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ON THE CLASS OF b-L-WEAKLY AND ORDER M-WEAKLY COMPACT OPERATORS

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Abstract. In this paper, we introduce and study new concepts of b-L-weakly and order M-weakly compact operators. As consequences, we obtain some characterizations of KB-spaces.

Keywords: L-weakly compact operator; M-weakly compact operator; b-order bounded operator; b-weakly compact operator; b-L-weakly compact operator; order M-weakly compact operator; KB-space

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1. Introduction and notation

Throughout this paper X and Y denote real Banach spaces, and E and F denote real Banach lattices. B_X is the closed unit ball of X and $\mathrm{sol}(A)$ denotes the solid hull of a subset A of a Banach lattice. We use the term operator between two Banach spaces to mean a bounded linear mapping. Let us recall some notions and results from [2] and [7]. E is called a KB-space (Kantorovich-Banach), if every increasing norm bounded sequence of E^+ is norm convergent. Note that every KB-space has order continuous norm. A nonempty bounded subset A of E is said to be L-weakly compact if $\lim \|x_n\| = 0$ for every disjoint sequence (x_n) contained in the solid hull of A. Note that every L-weakly compact subset $A \subset E$ is relatively weakly compact (see [7], Proposition 3.6.5).

Recall from [3] that a subset A of E is called b-order bounded if it is order bounded in the topological bidual E''. Note that every order bounded subset of E is b-order bounded, however, the converse is not true in general. But a Banach lattice E is said

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to have property (b) if each subset A of E is order bounded whenever it is b-order bounded. Note that every topological dual of a Banach lattice has property (b).

Based on this concept, the class of b-weakly compact operators is defined in [3]. In fact, an operator T from a Banach lattice E into a Banach space Y is called b-weakly compact if it maps each b-order bounded subset of E into a relatively weakly compact subset of Y. The space of b-weakly compact operators is bigger than the class of weakly compact operators, but smaller than the class of order weakly compact operators, which was introduced by Dodds in [5]. Also, an operator $T\colon E \to F$ is called b-order bounded if it maps b-order bounded subsets of E into b-order bounded subsets of F.

The classes of L-weakly and M-weakly compact operators were introduced by Meyer-Nieberg (see [6]). An operator T from X into F is called L-weakly compact if $T(B_X)$ is an L-weakly compact subset of F. An operator T from E into Y is called M-weakly compact if $\lim T(x_n) = 0$ holds for every norm bounded disjoint sequence (x_n) in E.

We introduce new classes of b-L-weakly and order M-weakly compact operators. An operator T from a Banach lattice E into a Banach lattice F is called b-L-weakly compact if it maps b-order bounded subsets of E into L-weakly compact subsets of F, and an operator T from a Banach lattice E into a Banach lattice F is called order M-weakly compact if for every disjoint sequence (x_n) in B_E and every order bounded sequence (f_n) of F' we have $f_n(T(x_n)) \to 0$.

Note that the class of b-L-weakly compact operators contains strictly that of L-weakly compact operators, and the class of order M-weakly compact operators contains strictly that of M-weakly compact operators. On the other hand, it is easy to see that every b-L-weakly compact operator is b-weakly compact but the converse is false in general. We begin by establishing a sequential characterization of b-L-weakly compact operators. As consequences, we give some interesting results. We know that the classes of L-weakly and M-weakly compact operators are in duality with each other (an operator T, between two Banach lattices, is L-weakly compact (or M-weakly compact) if and only if its adjoint T' is M-weakly compact (or L-weakly compact), see [7], Proposition 3.6.11). As we shall see, a similar result for the classes of b-L-weakly and order M-weakly compact operators are proved. Finally, we close this paper by presenting a necessary and sufficient condition on which every b-order bounded operator is b-L-weakly (or order M-weakly) compact.

In what follows:

- ▷ $L(X, Y)$ denotes the space of all operators from X into Y,
- ▷ $LW(X, F)$ denotes the space of all L-weakly compact operators from X into F,
- ▷ $MW(E, Y)$ denotes the space of all M-weakly compact operators from E into Y,
- ▷ $bLW(E, F)$ denotes the space of all b-L-weakly compact operators from E into F,

▷ $oMW(E,F)$ denotes the space of all order M-weakly compact operators from E into F.

For the theory of Banach lattices and operators, we refer the reader to the monographs [2], [7], [8].

2. Main results

We start by the following definitions.

Definition 2.1. An operator T from E into F is called b-L-weakly compact if it maps b-order bounded subsets of E into L-weakly compact subsets of F.

Definition 2.2. An operator T from E into F is called order M-weakly compact if for every disjoint sequence (x_n) in B_E and every order bounded sequence (f_n) of F', we have $f_n(T(x_n)) \to 0$.

Remark 2.1. Note that as the topological dual E' has always the property (b), in the previous definition one can replace "every order bounded sequence (f_n)" with "every b-order bounded sequence (f_n)".

Proposition 2.1. *The following assertions are equivalent:*

(1) *The identity operator* $\mathrm{Id}_E\colon E \to E$ *is b-L-weakly compact.*
(2) *Every b-order bounded subset of* E *is L-weakly compact.*
(3) E *is a KB-space.*

Proof. (1) $\Leftrightarrow$ (2): The proof is obvious.

(2) $\Rightarrow$ (3): According to Proposition 2.8 and Proposition 2.10 of [3], it suffices to show that each b-order bounded disjoint sequence of E is norm convergent to zero. Given such a sequence (x_n) of E, the set $A = \{x_n\colon n \in \mathbb{N}\}$ is b-order bounded, and so by (2) A is L-weakly compact. Thus $\|x_n\| \to 0$.

(3) $\Rightarrow$ (2): Let A be a b-order bounded subset of E and (x_n) a disjoint sequence in the solid hull of A. Note that the sequence (x_n) is b-order bounded. In fact, pick some $0 \leqslant x'' \in E''$ such that $|x| \leqslant x''$ for all $x \in A$. $|x_n| \leqslant |y_n|$ for some $y_n \in A$ and hence $|x_n| \leqslant |y_n| \leqslant x''$. So, $|x_n| \leqslant x''$ for all $n \in \mathbb{N}$, i.e. (x_n) is b-order bounded. Then, by Proposition 2.8 and Proposition 2.10 of [3], we have $\|x_n\| \to 0$ and hence A is L-weakly compact. □

Remark 2.2. Clearly, every L-weakly compact operator is b-L-weakly compact (it suffices to note that every b-order bounded subset of E is norm bounded), but the converse is not true in general. For instance, consider the operator $\mathrm{Id}_{\ell_1}\colon \ell_1 \to \ell_1$. Since ℓ_1 is a KB-space, Id_{ℓ_1} is b-L-weakly compact. On the other hand, B_{ℓ_1} is not relatively weakly compact and therefore is not L-weakly compact. Hence Id_{ℓ_1} is not L-weakly compact.

On the other hand, it is easy to see that every b-L-weakly compact operator is b-weakly compact (it suffices to note that every L-weakly compact subset is relatively weakly compact). The converse, however, need not be true. For instance, consider the operator $T\colon \ell_1 \to \ell_\infty$ defined by

$$\forall\,(\alpha_n) \in \ell_1, \quad T((\alpha_n)) = \Bigl(\sum_{n=1}^{\infty} \alpha_n\Bigr)(1,1,1,\ldots).$$

Clearly, T is a compact operator (it has rank one) and hence T is b-weakly compact.

Let $e = (1/n^2)_{n\in\mathbb{N}^*}$. The sequence (e_n) of standard unit vectors is a disjoint sequence in the solid hull of $T[0,e]$, $|e_n| \leqslant T(e)$. From $\|e_n\| = 1 \nrightarrow 0$ we see that T fails to be b-L-weakly compact.

Clearly, every M-weakly compact operator is order M-weakly compact (for every sequence (y_n) of F, if $\|y_n\| \to 0$, then $f_n(y_n) \to 0$ for every order bounded sequence (f_n) of F'), but the converse is not true in general. For instance, consider the operator Id_{c_0}. Since $\ell_1 = {c_0}'$ is a KB-space, Id_{c_0} is order M-weakly compact (see Corollary 2.2). And since $\mathrm{Id}_{\ell_1} = \mathrm{Id}'_{c_0}$ is not L-weakly compact, Id_{c_0} is not M-weakly compact.

The following lemmas are used throughout this paper.

Lemma 2.1 ([2], Theorem 5.63). *For any two nonempty bounded sets $A \subset E$ and $B \subset E'$, the following statements are equivalent:*

(1) *Every disjoint sequence in the solid hull of A converges uniformly to zero on B.*
(2) *Every disjoint sequence in the solid hull of B converges uniformly to zero on A.*

Lemma 2.2. *For every nonempty bounded subset $A \subset E$, the following assertions are equivalent:*

(1) *A is L-weakly compact.*
(2) *$f_n(x_n) \to 0$ for every sequence (x_n) of A and every disjoint sequence (f_n) of $B_{E'}$.*

Proof. Let A be a nonempty bounded subset of E. A is L-weakly compact if and only if $\|x_n\| \to 0$ holds for every disjoint sequence (x_n) of $\mathrm{sol}(A)$. Thus A is L-weakly compact if and only if every disjoint sequence (x_n) of $\mathrm{sol}(A)$ converges uniformly to zero on $B_{E'}$ (i.e. $\sup\{|f(x_n)|\colon f \in B_{E'}\} \to 0$). By Lemma 2.1, this is equivalent to saying that every disjoint sequence (f_n) of $B_{E'}$ converges uniformly to zero on A (i.e. $\sup\{|f_n(x)|\colon x \in A\} \to 0$).

Let us now prove the equivalence

$$\sup\{|f_n(x)|\colon\ x \in A\} \to 0 \text{ if and only if for each sequence } (x_n) \text{ of } A,\ f_n(x_n) \to 0.$$

Indeed, if $\sup\{|f_n(x)|\colon\ x \in A\} \to 0$, then for each sequence (x_n) of A

$$|f_n(x_n)| \leqslant \sup\{|f_n(x)|\colon\ x \in A\} \to 0.$$

Therefore $f_n(x_n) \to 0$.

Conversely, assume that $f_n(x_n) \to 0$ for each sequence (x_n) of A. Assume by way of contradiction that $\sup\{|f_n(x)|\colon\ x \in A\} \nrightarrow 0$. Then there exist some $\varepsilon > 0$ and a subsequence $(f_{\varphi(n)})$ of (f_n) satisfying $\sup\{|f_{\varphi(n)}(x)|\colon\ x \in A\} > \varepsilon$ for all $n \in \mathbb{N}$. Thus for each $n \in \mathbb{N}$, there exists some $x_{\varphi(n)} \in A$ with $|f_{\varphi(n)}(x_{\varphi(n)})| > \varepsilon$, from our hypothesis it follows that $f_{\varphi(n)}(x_{\varphi(n)}) \to 0$, which is impossible, and the proof of the lemma is finished. □

In a similar way we may prove the following result.

Lemma 2.3. *For every nonempty bounded subset $A \subset E'$ the following assertions are equivalent:*

(1) *A is L-weakly compact.*
(2) *$f_n(x_n) \to 0$ for every sequence (f_n) of A and every disjoint sequence (x_n) of B_E.*

The following results give a sequential characterization of b-L-weakly compact operators.

Theorem 2.1. *For an operator $T\colon\ E \to F$, the following statements are equivalent:*

(1) *T is b-L-weakly compact.*
(2) *For every b-order bounded sequence (x_n) of E and every disjoint sequence (f_n) of $B_{F'}$ we have $f_n(T(x_n)) \to 0$.*
(3) *For every b-order bounded sequence (x_n) of E and every disjoint sequence (f_n) of $B_{F'^+}$ we have $f_n(T(x_n)) \to 0$.*
(4) *For every b-order bounded sequence (x_n) of E^+ and every disjoint sequence (f_n) of $B_{F'}$ we have $f_n(T(x_n)) \to 0$.*
(5) *For every b-order bounded sequence (x_n) of E^+ and every disjoint sequence (f_n) of $B_{F'^+}$ we have $f_n(T(x_n)) \to 0$.*

P r o o f. (1) $\Leftrightarrow$ (2): Consider an operator $T\colon E \to F$. T is b-L-weakly compact if and only if for every b-order bounded subset $A \subset E$, $T(A)$ is L-weakly compact. By Lemma 2.2, this is equivalent to saying that for every b-order bounded subset $A \subset X$, $f_n(T(x_n)) \to 0$ for every sequence (x_n) of A and every disjoint sequence (f_n) of $B_{F'}$.

To conclude, it is sufficient to note the equivalence of the following assertions:

(i) For every b-order bounded subset $A \subset E$, $f_n(T(x_n)) \to 0$ for every sequence (x_n) of A and every disjoint sequence (f_n) of $B_{F'}$.

(ii) $f_n(T(x_n)) \to 0$ for every b-order bounded sequence (x_n) of E and every disjoint sequence (f_n) of $B_{F'}$.

(i) $\Rightarrow$ (ii): Let (x_n) be a b-order bounded sequence of E. It is sufficient to apply (i) to the set $A = \{x_n \colon n \in \mathbb{N}\}$.

(ii) $\Rightarrow$ (i): Let A be a b-order bounded subset of E. It is sufficient to note that every sequence (x_n) of A is b-order bounded.

The proof of the equivalence of (1) and (2) is finished.

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5): The proof is obvious. □

In the same way we prove the following result:

Theorem 2.2. *For an operator $\varphi\colon E' \to F'$ the following statements are equivalent:*

(1) *φ is b-L-weakly compact.*

(2) *$\varphi(f_n)(y_n) \to 0$ for every order bounded sequence (f_n) of E' and every disjoint sequence (y_n) of B_F.*

(3) *$\varphi(f_n)(y_n) \to 0$ for every order bounded sequence (f_n) of E'^+ and every disjoint sequence (y_n) of B_{F^+}.*

As a consequence, we obtain the following characterizations of KB-spaces.

Corollary 2.1. *For a Banach lattice E the following statements are equivalent:*

(1) $\mathrm{Id}_E \in bLW(E)$.

(2) *E is a KB-space.*

(3) *$f_n(x_n) \to 0$ for every b-order bounded sequence (x_n) of E and every disjoint sequence (f_n) of $B_{E'}$.*

Corollary 2.2. *For a Banach lattice E the following statements are equivalent:*

(1) $\mathrm{Id}_{E'} \in bLW(E')$.

(2) *E' is a KB-space.*

(3) $f_n(x_n) \to 0$ *for every order bounded sequence* (f_n) *of* E' *and every disjoint sequence* (x_n) *of* B_E.
(4) $\mathrm{Id}_E \in oMW(E)$.

Contrary to weakly compact operators (see [1]), we also deduce that the class of b-L-weakly (or order M-weakly) compact operators satisfies the domination problem.

Corollary 2.3. *Let* $S, T\colon E \to F$ *be two positive operators such that* $0 \leqslant S \leqslant T$. *Then* S *is b-L-weakly compact (or order M-weakly compact) whenever* T *is one.*

Proposition 2.2. *Let* E *and* F *be two Banach lattices. Then:*

(1) *The set of all b-L-weakly compact operators from* E *to* F *is a closed vector subspace of* $L(E,F)$.
(2) *The set of all order M-weakly compact operators from* E *to* F *is a closed vector subspace of* $L(E,F)$.

Proof. (1) Let $T_1, T_2 \in bLW(E,F)$, and $\alpha \in \mathbb{R}$. Let (x_n) be a b-order bounded sequence of E and (f_n) a disjoint sequence of $B_{F'}$. Since $T_1, T_2 \in bLW(E,F)$, it follows from Theorem 2.1, that

$$f_n((\alpha T_1 + T_2)(x_n)) = \alpha f_n(T_1(x_n)) + f_n(T_2(x_n)) \to 0.$$

Then $\alpha T_1 + T_2 \in bLW(E,F)$. Thus $bLW(E,F)$ is a vector subspace of $L(E,F)$. To see that it is also a closed vector subspace of $L(E,F)$, let T be in the closure of $bLW(E,F)$. Let (x_n) be a b-order bounded sequence of E and (f_n) a disjoint sequence of $B_{F'}$. We have to show that $f_n(T(x_n)) \to 0$. To this end, let $\varepsilon > 0$. Pick a b-L-weakly compact operator $S\colon E \to F$ with $\|T - S\| < \varepsilon$ and note that from the inequalities

$$\begin{aligned} |f_n(T(x_n))| &\leqslant |f_n((T-S)(x_n))| + |f_n(S(x_n))| \\ &\leqslant \|f_n\|\|T-S\|\|(x_n)\| + |f_n(S(x_n))| \end{aligned}$$

it follows that $\limsup |f_n(T(x_n))| \leqslant \varepsilon\|(x_n)\|$.

Since ε is arbitrary, we see that $f_n(T(x_n)) \to 0$ holds as desired.

(2) Clearly, $oMW(E,F)$ is a vector subspace of $L(E,F)$. To see that it is also a closed vector subspace of $L(E,F)$, let T be in the closure of $oMW(E,F)$. Assume that (x_n) is a disjoint sequence of B_E and (f_n) an order bounded sequence of F'. We have to show that $f_n(T(x_n)) \to 0$. To this end, let $\varepsilon > 0$. Pick an order M-weakly

compact operator $S\colon E \to F$ with $\|T - S\| < \varepsilon$ and note that from the inequalities

$$\begin{aligned}|f_n(T(x_n))| &\leqslant |f_n((T-S)(x_n))| + |f_n(S(x_n))| \\ &\leqslant \|f_n\|\|T-S\|\|x_n\| + |f_n(S(x_n))|\end{aligned}$$

it follows that $\limsup |f_n(T(x_n))| \leqslant \varepsilon\|f_n\| \leqslant \varepsilon\|(f_n)\|$.

Since ε is arbitrary, we see that $f_n(T(x_n)) \to 0$ holds as desired. □

The classes of L-weakly and M-weakly compact operators are in duality with each other. For the classes of b-L-weakly and order M-weakly compact operators, we have the following result.

Theorem 2.3. *Let E and F be two Banach lattices. Then the following statements hold:*

(1) *An operator $T\colon E \to F$ is order M-weakly compact if and only if its adjoint T' is b-L-weakly compact.*

(2) *For an operator $T\colon E \to F$, if its adjoint T' is order M-weakly compact then T is b-L-weakly compact.*

Proof. (1) Consider an operator $T\colon E \to F$. By Theorem 2.2, T' is b-L-weakly compact if and only if $T'(f_n)(x_n) \to 0$ for every order bounded sequence (f_n) of F' and every disjoint sequence (x_n) of B_E. This is equivalent to saying that $f_n(T(x_n)) \to 0$ for every order bounded sequence (f_n) of F' and every disjoint sequence (x_n) of B_E. In other words, $T'\colon F' \to E'$ is b-L-weakly compact if and only if $T\colon E \to F$ is order M-weakly compact.

(2) Let $T\colon E \to F$ be an operator such that T' is order M-weakly compact. Let (x_n) be a b-order bounded sequence of E and (f_n) a disjoint sequence of $B_{F'}$. Let $J\colon E \to E''$ be the canonical embedding of E into E''. Since $T'\colon F' \to E'$ is order M-weakly compact and the sequence $(J(x_n))$ of E'' is order bounded, then $J(x_n)(T'(f_n)) = f_n(T(x_n)) \to 0$. Hence T is b-L-weakly compact. □

Remark 2.3. However, in general:

$$T \text{ is b-L-weakly compact} \nLeftarrow T' \text{ is order M-weakly compact.}$$

Indeed, the Banach lattice $E = \ell^1(\ell_n^\infty)$ is a KB-space whose bidual E'' fails to have an order continuous norm (see [7], page 95). Therefore the identity operator of E is b-L-weakly compact, but $\mathrm{Id}'_E = \mathrm{Id}_{E'}$ is not order M-weakly compact.

We now present another characterization of KB-spaces.

Theorem 2.4. *A Banach lattice F is a KB-space if and only if for every Banach lattice E and b-order bounded operator $T\colon E \to F$, T is b-L-weakly compact.*

P r o o f. If the hypothesis on F is true then taking $E = F$ we see that the identity on E is b-L-weakly compact and thus, by Proposition 2.1, F is a KB-space. On the other hand, let $T\colon E \to F$ be a b-order bounded operator, and let A be a b-order bounded subset of E. Since T is b-order bounded, $T(A) \subset F$ is b-order bounded. If F is a KB-space, then by Proposition 2.1, $T(A)$ is L-weakly compact and so T is b-L-weakly compact. □

Theorem 2.5. *Let E and F be nonzero Banach lattices. Then the following assertions are equivalent:*

(1) *Every b-order bounded operator $T\colon E \to F$ is order M-weakly compact.*
(2) *E' is a KB-space.*

P r o o f. (1) ⇒ (2): Assume by the way of contradiction that E' is not a KB-space. We have to construct a b-order bounded operator $T\colon E \to F$ which is not order M-weakly compact. Since E' is not a KB-space (i.e. the norm of E' is not order continuous), it follows from [2], Theorem 4.14, that there exists some $f \in E'^{+}$ and there exists a disjoint sequence (f_n) in $[0, f]$ which does not converge to zero in norm. Pick some $c \in F^{+}$ and $g \in F'^{+}$ such that $g(c) = 1$.

Now, we consider the positive operator $T\colon E \to F$ defined by $T(x) = f(x)c$ for every $x \in E$. T is b-order bounded, on the other hand, we claim that T is not order M-weakly compact. By Theorem 2.3, it suffices to show that its adjoint $T'\colon F' \to E'$ is not b-L-weakly compact. Note that $T'(\varphi) = \varphi(c)f$ for every $\varphi \in F'$. In particular, $T'(g) = g(c)f = f$. So, $f \in T'([0, g])$. From $(f_n) \subset [0, f]$ it follows that (f_n) is a disjoint sequence in the solid hull of $T'([0, g])$. Since (f_n) is not norm convergent to zero, then T' is not b-L-weakly compact. Hence T is not order M-weakly compact. But this is in contradiction with our hypothesis (1). So, E' is a KB-space.

(2) ⇒ (1): Let $T\colon E \to F$ be a b-order bounded operator. By [4], Proposition 1, the adjoint operator $T'\colon F' \to E'$ is order bounded, and hence it is b-order bounded. Since E' is a KB-space, it follows from Theorem 2.4 that T' is b-L-weakly compact and so by Theorem 2.3 the operator T is order M-weakly compact. □

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