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ON THE LOCAL CONVERGENCE OF KUNG-TRAUB'S TWO-POINT METHOD AND ITS DYNAMICS

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Abstract. In this paper, the local convergence analysis of the family of Kung-Traub's two-point method and the convergence ball for this family are obtained and the dynamical behavior on quadratic and cubic polynomials of the resulting family is studied. We use complex dynamic tools to analyze their stability and show that the region of stable members of this family is vast. Numerical examples are also presented in this study. This method is compared with several widely used solution methods by solving test problems from different chemical engineering application areas, e.g. Planck's radiation law problem, natch distillation at infinite reflux, van der Waal's equation, air gap between two parallel plates and flow in a smooth pipe, in order to check the applicability and effectiveness of our proposed methods.

Keywords: local convergence; Kung-Traub's method; complex dynamics; parameter space; basins of attraction; stability

MSC 2020: 65F10, 65H04, 37P40, 37Fxx

1. Introduction

Multi-point iterative algorithms for approximation of simple roots of nonlinear equations with a suitable level of accuracy have vital importance in various branches of science and engineering [34], [42], [50]. Finding solutions of the equation $F(x) = 0$ is one of the oldest mathematical problems. In 1669, Sir Issac Newton discussed this problem which was later improved by Joseph Raphson in 1690. This algorithm is presently known as the Newton-Raphson method, or more commonly as Newton's method [41]. In the last decade, many modified iterative methods have been developed to improve these classical methods, see [2], [1], [6], [9], [11], [13], [14], [24], [23], [25], [26], [30], [33], [35], [37], [36], [43], [48], [49] and the references therein.

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In 1974 Kung and Traub proposed an optimal fourth-order method [34], [42]. Let $F\colon \mathcal{D} \subset \mathbb{X} \to \mathbb{Y}$ be a nonlinear Fréchet differentiable operator in an open convex domain $\mathcal{D}$. Let us assume that $F'(x_0)^{-1} \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$, where $\mathcal{L}(\mathbb{Y}, \mathbb{X})$ is the set of bounded linear operators from $\mathbb{Y}$ into $\mathbb{X}$. Let α be a simple real zero of a real function $F(x)$ and let x_0 be an initial approximation to α. Kung-Traub's two-point method can be represented by

$$\begin{cases} y_n = x_n - \dfrac{F(x_n)}{F'(x_n)}, \\ x_{n+1} = y_n - G_F(x_n), \end{cases} \tag{1.1}$$

where

$$G_F(x_n) = \frac{F(x_n)^2 F(y_n)}{F'(x_n)(F(y_n) - F(x_n))^2}. \tag{1.2}$$

This method requires 3 function evaluations and has the efficiency index $2^{2/3} \approx 1.587$. By the Kung-Traub conjecture [34], it is an optimal method. We recall that, according to this conjecture, the order of convergence of any multi-point method without memory cannot exceed the bound 2^{n-1}, where n is the number of function evaluations per iteration. Therefore, the optimal efficiency index would be $p^{1/n}$, where p is the order of convergence.

Some of the important problems in the study of iterative procedures are to find the estimates of the radii of convergence balls or to discuss of the local convergence analysis. There are many studies which deal with the local and semilocal convergence analyses of Newton-like methods such as [8], [24], [27], [28], [29], [33], [38]. Recently, Veiseh et al. [51] studied the local convergence and dynamical behavior of derivative-free Kung-Traub's method.

The purpose of the present paper is twofold. One of our intentions is to find the local convergence of Kung-Traub's method. The other aim is to analyze the stability of Kung-Traub's schemes on quadratic and cubic polynomials by using complex dynamical tools.

The present paper is organized as follows. In Section 2 we study the local convergence of the family (1.1) of iterative methods. We study the dynamical behavior of the rational operator associated with Kung-Traub's schemes on quadratic and cubic polynomials. The fixed, strange and critical points of this operator are obtained and the stability regions of the strange fixed points are discussed. In Section 4, some examples are given to show the efficiency of the local convergence theorem and conclusions are drawn in Section 5.

2. Local convergence

In this section, we present the local convergence analysis of the method (1.1). Let $U(x^*, r), \overline{U}(x^*, r)$ be open and closed balls in $\mathbb{R}$, respectively, with the center $x^* \in \mathbb{R}$ and radius $r > 0$. Let $L_0 > 0$, $L > 0$, and $M > 0$ be given parameters with $L_0 \leqslant L$. Let us introduce some functions and parameters for the local convergence analysis. We define function g_1 on the interval $[0, 1/L_0)$ by

$$g_1(x) = \frac{Lx}{2(1 - L_0 x)}. \tag{2.1}$$

Notice that for $r_1 = 2/(2L_0 + L)$, $0 < r_1 < 1/L_0$, we have $g_1(r_1) = 1$ and for any $x \in [0, r_1)$, it is $0 \leqslant g_1(x) < 1$. Consider on interval $[0, 1/L_0)$ the functions

$$\begin{aligned} P_1(x) &= \frac{L_0}{2} x + M g_1(x), \\ h_1(x) &= P_1(x) - 1. \end{aligned}$$

We have $h_1(0) = -1 < 0$ and $h_1(x) \to \infty$ as $x \to (1/L_0)^-$. By the intermediate value theorem, we can say that the function $h_1(x)$ has zeros in interval $(0, 1/L_0)$. Let r_2 be the smallest such zero. Define functions $q_1(x)$, $g_2(x)$, and $h_2(x)$ on interval $[0, r_2)$ by

$$\begin{aligned} q_1(x) &= 1 + \frac{M^3}{(1 - L_0 x)(1 - P_1(x))^2}, \\ g_2(x) &= q_1(x) g_1(x), \\ h_2(x) &= g_2(x) - 1. \end{aligned}$$

Then we observe that $h_2(0) = -1 < 0$ and $h_2(x) \to \infty$ as $x \to r_2^-$. That is, the function $h_2(x)$ has zeros on $(0, r_2)$. We denote the smallest such zero by r. Therefore,

$$0 < r < r_2 < r_1 < \frac{1}{L_0} \tag{2.2}$$

and for any $x \in [0, r)$

$$0 \leqslant g_1(x) < 1, \tag{2.3}$$

$$0 \leqslant g_2(x) < 1, \tag{2.4}$$

$$0 \leqslant P_1(x) < 1. \tag{2.5}$$

Theorem 2.1. *Let $F\colon \mathcal{D} \subseteq \mathbb{R} \to \mathbb{R}$ be a Fréchet differentiable function. Suppose that there exist $x^* \in \mathcal{D}$, $L_0 > 0$, $L > 0$ $(L_0 \leqslant L)$ and $M \geqslant 1$ such that for any*

$x, y \in \mathcal{D}$, we have

$$F(x^*) = 0,\ F'(x^*) \neq 0, \tag{2.6}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leqslant L_0\|x - x^*\|, \tag{2.7}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leqslant L\|x - y\|, \tag{2.8}$$

$$\|F'(x^*)^{-1}F'(x)\| \leqslant M, \tag{2.9}$$

and

$$\overline{U}(x^*, r) \subseteq \mathcal{D}, \tag{2.10}$$

where the radius r is given by (2.2). Then, the sequence $\{x_n\}$ generated by the method (1.1) for $x_0 \in U(x^*, r) - \{x^*\}$ is well defined, remains in $\overline{U}(x^*, r)$ for all $n = 0, 1, 2, \ldots$, and converges to x^*. Moreover, we have the error bounds

$$\|y_n - x^*\| \leqslant g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r \tag{2.11}$$

and

$$\|x_{n+1} - x^*\| \leqslant g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|. \tag{2.12}$$

Furthermore, for $\varrho \in [r, 2/L_0)$ the solution x^* is unique in $\overline{U}(x^*, \varrho) \cap \mathcal{D}$.

Proof. We will carry out the proof of (2.11) and (2.12) by mathematical induction. By the first sub-step of the method (1.1) for $n = 0$, we get

$$y_0 - x^* = x_0 - x^* - F'(x_0)^{-1}F(x_0). \tag{2.13}$$

Clearly, it also holds that

$$\begin{aligned} y_0 - x^* = x_0 - x^* - F'(x_0)^{-1}F(x_0) &= -F'(x_0)^{-1}F'(x^*) \\ &\times \int_0^1 F'(x^*)^{-1}[F'(x^* + \theta(x_0 - x^*)) - F'(x_0)](x_0 - x^*)\,\mathrm{d}\theta. \end{aligned} \tag{2.14}$$

Then

$$\begin{aligned} \|y_0 - x^*\| \leqslant {} & \|F'(x_0)^{-1}F'(x^*)\| \\ &\times \left\|\int_0^1 F'(x^*)^{-1}[F'(x^* + \theta(x_0 - x^*)) - F'(x_0)]\right\| \mathrm{d}\theta\|x_0 - x^*\|. \end{aligned} \tag{2.15}$$

Since $x_0 \in U(x^*, r) - x^*$, by the definition of r and also by (2.7) we have

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leqslant L_0\|x_0 - x^*\| < L_0 r < 1. \tag{2.16}$$

Moreover,

$$\begin{aligned} F'(x^*)^{-1}(F'(x_0) - F'(x^*)) &= F'(x^*)^{-1}F'(x_0) - F'(x^*)^{-1}F'(x^*) \\ &= F'(x^*)^{-1}F'(x_0) - I. \end{aligned}$$

By (2.14), we obtain

$$\begin{aligned} \|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| &\leqslant \|F'(x^*)^{-1}F'(x_0)\| + \|I\| \\ &\leqslant L_0\|x_0 - x^*\|. \end{aligned}$$

The Banach lemma on invertible functions implies that $F'(x_0) \neq 0$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leqslant \frac{1}{1 - L_0\|x_0 - x^*\|}. \tag{2.17}$$

Then y_0 is well defined by the first sub-step of the method (1.1) for $n = 0$. In view of relations (2.1), (2.3), (2.8), and (2.17) we obtain

$$\begin{aligned} \|y_0 - x^*\| &\leqslant \|F'(x_0)^{-1}F'(x^*)\| \\ &\quad\times \left\| \int_0^1 F'(x^*)^{-1}[F'(x^* + \theta(x_0 - x^*)) - F'(x_0)] \right\| \mathrm{d}\theta \, \|x_0 - x^*\| \\ &\leqslant \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} = g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned} \tag{2.18}$$

which proves (2.11) for $n = 0$ and $y_0 \in U(x^*, r)$.

Using the second sub-step of method (1.1) for $n = 0$, by (1.1) and (2.18) we have

$$\|x_1 - x^*\| \leqslant \|y_0 - x^*\| + \|G_F(x_0)\|, \tag{2.19}$$

where

$$\begin{aligned} \|G_F(x_0)\| &\leqslant \|F'(x_0)^{-1}F'(x^*)\| \, \|F'(x^*)^{-1}F(y_0)\| \\ &\quad\times \|F'(x^*)^{-1}F(x_0)\|^2 \|F'(x^*)(F(y_0) - F(x_0))^{-1}\|^2. \end{aligned} \tag{2.20}$$

By the hypothesis $F(x^*) = 0$, we can write

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*) \, \mathrm{d}\theta.$$

Consequently,

$$F'(x^*)^{-1}F(x_0) = F'(x^*)^{-1}(F(x_0) - F(x^*)) = \int_0^1 F'(x^*)^{-1}F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)\,\mathrm{d}\theta \tag{2.21}$$

and thus,

$$\|x^* + \theta(x_0 - x^*) - x^*\| = \theta\|x_0 - x^*\| \leqslant \|x_0 - x^*\| < r.$$

Hence, $x^* + \theta(x_0 - x^*) - x^* \in U(x^*, r)$. Therefore, using Eq. (2.9), we obtain

$$\|F'(x^*)^{-1}F(x_0)\| \leqslant M\|x_0 - x^*\|. \tag{2.22}$$

Similarly, since $y_0 \in U(x^*, r)$, we have by (2.18) that

$$\|F'(x^*)^{-1}F(y_0)\| \leqslant M\|y_0 - x^*\| \leqslant Mg_1(\|x_0 - x^*\|)\|x_0 - x^*\|. \tag{2.23}$$

Now, we show that $F(x_0) - F(y_0) \neq 0$. Using (2.5), (2.7), (2.9), (2.15), and (2.23), we can write

$$\begin{aligned} &\|(F'(x^*)(x_0 - x^*))^{-1}[F(x_0) - F(x^*) - F(x^*)(x_0 - x^*) - F(y_0)]\| \\ &\leqslant \frac{1}{\|x_0 - x^*\|}\Big[\Big\|\int_0^1 F'(x^*)^{-1}(F'(x_0 - \theta(x_0 - x^*)) - F'(x^*))\,\mathrm{d}\theta\Big\|\|x_0 - x^*\| \\ &\qquad + \|F'(x^*)^{-1}F(y_0)\|\Big] \\ &\leqslant \frac{1}{\|x_0 - x^*\|}\Big[\frac{L_0}{2}\|x_0 - x^*\|^2 + M\|y_0 - x^*\|\Big] \\ &\leqslant \frac{L_0}{2}\|x_0 - x^*\| + Mg_1(\|x_0 - x^*\|) = P_1(\|x_0 - x^*\|) \leqslant P_1(r) < 1. \end{aligned} \tag{2.24}$$

The Banach lemma on invertible functions and (2.24) imply that

$$\|F'(x^*)(F(y_0) - F(x_0))^{-1}\| \leqslant \frac{1}{\|x_0 - x^*\|(1 - P_1(\|x_0 - x^*\|))}. \tag{2.25}$$

Hence, by (2.22), (2.23) and (2.25), x_1 is well defined. Using (2.20), (2.22), (2.23), and (2.25), we can write

$$\begin{aligned} \|G_F(x_0)\| &\leqslant \|F'(x_0)^{-1}F'(x^*)\|\,\|F'(x^*)^{-1}F(y_0)\| \\ &\qquad \times \|F'(x^*)^{-1}F(x_0)\|^2\|F'(x^*)(F(y_0) - F(x_0))^{-1}\|^2 \\ &\leqslant \frac{M^3\|y_0 - x^*\|}{(1 - L_0\|x_0 - x^*\|)(1 - p_1(\|x_0 - x^*\|))^2}. \end{aligned} \tag{2.26}$$

Using (2.4), (2.19), and (2.26), we can write

$$
\begin{aligned}
(2.27)\qquad \|x_1 - x^*\| &\leqslant \|y_0 - x^*\| + \|G_F(x_n)\| \\
&\leqslant \|y_0 - x^*\| + \frac{M^3\|y_0 - x^*\|}{(1 - L_0\|x_0 - x^*\|)(1 - p_1(\|x_0 - x^*\|))^2} \\
&\leqslant \|y_0 - x^*\|\Big(1 + \frac{M^3}{(1 - L_0(\|x_0 - x^*\|))(1 - p_1(\|x_0 - x^*\|))^2}\Big) \\
&\leqslant g_1(\|x_0 - x^*\|)q_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&\leqslant g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leqslant \|x_0 - x^*\| < r < 1,
\end{aligned}
$$

which proves (2.12) for $n = 0$ and $x_1 \in U(x^*, r)$.

By replacing x_1 and y_0 by x_{k+1} and y_k in the preceding estimates, we arrive at the estimates (2.11) and (2.12). Using the estimate $\|x_{k+1} - x^*\| < \|x_k - x^*\| < r$, we deduce that $x_{k+1} \in U(x^*, r)$ and $\lim\limits_{k\to\infty} x_k = x^*$.

We prove the uniqueness of x^* by considering $T = \int_0^1 F'(y^* + \theta(x^* - y^*))\,\mathrm{d}\theta$. Using (2.7), we get

$$
\begin{aligned}
(2.28)\qquad & \|F'(x^*)^{-1}(T - F'(x^*))\| \\
&= \Big\|\int_0^1 F'(x^*)^{-1}(F'(y^* + \theta(x^* - y^*)) - F'(x^*))\,\mathrm{d}\theta\Big\| \\
&\leqslant \int_0^1 L_0\|y^* + \theta(x^* - y^*) - x^*\|\,\mathrm{d}\theta \\
&= L_0\int_0^1 (1 - \theta)\|y^* - x^*\| \leqslant \frac{L_0}{2}\varrho < 1.
\end{aligned}
$$

It follows from (2.28) and the Banach lemma on invertible functions that T is invertible. Finally, from the identity $0 = F(x^*) - F(y^*) = T(x^* - y^*)$, we conclude that $x^* = y^*$. This proves the statement. □

Remark 2.1. (1) According to (2.6) and

$$
\begin{aligned}
\|F'(x^*)^{-1}F'(x)\| &= \|I + F'(x^*)^{-1}(F'(x) - F'(x^*))\| \\
&\leqslant 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \\
&\leqslant 1 + L_0\|x - x^*\|,
\end{aligned}
$$

the condition (2.9) can be removed and M can be replaced by

$$M(x) = 1 + L_0 x$$

or simply by $M(x) = M = 2$, since $x \in [0, 1/L_0)$.

(2) The radius r_1 was shown in [5], [7], [10] to be the convergence radius of Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad n = 0, 1, 2, \dots,$$

under the conditions (2.6) and (2.7). By (2.2) we find that the convergence radius r of the method (1.1) cannot be larger than the convergence radius r_1 of second order Newton's method.

Moreover, based on [7], [10], r_1 is at least as little as the convergence ball given by Rheinboldt [44] and Traub [50],

$$r_R = \frac{2}{3L}.$$

By the initial assumption $L_0 < L$ in Theorem 2.1, we have $r_R < r_1$ and $r_R/r_1 \to 1/3$ as $L_0/L \to 0$. Therefore, the radius of our convergence ball r_1 is at most three times greater than r_R.

3. Dynamical analysis

The dynamical properties of the rational function associated with an iterative method give us important information about its stability and reliability. In this section, we study the dynamical behavior of an operator associated with the family (1.1) for quadratic and cubic polynomials. First, we recall some complex dynamical concepts that we use in this paper. For more information about these concepts, see [3], [4], [19], [21], [22], [20], [29], [38], [40], [46]. Let the nonlinear function F is defined on the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. Let us assume that a fixed-point iteration function acts on a generic polynomial $p(z)$, so a rational function $R(z) = H(z)/Q(z)$ on $\mathbb{C}_\infty$ is obtained, where the polynomials $H(z)$ and $Q(z)$ are coprime and not both zero. So, for any given rational function $R\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ the *orbit of a point* $z_0 \in \mathbb{C}_\infty$ is defined as

$$\{z_0, R(z_0), R^2(z_0), R^3(z_0), \dots, R^n(z_0)\},$$

where $R^k(z)$ is the kth composition of R.

A rational map R divides $\mathbb{C}_\infty$ into two subsets, that are known as the *Fatou set* and *Julia set*. The Fatou set, denoted by $\mathcal{F}(R)$, is defined as the set of points $z_0 \in \mathbb{C}_\infty$ such that the family of iterates R^n is a normal family in some neighborhood U_{z_0} of z_0. The Julia set, $\mathcal{J}(R)$, is the complement of the Fatou set, $\mathcal{J}(R) = \mathbb{C}_\infty - \mathcal{F}(R)$.

The basin of attraction of a fixed point α of a rational map R is the set

$$\mathcal{C}(\alpha) = \{z \in \mathbb{C}_\infty\colon\ R^n(z_0) \to \alpha,\ n \to \infty\},$$

that is used as the initial estimation converges to α. It is well-known that $\mathcal{C}(\alpha) \subset \mathcal{F}(R)$ and $\mathcal{J}(R) = \partial\mathcal{C}(\alpha)$, see [15].

Let $R_p(z)$ is the fixed point operator of Kung-Traub's method. An $\alpha \in \mathbb{C}_\infty$ is a fixed point of R if $R_p(\alpha) = \alpha$. The fixed points different from those associated with the roots of the polynomial $p(z)$ are called *strange fixed points*. A fixed point α is called an *attractor* if $|R_p'(\alpha, r)| < 1$, *superattractor* if $|R_p'(\alpha, r)| = 0$, *repulsor* if $|R_p'(\alpha, r)| > 1$ and *parabolic* if $|R_p'(\alpha, r)| = 1$.

A point z_0 is a *critical point* of the rational map R_p if $R_p'(z_0) = 0$. If the iterative method has the order of convergence at least two, the roots of $p(z)$ are superattracting fixed points. Therefore, any superattracting fixed point is a critical point. If a critical point is different from those associated with the roots of the polynomial $p(z)$, it is called a *free critical point*. The following theorem is a key fact to be used in the definition and interpolation of parameter planes [8].

Theorem 3.1 (see [27], [28], [32]). *Let R_p be a rational function. The immediate basin of attraction attracting fixed or periodic points holds, at least, a critical point.*

Theorem 3.2 (Scaling theorem [3]). *Let $F(z)$ be an analytic function and let $A(z) = \alpha z + \gamma$, with $\alpha \neq 0$, be an affine map. Let $h(z) = \lambda(F \circ A)(z)$ with $\lambda \neq 0$. Let $R_p(z)$ be the fixed point operator of Kung-Traub's method. Then $(A \circ R_h \circ A^{-1})(z) = R_F(z)$, that is, R_F is affine conjugated to R_G by A.*

3.1. Dynamics of Kung-Traub's method on quadratic polynomials. In the following, we apply the fixed point operator associated to Kung-Traub's method (1.1) on polynomial $p(z) = (z - r)(z - 1)$. The rational operator associated to the family of iterative schemes is

$$\begin{aligned} R_p(z, r) = \big[&r + 3r^2 + 5r^3 + 3r^4 + r^5 + (-8r - 22r^2 - 22r^3 - 8r^4)z \\ &+ (27r + 51r^2 + 27r^3)z^2 + (-40r - 40r^2)z^3 \\ &+ (-2 + 19r - 2r^2)z^4 + 6z^5 + 6rz^5 - 5z^6\big] \\ &\times (1 + r - 2z)^{-1}(1 + r + r^2 - 3z - 3rz + 3z^2)^{-2}, \end{aligned} \tag{3.1}$$

where $z \in \mathbb{C}$ and $r \in \mathbb{C}$. For specific values of parameter r, the rational function $R_p(z, r)$ is simpler:

▷ If $r = 1$, the fixed-point operator is

$$R_p(z, 1) = \frac{13 + 5z}{18}.$$

▷ If $r=-1$, the fixed-point operator can be expressed as

$$R_p(z,-1)=\frac{1+3z^2+23z^4+5z^6}{2z(1+3z^2)^2}.$$

Due to the fourth order convergence of the method, the roots of $p(z)$, i.e., $z=1$ and $z=r$, are superattracting fixed points. Furthermore, there are some strange fixed points of $R_p(z,r)$, which are analyzed in the following. The strange fixed points of $R_p(z,r)$ are roots of the polynomial

$$\begin{aligned}S(z)=&-1-3r-5r^2-3r^3-r^4+(7+19r+19r^2+7r^3)z\\&-(20+38r+20r^2)z^2+(26+26r)z^3-13z^4.\end{aligned}$$

We denote by $s_i(r)$, $i=1,2,3,4$, and

$$\begin{aligned}s_1(r)&=\frac{1}{52}\left(26+26r-\sqrt{-52-104\sqrt{3}\sqrt{-(-1+r)^4}+104r-52r^2}\right),\\s_2(r)&=\frac{1}{52}\left(26+26r+\sqrt{-52-104\sqrt{3}\sqrt{-(-1+r)^4}+104r-52r^2}\right),\\s_3(r)&=\frac{1}{52}\left(26+26r-\sqrt{-52+104\sqrt{3}\sqrt{-(-1+r)^4}+104r-52r^2}\right),\\s_4(r)&=\frac{1}{52}\left(26+26r+\sqrt{-52+104\sqrt{3}\sqrt{-(-1+r)^4}+104r-52r^2}\right).\end{aligned}$$

Then there are four different strange fixed points except in the case of $r=1$. If $r=1$, then $s_1(r)=s_2(r)=s_3(r)=s_4(r)=1$.

In order to analyze the stability of each one of these strange fixed points, we define the stability function of each fixed point as $S_i(r)=|R_p'(s_i(r),r)|$ for $i=1,\ldots,4$. We obtain that these fixed points are repulsive, that is, $|R_p'(s_i(r);r)|>1$ for $i=1,2,3,4$ and for all $r\in\mathbb{C}$.

However, it is possible that the convergence of the iterative schemes leads us to other attracting elements, such as periodic orbits. In order to detect this kind of behavior, the analysis of the critical points is necessary [8]. The critical points of the family are the solution of $R_p'(z,r)=0$, where

$$R_p'(z,r)=\frac{-2(r-z)^3(-1+z)^3(4+4r^2+r(7-15z)-15z+15z^2)}{(1+r-2z)^2(1+r+r^2-3z-3rz+3z^2)^3}.$$

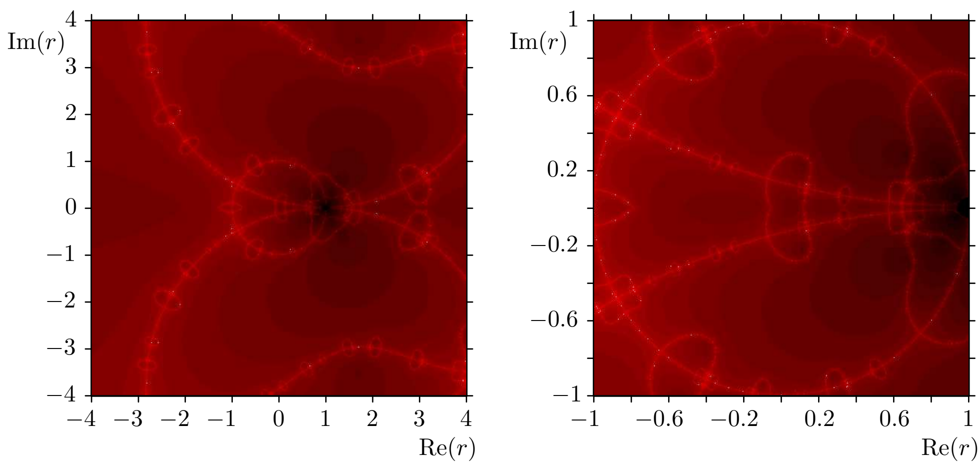
By attention to the fourth order convergence of the method, the roots of $p(z)$ are also critical points, hence, the zeros of $R_p'(z,r)$, $z=1$ and $z=r$, are also free critical points. Furthermore, the free critical points are

$$c_1(r) = 0.5 - \frac{\sqrt{-(-1+r)^2}}{2\sqrt{15}} + \frac{r}{2},$$
$$c_2(r) = 0.33\big(15 + \sqrt{15}\sqrt{-(-1+r)^2} + 15r\big).$$

If $r = 1$, $c_1(1) = c_2(1) = 0$, then there is only one free critical point.

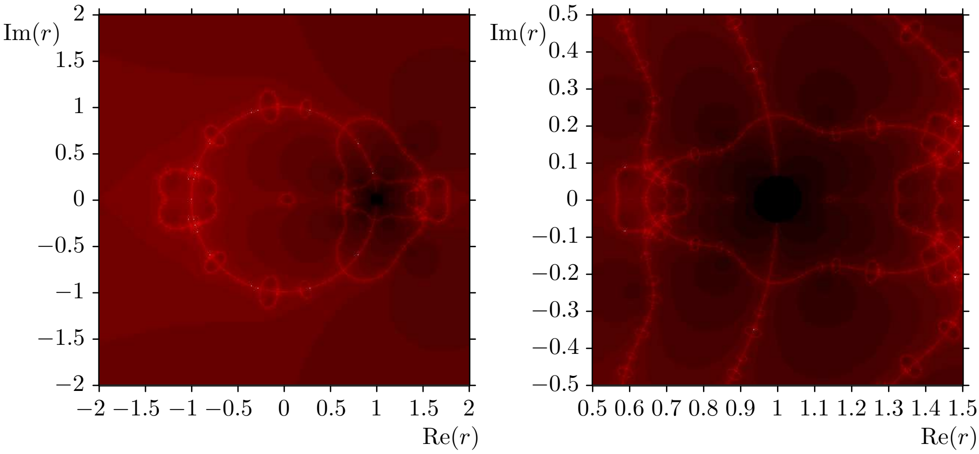
3.1.1. Parameter and dynamical planes. The aim of this section is to analyze the dynamical behavior of the rational operator (3.1) in different parameter planes. Applying this operator on free critical points as initial estimation, these parameter planes are obtained. Then, if this critical point goes to the basin of attraction of any of the roots of $p(z)$, it is drawn by the red and black color. Otherwise, it is plotted in white. Each value of the parameter r is selected in a mesh of 1000×1000 points and the tolerance of 10^{-3}. In this representation, we have used the software described in [17] to obtain the parameter planes and the different dynamical planes.

In Figures 1 and 2, the parameter plane associated to $c_1(r)$ and $c_2(r)$ is shown, respectively. Each value of the parameter r is selected in a mesh of 1000×1000 points and the tolerance of 10^{-3}. Then, we draw a point in the red and black color if the iteration of the method converges to the fixed-point, and in white in any other case, with a maximum of 20 iterations. After 200 iterations, we obtain Figure 3. We can observe that the behavior of the family is stable and also that it shows the convergence behavior for the most complex values of r.



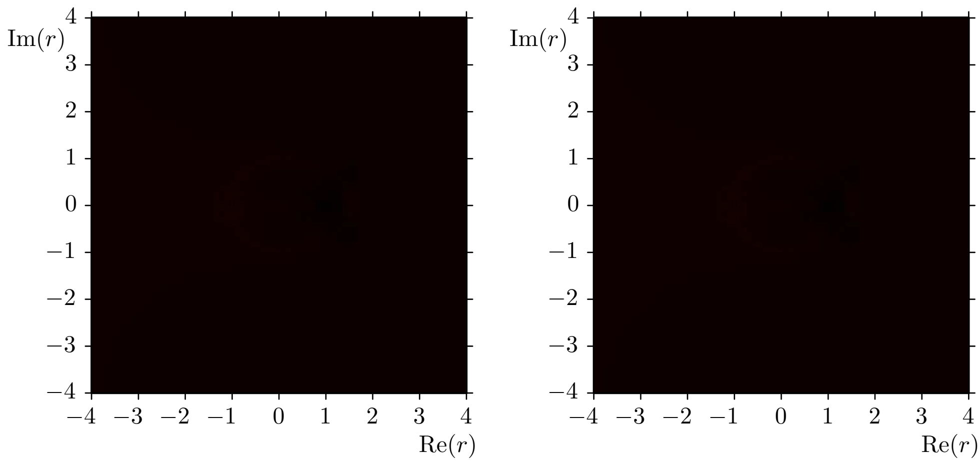
(a) $z_0 = c_1(r)$ (b) $z_0 = c_1(r)$, a detail

Figure 1. Parameter planes corresponding to the free critical point $c_1(r)$.



(a) $z_0 = c_2(r)$ (b) $z_0 = c_2(r)$, a detail

Figure 2. Parameter planes corresponding to the free critical point $c_2(r)$.



(a) Critical point $cr_1(r)$. (b) Critical point $cr_2(r)$.

Figure 3. Parameter planes corresponding to the free critical points after 200 iterations.

In order to analyze the behavior of Kung-Traub's method in this case, a value of parameter r is chosen and the associated dynamical plane is obtained. Now, we draw some dynamical planes corresponding to the selected values of parameter r, see Figure 4. We have painted in sky blue and violet the regions corresponding to the area of convergence to the roots and in black the zones with no convergence to the roots if they exist. These pictures have been also generated by using the software in [17], with a mesh of 500×500 points, the maximum iteration number of 50, and 10^{-3} as tolerance.

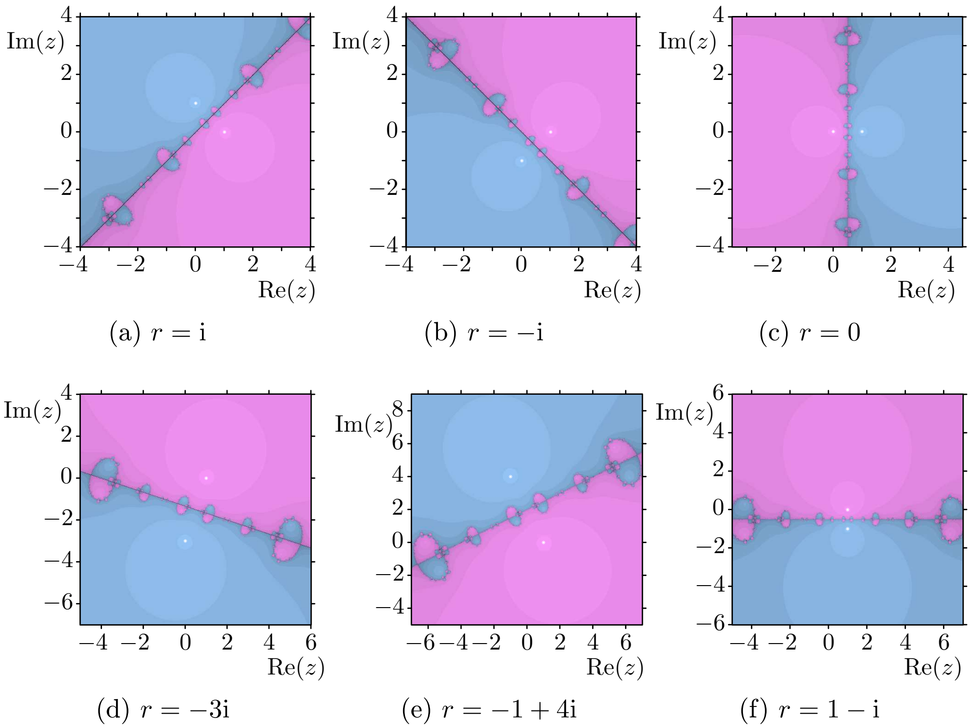


Figure 4. Dynamical planes associated to the values of r.

3.2. Dynamics of Kung-Traub's method on cubic polynomials. Let us consider $p(z) = (z - r)(z - 1)(z + 1)$, an arbitrary cubic polynomial with a given parameter $r \in \mathbb{C}$.

Theorem 3.3 (see [8], [45]). *Let $q(z)$ be any cubic polynomial with simple roots. Then it can be parametrized by means of an affine map to $p(z) = (z-1)(z-r)(z+1)$, $r \in \mathbb{C}$.*

The fixed-point operator corresponding to the family (1.1), when applied to $p(z)$, is

$$R_p(z, r) = \frac{T(z, r)}{(1 + 2rz - 3z^2)Q(z, r)}, \tag{3.2}$$

where

$$\begin{aligned} T(z, r) = r &+ (10r^2 + 2r^4)z + (-17r + 44r^3 + 4r^5 + 4r^7)z^2 \\ &+ (-156r^2 + 122r^4 - 16r^6)z^3 + (133r - 644r^3 + 252r^5 + 12r^7)z^4 \\ &+ (-2 + 1070r^2 - 1460r^4 + 136r^6)z^5 \end{aligned}$$

$$
\begin{aligned}
&+(-601r+3444r^3-1428r^5+92r^7)z^6\\
&+(14-3656r^2+4708r^4-800r^6)z^7\\
&+(1463r-7348r^3+2852r^5+20r^7)z^8\\
&+(4+5454r^2-5214r^4-216r^6)z^9+(-1327r+4800r^3+1008r^5)z^{10}\\
&+(-228-1388r^2-2638r^4)z^{11}+(-925r+4184r^3)z^{12}\\
&+(590-4022r^2)z^{13}+2169rz^{14}-506z^{15},
\end{aligned}
$$

$$
\begin{aligned}
Q(z,r)=\,&(1+(4r+2r^3)z+(-7+2r^2)z^2+(-18r+6r^3)z^3\\
&+(17-26r^2)z^4+38rz^5-19z^6)^2.
\end{aligned}
$$

Some simpler forms of the fixed point operator (3.2) for specific values of parameter r are:

▷ If $r=1$ then

$$R_p(z,r)=\frac{1+19z+147z^2+615z^3+1601z^4+2537z^5+2417z^6+1373z^7+506z^8}{(1+3z)(1+9z+19z^2+19z^3)^2}.$$

▷ If $r=-1$ then

$$R_p(z,r)=\frac{1-19z+147z^2-615z^3+1601z^4-2537z^5+2417z^6-1373z^7+506z^8}{(-1+3z)(-1+9z-19z^2+19z^3)^2}.$$

▷ If $r=0$ then

$$R_p(z,r)=\frac{2(z^5-7z^7-2z^9+114z^{11}-295z^{13}+253z^{15})}{(-1+3z^2)(-1+7z^2-17z^4+19z^6)^2}.$$

Some dynamical planes corresponding to the fixed points $r=-1$, $r=1$, and $r=0$ appear in Figure 6 (a), (b), and (c), respectively. It can be seen that we have to mention the case $r=1$ and $r=-1$, where one of the roots has multiplicity two, and there exist only two wide basins on attraction.

▷ If $r=-1$, the superattracting fixed points are the roots of $p(z)=z^3+z^2-z-1$, i.e. $z_1=1$ and $z_2=-1$.

▷ If $r=1$, the superattracting fixed points are the roots of $p(z)=z^3-z^2-z+1$, i.e. $z_1=1$ and $z_2=-1$.

▷ If $r=0$, the superattracting fixed points are the roots of $p(z)=z^3-z$, i.e. $z_1=0$, $z_2=-1$ and $z_3=1$.

These superattracting fixed points are plotted with white star points in Figure 6 (a), (b) and (c), respectively.

As the fixed points satisfy $R_p(z,r)=z$, it can be checked that the roots of $p(z)$, i.e. $z=r$, $z=1$, and $z=-1$, are the superattracting fixed points and also the strange fixed points of $R_p(z,r)$ are the roots of polynomial $S(z) = 1+(10r+2r^3)z+(-16+42r^2+4r^4+4r^6)z^2+(-140r+104r^3-16r^5)z^3+(111-528r^2+200r^4+8r^6)z^4+(828r-1092r^3+96r^5)z^5+(-428+2464r^2-972r^4+52r^6)z^6+(-2496r+3144r^3-464r^5)z^7+(951-4872r^2+1728r^4)z^8+(3722r-3438r^3)z^9+(-1132+3854r^2)z^{10}-2308rz^{11}+577z^{12}$, which we denote by $s_i(r)$, $i=1,2,\ldots,12$. To analyze the stability of the strange fixed points, the first derivative of $R_p(z,r)$ must be calculated. It is

$$R'_p(z,r) = \frac{2(r-z)^3(-1+z)^3(1+z)^3\Omega(z,r)}{(1+2rz-3z^2)^2\gamma(z,r)}, \tag{3.3}$$

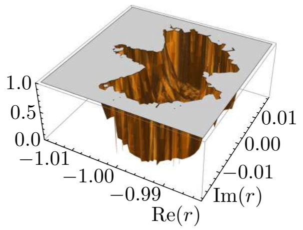
where

$$\begin{aligned}
\Omega(z,r) = {} & r - 5z + (16r^2 - 2r^4)z + (-74r + 76r^3 + 4r^5)z^2 \\
& + (50 - 352r^2 + 36r^4 + 48r^6 + 8r^8)z^3 + (375r - 136r^3 - 604r^5 + 48r^7)z^4 \\
& + (3 - 648r^2 + 3584r^4 - 1456r^6 + 120r^8)z^5 \\
& + (2376r - 12176r^3 + 10140r^5 - 1712r^7)z^6 \\
& + (-1872 + 23400r^2 - 36260r^4 + 10752r^6)z^7 \\
& + (-23325r + 76056r^3 - 38852r^5)z^8 + (9333 - 94648r^2 + 88322r^4)z^9 \\
& + (64834r - 129228r^3)z^{10} + (-18858 + 118696r^2)z^{11} \\
& - 62491rz^12 + 14421z^{13}, \\
\gamma(z,r) = {} & (1 + (4r + 2r^3)z + (-7 + 2r^2)z^2 + (-18r + 6r^3)z^3 \\
& + (17 - 26r^2)z^4 + 38rz^5 - 19z^6)^3.
\end{aligned}$$

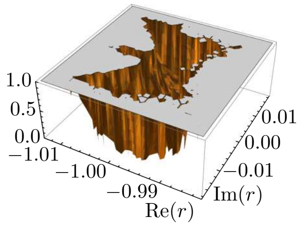
It can be checked that $z=\infty$ is a fixed point of $R_p(z,r)$ for every value of r. The stability of the other fixed points is more complicated and will be shown in a separate way. We use the graphical tools of software Mathematica in order to obtain the regions of stability of each of the strange fixed points. We evaluate numerically $z \in \mathbb{C}$ such that $|R'_p(s_i(r),r)| < 1$ for all $i=1,2,\ldots,12$ in the complex plane. In Figure 5, the stability region of the strange fixed points (in the saddle brown color) can be observed.

Taking into account these regions, the following statements summarize the behavior of the strange fixed points.

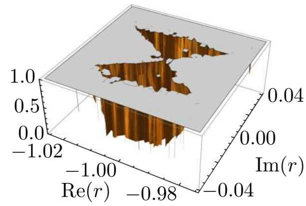
- ▷ $s_i(r)$, $i=1,2,3,4,5,6,7,11,12$, can be attractor (even parabolic) in different small areas of the complex plane.
- ▷ $s_i(r)$, $i=8,9,10$, are repulsive for all $r \in \mathbb{C}$.



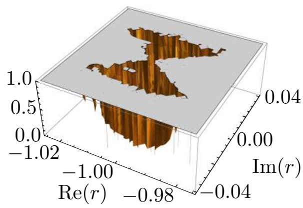
(a) $|R_p'(s_1(r), r)|$



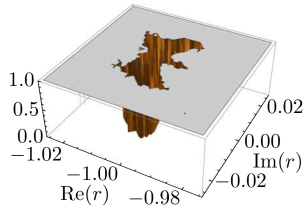
(b) $|R_p'(s_2(r), r)|$



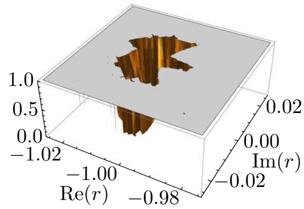
(c) $|R_p'(s_3(r), r)|$



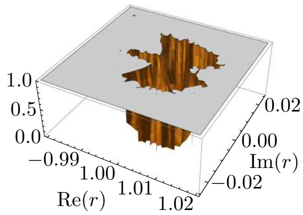
(d) $|R_p'(s_4(r), r)|$



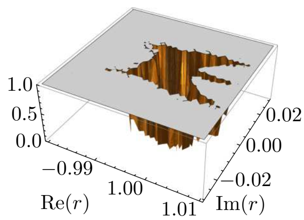
(e) $|R_p'(s_5(r), r)|$



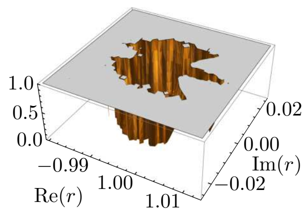
(f) $|R_p'(s_6(r), r)|$



(g) $|R_p'(s_7(r), r)|$



(h) $|R_p'(s_{11}(r), r)|$

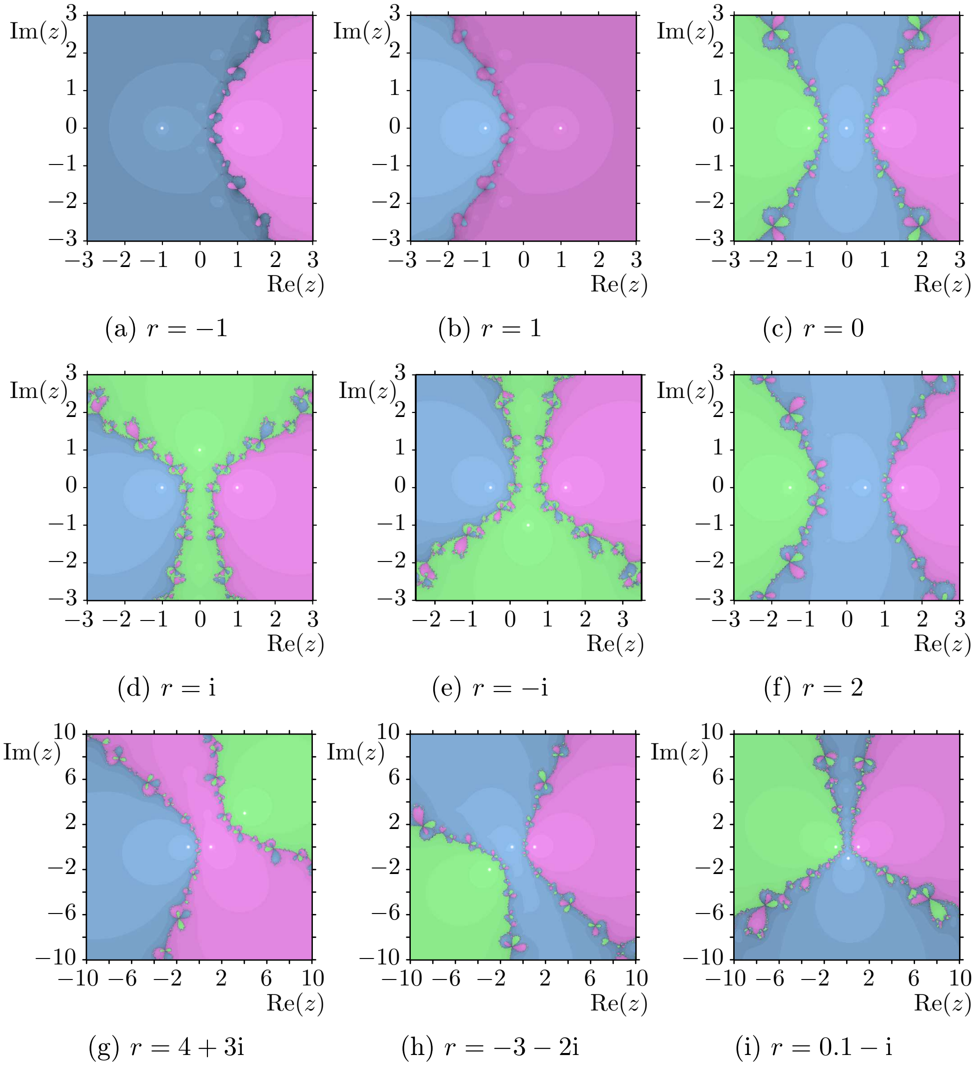


(i) $|R_p'(s_{12}(r), r)|$

Figure 5. Stability functions for strange fixed points in different areas.

3.2.1. Study of the critical points and parameter spaces. In this section, the critical points will be calculated and the parameter spaces associated with the free critical points will be shown. Critical points can be obtained by solving the equation $R_p'(z, r) = 0$, where $R_p'(z, r)$ is described in (3.3). It is clear that $z = -1$, $z = 1$ and $z = r$ are critical points, which are related to the roots of the polynomial $p(z)$, and also there are 13 critical points which are not related to the roots. These points are called free critical points.

We draw some dynamical planes associated with selected values of the parameter r in Figure 6. We have painted them as sky blue, light green, and violet regions corresponding to the areas of convergence to the roots and black zones with no convergence to the roots, if they exist. The superattracting fixed points are plotted with white star points. These pictures have been also generated by using the software in [17], with a mesh of 500×500 points, the maximum iteration number of 50, and 10^{-3} as tolerance.



(a) $r = -1$ (b) $r = 1$ (c) $r = 0$

(d) $r = \mathrm{i}$ (e) $r = -\mathrm{i}$ (f) $r = 2$

(g) $r = 4 + 3\mathrm{i}$ (h) $r = -3 - 2\mathrm{i}$ (i) $r = 0.1 - \mathrm{i}$

Figure 6. Dynamical planes associated to the values of r.

Finally, it can be concluded that the behavior of Kung-Traub's method on cubic polynomials is very stable.

4. Numerical experiments

In this section, we will present some numerical experiments using Kung-Traub's method, compare these results with the other schemes and show that this method is useful for all practical problems that lead to solving nonlinear equations.

For comparison, we consider some of the following existing fourth-order iterative methods proposed by researchers:

CM_4: The fourth order method proposed by Chun [18]

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(x_n)^2}{f(x_n)^2 - 2f(x_n)f(y_n) + 2f(y_n)^2} \dfrac{f(y_n)}{f'(x_n)}. \end{cases}$$

MM_4: The fourth order method proposed by Maheswari [39]

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n + \dfrac{1}{f'(x_n)} \Big[\dfrac{f(x_n)^2}{f(y_n) - f(x_n)} - \dfrac{f(y_n)^2}{f(x_n)} \Big]. \end{cases}$$

KM_4: The fourth order method proposed by King [33]

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2) f(y_n)} \dfrac{f(y_n)}{f'(x_n)}, \end{cases}$$

where $\beta \in \mathbb{R}$. For the computation we consider $\beta = 1$ in the above scheme.

TOM_4: The fourth-order method of Traub and Ostrowski (also known as Ostrowski method) [41]

$$\begin{cases} y_n = \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - y(x_n) \Big(1 + \dfrac{f(x_n - y(x_n))}{f(x_n) - 2f(x_n - y(x_n))} \Big). \end{cases}$$

Now, we compare the proposed method with the existing methods which are denoted by CM_4, MM_4, KM_4, and TOM_4. All computations have been performed using the programming package Mathematica 10 with Intel(R) Core i5-2500 CPU @3.30 GHz, RAM 16GB. In addition, $a(-b)$ stands for $a \times 10^{-b}$ in Tables 1–5. Moreover, we display the number of iteration indexes n, approximate zeros (x_n), the absolute residual error of the corresponding function $|f(x_n)|$, errors $|x_n - x^*|$, the mean elapsed time (e-time) and, finally, the computational order of convergence ϱ_c (COC) [31] using the formula

$$\varrho_c = \frac{\log(|f(x_n)/f(x_{n-1})|)}{\log(|f(x_{n-1})/f(x_{n-2})|)}, \quad n = 2, 3, \ldots$$

We calculate the computational order of convergence, computational order of convergence constant, and other constants up to several significant digits (minimum 2000 significant digits) to minimize the roundoff error. Due to the page limitation, we have displayed the values of x_n up to 20 significant digits only. From the results displayed in Tables 1–5, it can be concluded that the convergence of the tested multi-point methods is remarkably fast and the convergence behavior of the considered multi-point methods strongly depends on the structure of nested functions and the accuracy of initial approximations.

Example 4.1 (see [12]). We consider following Planck's radiation law problem which determines the energy density within an isothermal blackbody,

$$\varphi(\lambda) = \frac{8\pi ch\lambda^{-5}}{\exp(ch/\lambda kT) - 1},$$

where λ is the wavelength of the radiation, t is the absolute temperature of the blackbody, k is Boltzman's constant, h is the Planck's constant and c is the speed of light. We are going to determine the wavelength λ which corresponds to the maximum energy density $\varphi(\lambda)$. Therefore, we have

$$\varphi'(\lambda) = \frac{8\pi ch\lambda^{-6}}{\exp(ch/\lambda kT) - 1}\left(\frac{(ch/\lambda kT)\exp(ch/\lambda kT)}{\exp(ch/\lambda kT) - 1} - 5\right).$$

Hence, the maximum value of φ occurs when

$$\frac{(ch/\lambda kT)\exp(ch/\lambda kT)}{\exp(ch/\lambda kT) - 1} = 5.$$

By considering $x = ch/\lambda kT$, we obtain

$$1 - \frac{x}{5} = \mathrm{e}^{-x}. \tag{4.1}$$

Define

$$F(x) = \mathrm{e}^{-x} + \frac{x}{5} - 1. \tag{4.2}$$

As argued in [18], the left-hand side of equation (4.1) is zero for $x = 5$, moreover $\mathrm{e}^{-5} = 6.7410 \times 10^{-3}$. Therefore, it is expected that a root of the equation $F_1(x) = 0$ might occur near $x = 5$. The approximate root of Eq. (4.2) is given by $x^* \approx 4.96511423174427630369$. Under the assumptions in Section 2, $L > 0$, $L_0 > 0$, $M > 0$ such that $L_0 < L$. Let us consider $L_0 = L = 5.2$ and $M = 4.14$. The parameters are

$$r_1 = 0.128205, \quad r_2 = 0.0561651, \quad r = 0.0045005.$$

We observe that (2.2) holds, and by Figure 7(a) for all $x \in [0, 0.00450058)$, we can see

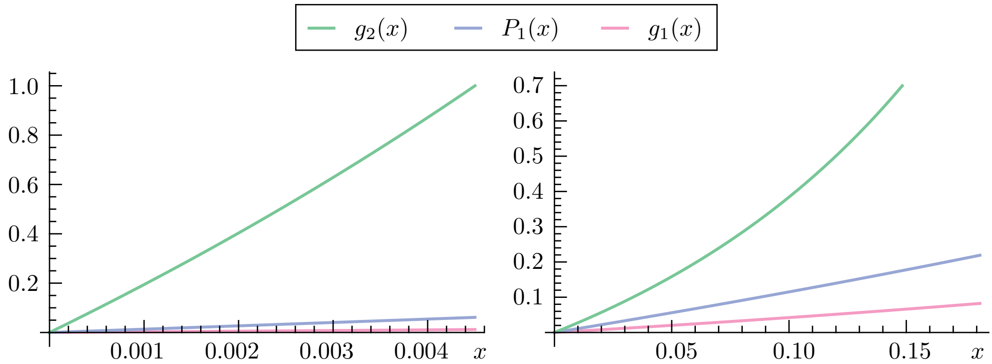
$$0 \leqslant g_1(x) < 1, \quad 0 \leqslant g_2(x) < 1, \quad 0 \leqslant P_1(x) < 1.$$

So we can ensure the convergence of the method (1.1) by Theorem 2.1.

The numerical results obtained for the problem are shown in Table 1. We show the number of iterations, the residual of the function at the last iteration, $\|f(x_n)\|$, the difference between the last iteration and the preceding one, $\|x_n - x_{n-1}\|$, the mean elapsed time after n executions and the computational order of convergence. As you can see, the mean elapsed time of Kung-Traub's two-point method is better than that of the other algorithms.

I. M.	n	x_n	$\|f(x_n)\|$	$\|x_{n+1} - x_n\|$	e-time	ϱ_c
	0	4	1.81684(−1)	0.96511		
Our method	1	4.96551894878152507 85	7.81202(−5)	4.0472(−4)		
	2	4.96511423174427630 69	6.24681(−19)	1.7443(−9)		
	3	4.96511423174427630 37	0.00000(−30)	1.7443(−9)	0.3588	4
	0	4	1.81684(−1)	0.96511		
CM_4	1	4.96556298561005389181	8.66204(−5)	4.4876(−4)		
	2	4.96511423174427630 88	9.903611(−19)	1.7443(−9)		
	3	4.96511423174427630 37	0.00000(−30)	1.7443(−9)	0.4212	4
	0	4	1.81684(−1)	0.96511		
KM_4	1	4.96556637304409586579	8.72743(−5)	4.5214(−4)		
	2	4.96511423174427630 90	1.02059(−18)	1.7443(−9)		
	3	4.96511423174427630 37	0.00000(−30)	1.7443(−9)	0.3588	4
	0	4	1.81684(−1)	0.96511		
MM_4	1	4.96561555608307271357	9.67679(−5)	5.0133(−4)		
	2	4.96511423174427631 21	1.61432(−18)	1.7443(−9)		
	3	4.96511423174427630 37	0.00000(−30)	1.7443(−9)	0.3432	4
	0	4	1.81684(−1)	0.96511		
TOM_4	1	4.96547479918333196894	6.95982(−5)	3.6057(−4)		
	2	4.96511423174427630 56	3.74324(−19)	1.7443(−9)		
	3	4.96511423174427630 37	0.00000(−30)	1.7443(−9)	0.3496	4

Table 1. Convergence behavior of different methods for Example 4.1.



(a) Example 4.1 (b) Example 4.2

Figure 7. Graph of the functions $g_1(t)$, $g_2(t)$ and $P_1(t)$.

E x a m p l e 4.2 ([12]). In the study of the trajectory of an electron in the air gap between two parallel plates it is given by

$$x(t) = x_0 + \left(v_0 + e\frac{E}{m\omega}\sin(\omega t_0 + \alpha)\right)(t - t_0) + e\frac{E_0}{m\omega^2}(\cos(\omega t + \alpha) + \sin(\omega t + \alpha)).$$

In the above expression, e and m are the charge and the mass of the electron at rest, x_0 and v_0 are the position and velocity of the electron at time t_0, and $E_0 \sin(\omega t + \alpha)$ is the RF electric field between the plates. By using the particular values, we obtain the nonlinear function

$$F(x) = x - \frac{1}{2}\cos x + \frac{\pi}{4}. \tag{4.3}$$

The approximate root of the (4.3) is given by $x^* = -0.3090932715417949$. Now, let us choose $L = 0.79$, $L_0 = 0.78$ and $M = 1.77$. Therefore, we obtain

$$r_1 = 0.851064, \quad r_2 = 0.592049, \quad r = 0.184732.$$

Hence, our convergence ratio is $r = 0.184732$, inequalities (2.2) hold and by Figure 7(b) for all $x \in [0, 0.184732)$ we obtain

$$0 \leqslant g_1(x) < 1, \quad 0 \leqslant g_2(x) < 1, \quad 0 \leqslant P_1(x) < 1.$$

So we can ensure the convergence of the method (1.1) by Theorem 2.1.

The numerical results obtained for the problem are shown in Table 2. The results presented in this table show that the mean elapsed time of Kung-Traub's two-point method is better than other algorithms.

I. M.	n	x_n	$\|f(x_n)\|$	$\|x_{n+1} - x_n\|$	e-time	ϱ_c
	0	0	2.85398(−1)	3.0909(−1)		
Our method	1	−0.3088262228684788345	2.26448(−4)	2.6705(−5)		
	2	−0.3090932715417947701	1.54857(−16)	1.8265(−16)		
	3	−0.3090932715417949527	0.00000(−30)	3.0463(31)	0.2964	4
	0	0	2.85398(−1)	3.0909(−1)		
CM_4	1	−0.3086907215065416742	3.41361(−4)	4.0255(−4)		
	2	−0.3090932715417934281	1.29270(−15)	1.5246(−15)		
	3	−0.3090932715417949527	0.00000(−30)	3.0463(−31)	0.3432	4
	0	0	2.85398(−1)	3.0909(−1)		
KM_4	1	−0.3087085655420003151	3.26228(−4)	3.8471(−4)		
	2	−0.3090932715417936811	1.07821(−16)	1.2716(−15)		
	3	−0.3090932715417949527	0.00000(−30)	3.0463(−31)	0.4524	4
	0	0	2.85398(−1)	3.0909(−1)		
MM_4	1	−0.3085992461850700239	4.18943(−4)	4.9403(−4)		
	2	−0.3090932715417901763	4.04992(−15)	4.7764(−14)		
	3	−0.3090932715417949527	0.00000(−30)	3.0463(−31)	0.3900	4
	0	0	2.85398(−1)	3.0909(−1)		
TOM_4	1	−0.3089633099776279397	1.10198(−4)	1.2996(−4)		
	2	−0.3090932715417949488	3.32863(−18)	3.9257(−18)		
	3	−0.3090932715417949527	0.00000(−30)	3.0463(−31)	0.3120	4

Table 2. Convergence behavior of different methods for Example 4.2.

Example 4.3 ([47]). Consider an equation describing a natch distillation at infinite reflux

$$F(x) = \frac{1}{63}\ln x + \frac{64}{63}\ln\frac{1}{1-x} + \ln(0.95 - x) - \ln(0.9),$$

where x is a mole fraction. This equation has two zeros, one of which being $\alpha = 0.03621008$ and the other one $\alpha = 0.5$. However, our desired root is $x^* = 0.5$.

We show the numerical results for this problem in Table 3. The number of iterations, the residual of the function at the last iteration, $\|f(x_n)\|$, the difference between the last iteration and the preceding one, $\|x_n - x_{n-1}\|$, the mean elapsed time after n executions and the computational order of convergence are shown in this table.

I. M.	n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_{n+1}-x_n\rvert$	e-time	ϱ_c
	0	0.7	6.35118(−2)	0.2000		
Our	1	0.5230656551301610113	3.92255(−3)	2.3066(−2)		
method	2	0.5000860886425302354	1.36652(−6)	8.6089(−6)		
	3	0.49999999999999900080	1.11022(−16)	9.9920(−16)	0.4056	Indeterminate
	0	0.7	6.351186(−2)	0.2000		
CM_4	1	0.5346337216679872828	6.10105(−3)	3.4634(−2)		
	2	0.5000709006740385876	1.12564(−5)	7.0901(−5)		
	3	0.50000000000000099920	2.22044(−16)	9.9920(−16)	0.4680	Indeterminate
	0	0.7	6.35118(−2)	0.2000		
KM_4	1	0.5296947994821256112	5.15261(−3)	2.9695(−2)		
	2	0.5000370893138897932	5.887838(−6)	3.7089(−5)		
	3	0.50000000000000011102	1.11022(−16)	1.1102(−16)	0.4524	Indeterminate
	0	0.7	6.35118(−2)	0.2000		
MM_4	1	0.53427634940734014535	6.03148(−3)	3.4276(−2)		
	2	0.5000856794431518537	1.36033(−5)	8.5679(−5)		
	3	0.50000000000000366374	7.77156(−16)	3.6637(−15)	0.53040	Indeterminate
	0	0.69999999999999995559	1.81684(−1)	0.02		
TOM_4	1	0.5057229848007834816	9.239479(−4)	5.7230(−3)		
	2	0.5000000092888765568	1.47442(−19)	9.2889(−10)		
	3	0.49999999999999938938	0.00000(−30)	6.1062(−16)	0.9360	Indeterminate

Table 3. Convergence behavior of different methods for Example 4.3.

Example 4.4 (see [16]). Van der Waal's equation is written as

$$\Big(P+\frac{an^2}{V^2}\Big)(V-nb)=nRT.$$

Here P, V, and T are the measured pressure, volume and temperature. The constants a and b are chosen to give the best agreement with the experiment for each gas. To determine the volume V, we must solve the nonlinear equation

$$PV^3-(nbP+nRT)V^2+an^2V-an^2b=0. \tag{4.4}$$

Using the specific value of constants a and b of a particular gas, we can find values of n, P and T so that equation (4.4) has three roots. Therefore, we obtain

$$F(x)=0.986x^3-5.181x^2+9.067x-5.289. \tag{4.5}$$

This equation has three zeros, two of which are complex and the third one is real zero, with $x^*=1.9298462428478316$. The results obtained by Kung-Traub's method, the

Chun method, the Maheswari method and the King iteration method are shown in Table 4.

I. M.	n	x_n	$\|f(x_n)\|$	$\|x_{n+1} - x_n\|$
	0	2.2000000000000001776	8.12880(−2)	0.2701
Our method	1	1.9866202887880495264	6.78776(−3)	5.6774(−2)
	2	1.9311952631187849772	1.17560(−4)	1.3490(−3)
	3	1.9298462440869483814	1.070983(−10)	1.2391(−9)
	4	1.9298462428478628805	0.00000(−30)	3.1308(−14)
	0	2.2000000000000001776	8.12880(−2)	0.2701
CM_4	1	1.9965086800564135761	8.39793(−3)	6.6662(−2)
	2	1.93287248273252010300	2.66421(−4)	3.0262(−3)
	3	1.9298462907807827360	4.14293(−9)	4.7933(−9)
	4	1.9298462428478639907	1.77635(−15)	3.2419(−14)
	0	2.2000000000000001776	6.35118(−2)	0.2701
KM_4	1	1.9925176746557597962	7.73132(−3)	6.2671(−2)
	2	1.9321635282056297545	2.03132(−4)	2.3173(−3)
	3	1.9298462594285603622	1.43311(−10)	1.6581(−8)
	4	1.9298462428478753150	1.77635(−15)	4.3743(−14)
	0	2.2000000000000001776	6.35118(−2)	0.2000
MM_4	1	1.9967610746538029254	8.44086(−3)	6.6915(−2)
	2	1.93310452583249192671	2.87253(−4)	3.2583(−3)
	3	1.92984632830805025172	7.38649(−9)	8.5460(−8)
	4	1.9298462428478524444	1.77635(−15)	2.0872(−14)
	0	2.2000000000000001776	8.1288(−2)	0.27015
TOM_4	1	1.97318900342835856776	4.81741(−3)	4.3343(−2)
	2	1.9301427123462959745	2.56708(−5)	2.9647(−3)
	3	1.9298462428490692488	1.06581(−13)	1.2377(−11)
	4	1.9298462428478559971	1.776356(−15)	2.4425(−14)

Table 4. Convergence behavior of different methods for Example 4.4.

E x a m p l e 4.5 (see [47]). Consider the equation

$$F(x) = ax^2 + bx^{7/4} - c,$$

where a, b and c are known positive constants. The equation comes from the analysis of flow in a smooth pipe, where x is the liquid velocity, x^2 comes from the velocity head, $x^{7/4}$ from the friction loss due to the pipe friction factor, and the constant term from the gravity head. Let us consider

$$a = 200, \quad b = 40, \quad c = 200.$$

Our required zero to this problem is $x^* = 0.842524$.

I. M.	n	x_n	$\|f(x_n)\|$	$\|x_{n+1} - x_n\|$	ϱ_c
	0	1	80.00000	1.5748(−1)	
Our method	1	0.8426760744909584801	7.07152(−2)	1.5207(−4)	
	2	0.8425243283818484555	9.41650(−14)	3.2838(−7)	
	3	0.8425243283818482534	0.00000(−30)	3.2838(−7)	4
	0	1	80.00000	1.5748(−1)	
CM_4	1	0.8427439691431298825	1.023588(-1)	2.1997(-4)	
	2	0.8425243283818495797	6.180003(−13)	3.2838(−7)	
	3	0.8425243283818482534	0.00000(−30)	3.2838(−7)	4
	0	1	80.00000	1.5748(−1)	
KM_4	1	0.8427350707328158400	9.82114(−2)	2.1107(−4)	
	2	0.8425243283818493774	5.23746(−13)	3.2838(−7)	
	3	0.8425243283818482534	0.00000(-30)	3.2838(−7)	4
	0	1	80.00000	1.5748(−1)	
MM_4	1	0.8427899076530581909	1.23770(−1)	2.6591(−3)	
	2	0.8425243283818520266	1.758170(−12)	3.2838(−7)	
	3	0.8425243283818482534	0.00000(−30)	3.2838(−7)	4
	0	1	80.00000	1.5748(−1)	
TOM_4	1	0.8426073941144693251	3.870796(−2)	8.3394(−4)	
	2	0.8425243283818482626	4.26549(−15)	3.2838(−7)	
	3	0.8425243283818482534	0.00000(−30)	3.2838(−7)	4

Table 5. Convergence behavior of different methods for Example 4.5.

5. Conclusion

In this paper, the local convergence analysis of the family of Kung-Traub's two-point method is studied. The dynamical behavior of Kung-Traub's method on quadratic and cubic polynomials is analyzed. All the fixed and critical points and their dynamical behavior are obtained. In the dynamical study, it has been shown that the behavior of Kung-Traub's method on quadratic and cubic polynomials is remarkably stable. The numerical results confirmed that Kung-Traub's method converged.

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