]$ and consider the matrix $Y(s) = Y + sX$. For $s = 0$, $Y \in \mathrm{H}_{++,l}$. For $s \gg 1$, $Y(s)$ has

a negative eigenvalue. Hence, there exists $s_0 > 0$ such that $Y(s_0) \in H_{+,l}$ and at least one eigenvalue of $Y(s_0)$ is 0. Note that $Y(s_0) \neq 0$ as Y can not be proportional to X. As $\mathcal{C}(t_0)(Y(s_0)) = \mathcal{C}(t_0)(Y) = -W$, it follows that $Y(s_0)$ is optimal. As rank $Y(s_0) < l$, we deduce that Y is not minimal weak optimal, contrary to our assumptions.

(4) Let Y be minimal weak optimal of rank m. Assume as in 2 that we restricted ourselves to $\mathcal{A}', \mathcal{B}' \colon H_m \to H_m$. Part (2) yields that rank $\mathcal{C}(t_0) < m^2$. Let $X \neq 0$ satisfy $C(t_0)(X) = 0$. If X is indefinite as in the proof of (2), we deduce that Y is not minimal weak optimal, contrary to our assumption. If X or $-X$ is positive semidefinite, then either X or $-X$ is a GPF-vector. We claim that Y is proportional to X. Otherwise, assuming that X is positive semidefine, by considering $Y - sX$ we deduce that Y is not optimal. Hence, Y is a GPF-vector. Moreover, the above arguments yield that $\dim \ker \mathcal{C}(t_0) = 1$, i.e. rank $\mathcal{C}(t_0) = m^2 - 1$.

(5) Assume that $Y' \in H_{+,n}$ is minimal weak optimal with rank $l < m$. So $(\mathcal{A} - \varrho(A,B)\mathcal{B})(Y') \leqslant 0$. By part 1 we know that $(\mathcal{A} - \varrho(A,B)\mathcal{B})(Y')$ has at least one zero eigenvalue. If $l = 1$, the claim 5 of the theorem trivially holds.

Assume that $l > 1$. Consider all minimal weak optimal $\widetilde{Y}$ such that range $\widetilde{Y} = $ range Y'. Let Y be minimal weak optimal satisfying: range $Y = $ range Y' and $(\mathcal{A} - \varrho(A,B)\mathcal{B})(Y)$ has the maximum number of zero eigenvalues. Assume that this maximum is p. We claim that $p \geqslant l$. Suppose not. As in the proof of part (2) we restrict ourselves to $\mathcal{A}', \mathcal{B}' \colon H_l \to H_m$. So $t_0 = \varrho(\mathcal{A}', \mathcal{B}')$. By abusing the notation we assume that $Y', \widetilde{Y}, Y \in H_{+,l}$. Let $W = -C(t_0)(Y)$. Then $\mathbb{C}^m = \mathbf{U}_1 \oplus \mathbf{U}_2$, where $\mathbf{U}_1 = $ range W, $\mathbf{U}_2 = \mathbf{U}_1^{\perp} = \ker W$. Note that $W|\mathbf{U}_1$ is positive definite. Let $P_{\mathbf{U}_2}$ be the orthogonal projection of $\mathbb{C}^m$ on $\mathbf{U}_2$. Let $\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}} \colon H_l \to H_p$, where we identify H_p with $P_{\mathbf{U}_2} H_m P_{\mathbf{U}_2}$ and $\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}$ with $P_{\mathbf{U}_2} \mathcal{A} P_{\mathbf{U}_2}, P_{\mathbf{U}_2} \mathcal{B} P_{\mathbf{U}_2}$, respectively. Clearly, $\hat{\varrho}(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}) \leqslant \hat{\varrho}(\mathcal{A}', \mathcal{B}')$. As in the proof of part (5) of Theorem 5.1 we claim that $\hat{\varrho}(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}) = \hat{\varrho}(\mathcal{A}', \mathcal{B}')$. Suppose not. Then there exists $U \in H_{+,l} \setminus \{0\}$ such that $r(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, U) = t_1 < t_0$. Clearly, Y and U are linearly independent. As $Y \in H_{++,l}$, we can assume that $Y \geqslant U$. Let $Y(f) = (1 - f)Y + fU$. As in the proof of part (5) of Theorem 5.1 we claim that there exists $\varepsilon > 0$ such that $r(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, (1 - f)Y + fU)) \leqslant t_0 - \varepsilon f$ for $f \in [0,1]$. We can choose $\varepsilon = (t_0 - t_1)\lambda'$, where λ' is the smallest positive eigenvalue of $\widetilde{\mathcal{B}}(Y)^{-1}\widetilde{\mathcal{B}}(U)$. (Note that the eigenvalues of $\widetilde{\mathcal{B}}(Y)^{-1}\widetilde{\mathcal{B}}(U)$ are the eigenvalues of the Hermitian matrix $\widetilde{\mathcal{B}}(Y)^{-1/2}\widetilde{\mathcal{B}}(U)\widetilde{\mathcal{B}}(Y)^{-1/2}$.) We now claim that there exists small positive δ such that we have the inequality

$$(32) \qquad r(\mathcal{A}', \mathcal{B}', (1 - f)Y + fU)) \leqslant t_0 - \frac{1}{2}\varepsilon f, \quad f \in [0, \delta].$$

For that, it is enough to show that $\lambda(\mathcal{A}'((1 - f)Y + fU), \mathcal{B}'((1 - f)Y + fU)) \leqslant t_0 - \frac{1}{2}\varepsilon f$ for $f \in [0, \delta]$. We use the first variation formula for a geometrically simple

594

eigenvalue t_0 of the Rayleigh ratio on the right-hand side of (22), where $A = A(f) = \mathcal{A}'((1-f)Y + fU))$ and $B = B(f) = \mathcal{B}'((1-f)Y + fU))$, as a function in f. The standard variation formula for a geometrically simple eigenvalue [12], §3.8 yields that it is enough to consider the first variation of the Rayleigh quotient given by the right-hand side of (22) for $\mathbf{x} \in \mathbf{U}_2 \setminus \{\mathbf{0}\}$. This is equivalent to considering $\lambda(\widetilde{\mathcal{A}}((1-f)Y + fU), \widetilde{\mathcal{B}}((1-f)Y + fU))$. The inequality

$$\lambda(\widetilde{\mathcal{A}}((1-f)Y + fU), \widetilde{\mathcal{B}}((1-f)Y + fU)) \leqslant t_0 - \varepsilon f$$

for $f \in [0, \delta]$ yields that

(33) $$\frac{\mathrm{d}}{\mathrm{d}f}\lambda(\mathcal{A}'((1-f)Y + fU), \mathcal{B}'((1-f)Y + fU))|_{f=0^+} \leqslant -\varepsilon.$$

Recall Rellich's theorem that the eigenvalues of analytic functions of Hermitian matrices are analytic in the neighborhood of $\mathbb{R}$, see [12], 4.17, (when we do not insist on ordering of the eigenvalues). Hence $\lambda(\mathcal{A}'((1-f)Y + fU), \mathcal{B}'((1-f)Y + fU))$ is analytic in $[0, \delta]$ for some $\delta > 0$. That is, it has convergent Taylor series at $f = 0$. In particular, it is in the class $C^2[0, \delta]$. Hence (33) yields (32) for small enough positive δ.

Clearly, (32) contradicts our assumption that $\hat{\varrho}(\mathcal{A}', \mathcal{B}') = t_0$. Therefore $\hat{\varrho}(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}) = t_0 = r(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, Y)$. Let $\widetilde{\mathcal{C}}(t) = \widetilde{\mathcal{A}} - t\widetilde{\mathcal{B}}$. So $\widetilde{\mathcal{C}}(t_0)\colon \mathrm{H}_l \to \mathrm{H}_p$. Our assumption that $l > p$ yields that $\ker \widetilde{\mathcal{C}}(t_0) \geqslant l^2 - p^2 \geqslant 3$. In particular, there exists an indefinite $\widetilde{X} \in \mathrm{H}_l$ such that $\widetilde{\mathcal{C}}(t_0)(\widetilde{X}) = 0$. Identify $\widetilde{X}$ with $X \in \mathrm{H}_n$, where range $X \subseteq$ range Y. The assumption that $\widetilde{\mathcal{C}}(t_0)(\widetilde{X}) = 0$ is equivalent to range $\mathcal{C}(t_0)(X) \subseteq$ range $\mathcal{C}(t_0)(Y) =$ range W. Let $Y(s) = Y + sX$. Then $\mathcal{C}(t_0)(Y(s)) = -W + s\mathcal{C}(t_0)(X)$. As range $\mathcal{C}(t_0)(X) \subseteq$ range $\mathcal{C}(t_0)(Y) =$ range W, it follows that $-W + s\mathcal{C}(t_0)(X) \leqslant 0$ for $s \geqslant 0$ if and only if $s \in [0, s_1]$, where $s_1 \in (0, \infty]$. Let $s_0 > 0$ be the largest $s > 0$ such that $Y(s) \geqslant 0$. So rank $Y(s_0) \in [l-1]$. Note that $Y(s)$ is optimal for $s \in [0, \min(s_0, s_1)]$. We claim that $s_1 < s_0$. Suppose not. Then $Y(s_0)$ is weak optimal, range $Y(s_0) \subset$ range Y and rank $Y(s_0) \leqslant l-1$. Hence Y is not minimal weak optimal, contrary to our assumptions. Consider a minimal weak optimal $Y(s_1)$. Note that rank $(-W + s_1\mathcal{C}(t_0)(X)) < $ rank $-W = m - p$. Hence, $\ker \mathcal{C}(t_0)(Y(s_1) \geqslant p + 1$, contrary to the choice of Y. Therefore, $p \geqslant l$. $\qquad \square$

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