Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 70 (2020), No. 3, 891–903

Persistent URL: `http://dml.cz/dmlcz/148335`

Czechoslovak Mathematical Journal, 70 (145) (2020), 891–903

SOME RESULTS ON POINCARÉ SETS

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Received January 1, 2019. Published online March 27, 2020.

Abstract. It is known that a set H of positive integers is a Poincaré set (also called intersective set, see I. Ruzsa (1982)) if and only if $\dim_{\mathcal{H}}(X_H) = 0$, where

$$X_H := \left\{ x = \sum_{n=1}^{\infty} \frac{x_n}{2^n} \colon \ x_n \in \{0,1\},\ x_n x_{n+h} = 0 \text{ for all } n \geqslant 1,\ h \in H \right\}$$

and $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension (see C. Bishop, Y. Peres (2017), Theorem 2.5.5). In this paper we study the set X_H by replacing 2 with $b > 2$. It is surprising that there are some new phenomena to be worthy of studying. We study them and give several examples to explain our results.

Keywords: Poincaré set; homogeneous set; Hausdorff dimension

MSC 2020: 37B20, 11A07

1. Introduction

Let $\mathbb{N} := \{1, 2, 3, \ldots\}$. Recall that for $S \subseteq \mathbb{N}$ the upper density of S is defined by

$$\bar{d}(S) = \limsup_{n\to\infty} \frac{\#(S \cap \{1, 2, \ldots, n\})}{n},$$

where $\#A$ is the cardinality of a set A. Following Bishop and Peres (see [2]) we call $H \subseteq \mathbb{N}$ a *Poincaré set* if for every $S \subseteq \mathbb{N}$ with positive upper density we have $(S - S) \cap H \neq \emptyset$, i.e., there is an $h \in H$ such that $(S + h) \cap S \neq \emptyset$. Furstenberg in 1981 (see [6]) gave the following equivalent characterization:

Theorem 1.1. *$H \subseteq \mathbb{N}$ is a Poincaré set if and only if for any measure preserving system $(X, \mathcal{B}, \mu, T)$ and any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $n \in H$ such that $\mu(A \cap T^{-n}A) > 0$.*

This work was supported by the National Natural Science Foundation of China 11801035.

DOI: 10.21136/CMJ.2020.0001-19

By Theorem 1.1, Poincaré sets are also called *sets of recurrence*.

The study of classification of Poincaré sets has a long history. The fundamental work on this issue is due to Sárközy and Furstenberg. In the late 1970s, Sárközy, see [12] and Furstenberg, see [5], [6] independently proved the following result by using different methods, widely known today as Sárközy's theorem, which had previously been conjectured by Lovász.

Theorem 1.2. *If $H \subseteq \mathbb{N}$ is a set of positive upper density, then there are two distinct elements of H whose difference is a perfect square.*

Clearly the set $\{n^2\colon\ n \in \mathbb{N}\}$ is a Poincaré set by Theorem 1.2. Sárközy subsequently proved in [13], [14] that the sets $\{n^k\colon\ n \in \mathbb{N}\}$ for all $k \in \mathbb{N}$, $\{n^2 - 1\colon\ n > 1\}$ and $\{p \pm 1\colon\ \text{prime}\}$ are Poincaré sets. However, he also proved that the sets $2\mathbb{N} - 1$ and $\{n^2 + 1\colon\ n \in \mathbb{N}\}$ are not Poincaré sets in the same papers.

After the original work of Sárközy and Furstenberg, the Poincaré sets have been investigated in a variety of different mathematical fields and fascinating directions. For example, Kamae and Mendès France in [8] gave several criteria for Poincaré sets. Actually, their work was motivated by a different, stronger notion than Poincaré sets that they called *van der Corput sets*. They showed that any van der Corput set is also a Poincaré set. Subsequently, Ruzsa in [11] gave some further characterizations for van der Corput sets. Extensive accounts of van der Corput sets can be found in [1], [10] and the references therein. However, Bourgain in [3] proved the converse is not true by constructing a Poincaré set which is not a van der Corput set. Lê in [9] provides an excellent and detailed exposition on this subject.

Furstenberg in [2] also gave a new method connecting number theory and fractal geometry to check whether a set is a Poincaré set. He proved the following:

Theorem 1.3. *$H \subseteq \mathbb{N}$ is a Poincaré set if and only if $\dim_{\mathcal{H}}(X_H) = 0$, where*

$$X_H := \left\{x = \sum_{n=1}^{\infty} \frac{x_n}{2^n}\colon\ x_n \in \{0,1\},\ x_n x_{n+h} = 0 \text{ for all } n \geqslant 1,\ h \in H\right\},$$

and $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension.

It is natural to start with the simplest generalization and therefore ask: what will happen if we change 2 to an integer greater than 2 in the set X_H? We find that the result is unchanged in this case and the proof is similar. However, if we consider a more general set based on this situation, we will get some unexpected and interesting results.

Let $H \subseteq \mathbb{N}$. Let $b \geqslant 2$ be an integer and t an integer with $0 \leqslant t \leqslant b-1$. Define

$$X_{H,t}^b := \left\{ x = \sum_{n=1}^{\infty} \frac{x_n}{b^n} \colon\ x_n \in \{0,1,\ldots,b-1\},\ x_n x_{n+h} \equiv t \pmod b \text{ for all } n \geqslant 1,\ h \in H \right\}.$$

Proposition 1.4. *Suppose that H is a Poincaré set and the Quadratic Congruence Equation $y^2 \equiv t \pmod b$ has at most one solution. Then $\dim_{\mathcal{H}}(X_{H,t}^b) = 0$.*

Theorem 1.5. *If $t = 0$ and $\dim_{\mathcal{H}}(X_{H,t}^b) = 0$, then H is a Poincaré set.*

More generally, let I be an ideal of K, where $K = \mathbb{Z}_b$ or $\mathbb{Z}$. By abuse of notation, we will often not distinguish $\mathbb{Z}_b$ and $\{0,1,\ldots,b-1\}$. Define

$$X_{H,I}^b := \left\{ x = \sum_{n=1}^{\infty} \frac{x_n}{b^n} \colon\ x_n \in \{0,1,\ldots,b-1\},\ x_n x_{n+h} \in I \text{ for all } n \geqslant 1,\ h \in H \right\}.$$

Theorem 1.6. *If $\#(I \cap \mathbb{Z}_b) < b$ and $\dim_{\mathcal{H}}(X_{H,I}^b) \leqslant \log_b \#(I \cap \mathbb{Z}_b)$, then H is a Poincaré set.*

For the organization of the paper, we first devote Section 2 to introduce some basic lemmas and propositions which will be used in the following sections. In Section 3, we prove Proposition 1.4 and Theorem 1.5. The proof of Theorem 1.6 is presented in Section 4. Finally, some examples are given to explain our theory in Section 6.

2. Preliminaries

In this section we first recall two important lemmas in fractal geometry: Billingsley's Lemma and Furstenberg's Lemma (see for example [2], [4]), and then we give several results on some sets defined by digit frequency, which will be used in the following sections.

Lemma 2.1 (Billingsley's Lemma)**.** *Let $A \subseteq [0,1]$ be a Borel set and let μ be a finite Borel measure on $[0,1]$. Suppose $\mu(A) > 0$. If*

$$\alpha_1 \leqslant \liminf_{n\to\infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|} \leqslant \beta_1$$

for all $x \in A$, where $I_n(x)$ is the nth generation, half-open b-adic interval of the form $[(j-1)/b^n, j/b^n)$ containing x and $|I_n(x)|$ denotes the Lebesgue measure of $I_n(x)$, then

$$\alpha_1 \leqslant \dim_{\mathcal{H}}(A) \leqslant \beta_1.$$

Let $b \geqslant 2$ be an integer, and define a map $T_b\colon\ [0,1] \to [0,1]$ by

$$T_b(x) \equiv bx \pmod 1.$$

Definition 2.2. A compact set $K \subseteq [0,1]$ is called a *b-homogeneous set* if $T_b(K) = K$.

The next classical lemma implies that the Minkowski (box-counting) dimension of a homogeneous set agrees with its Haudorff dimension. One may refer to [2] and [4] for details.

Lemma 2.3 (Furstenberg's Lemma)**.** *Let $b > 1$ be an integer and let $K \subseteq [0,1]$ be a b-homogeneous set. Then $\dim_{\mathcal{H}}(K) = \dim_{\mathcal{M}}(K)$, where $\dim_{\mathcal{M}}$ denotes the Minkowski dimension.*

Fix a set $E \subsetneqq \{0,1,\ldots,b-1\}$ and a real number $p \in [0,1]$. Let $\{x_n\}$ be the b-ary expansion of the real number $x \in [0,1]$. Let

$$A_{E,p} := \Big\{x \in [0,1]\colon\ \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_E(x_k) = p\Big\}, \tag{2.1}$$

where χ_E is the characteristic function of x_k, i.e., $\chi_E(x_k) = 1$ for $x_k \in E$, and $\chi_E(x_k) = 0$ for $x_k \notin E$.

Lemma 2.4 ([2])**.** *Let $0 < p < 1$. We have $\dim_{\mathcal{H}}(A_{E,p}) = -p\log_b(p/\#E) - (1-p)\log_b((1-p)/(b-\#E))$.*

Here, we give a slight generalization of the above result.

Proposition 2.5. *If $\#E = b-1$ and $p = 0$, then $\dim_{\mathcal{H}}(A_{E,0}) = 0$.*

Proof. Let

$$\widetilde{A}_{E,p} := \Big\{x \in [0,1]\colon\ \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_E(x_k) \leqslant p\Big\},$$

thus clearly $A_{E,p} \subseteq \widetilde{A}_{E,p}$. By Lemma 2.4, $\dim_{\mathcal{H}}(A_{E,\#E/b}) = 1$. Consequently, $\dim_{\mathcal{H}}(\widetilde{A}_{E,p}) = 1$ if $p \geqslant \#E/b$. On the other hand if $0 < p < \#E/b$, by Lemma 2.4 again it follows that

$$-p\log_b\Big(\frac{p}{\#E}\Big) - (1-p)\log_b\Big(\frac{1-p}{b-\#E}\Big) = \dim_{\mathcal{H}}(A_{E,p}) \leqslant \dim_{\mathcal{H}}(\widetilde{A}_{E,p}).$$

Define a Borel measure on $[0,1]$ by

$$\mu_{E,p}(I_n(x)) = \Big(\frac{p}{\#E}\Big)^{\sum_{k=1}^n \chi_E(x_k)} \Big(\frac{1-p}{b-\#E}\Big)^{\sum_{k=1}^n (1-\chi_E(x_k))},$$

where $I_n(x)$ is defined as in Lemma 2.1 (in fact $\mu_{E,p}(A_{E,p}) = 1$, see the proof of Lemma 2.4 in [2]). Therefore,

$$\begin{aligned}\frac{\log \mu_{E,p}(I_n(x))}{\log |I_n(x)|} \\ &= \frac{n^{-1}\sum_{k=1}^n \chi_E(x_k)\log(\#E/(b-\#E)\cdot(1-p)/p) + \log(b-\#E)/(1-p)}{\log b}.\end{aligned}$$

Since $0 < p < \#E/b$, it follows that $\log(\#E/(b-\#E)\cdot(1-p)/p) > 0$, and for $x \in \widetilde{A}_{E,p}$ we get

$$\begin{aligned}\liminf_{n\to\infty} \frac{\log \mu_{E,p}(I_n(x))}{\log |I_n(x)|} &\leqslant \frac{p\log(\#E/(b-\#E)\cdot(1-p)/p) + \log(b-\#E)/(1-p)}{\log b} \\ &= -p\log_b\Big(\frac{p}{\#E}\Big) - (1-p)\log_b\Big(\frac{1-p}{b-\#E}\Big).\end{aligned}$$

Therefore, for $0 < p < \#E/b$, Billingsley's Lemma implies

$$\dim_{\mathcal{H}}(A_{E,p}) = \dim_{\mathcal{H}}(\widetilde{A}_{E,p}) = -p\log_b\Big(\frac{p}{\#E}\Big) - (1-p)\log_b\Big(\frac{1-p}{b-\#E}\Big).$$

Since $A_{E,0} \subseteq \widetilde{A}_{E,p}$ for all $0 < p < 1$ and $\#E = b-1$, we have

$$0 \leqslant \dim_{\mathcal{H}}(A_{E,0}) \leqslant \dim_{\mathcal{H}}(\widetilde{A}_{E,p}) = -p\log_b\Big(\frac{p}{b-1}\Big) - (1-p)\log_b(1-p).$$

The desired result then follows by letting $p \to 0$. □

3. Proof of Proposition 1.4 and Proof of Theorem 1.5

In this section we prove Proposition 1.4 and Theorem 1.5.

Proof of Proposition 1.4. We prove the result by the following two cases.

Case I: Suppose the equation $y^2 \equiv t \pmod b$ has no root. Let $a \in \{0,1,\dots,b-1\}$. Then $a^2 \not\equiv t \pmod b$. We claim that for any $x \in X_{H,t}^b$ with $x = \sum\limits_{n=1}^{\infty} x_n/b^n$, we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \chi_{\{a\}}(x_k) = 0. \tag{3.1}$$

If (3.1) is not true, then there exists an $x \in X^b_{H,t}$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{\{a\}}(x_k) =: \alpha > 0.$$

Let $S_a := \{k \in \mathbb{N}\colon\ \chi_{\{a\}}(x_k) = 1\}$. Then

$$\bar{d}(S_a) = \limsup_{n\to\infty} \frac{\#(S_a \cap \{1, 2, \ldots, n\})}{n} = \alpha > 0.$$

Since H is a Poincaré set, we have $(S_a - S_a) \cap H \neq \emptyset$. Thus there exist $s_1, s_2 \in S_a$ such that $s_1 - s_2 = h \in H$. From the definition of S_a, we have $x_{s_1} = x_{s_2} = a$. But $x_{s_2} x_{s_2+h} = a^2 \not\equiv t \pmod b$, which contradicts the fact that $x \in X^b_{H,t}$. So the claim is true.

Next we only need to prove that $X^b_{H,t} = \emptyset$. Assume that $X^b_{H,t} \neq \emptyset$. Then there is an $x' \in X^b_{H,t}$. By (3.1), we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{\{0,1,\ldots,b-1\}}(x'_k) = \sum_{a\in\{0,1,\ldots,b-1\}} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_a(x'_k) = 0.$$

But

$$\frac{1}{n} \sum_{k=1}^{n} \chi_{\{0,1,\ldots,b-1\}}(x'_k) = 1 \quad \text{for all } n,$$

and it follows that $0 = 1$ by letting $n \to \infty$. This is a contradiction. Hence $\dim_{\mathcal{H}}(X^b_{H,t}) = \dim_{\mathcal{H}}(\emptyset) = 0$.

Case II: Suppose the equation $y^2 \equiv t \pmod b$ has exactly one solution. Let $a_1 \in \{0, 1, \ldots, b-1\}$ with $a_1^2 \equiv t \pmod b$. Denote $E = \{0, 1, \ldots, b-1\} \setminus \{a_1\}$. For any $x \in X^b_{H,t}$, by (3.1) we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(x_k) = \sum_{a\in E} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_a(x_k) = 0.$$

Then we have $X^b_{H,t} \subseteq A_{E,0}$. Since $\#E = b - 1$, by Proposition 2.5, we have $\dim_{\mathcal{H}}(X^b_{H,t}) = \dim_{\mathcal{H}}(A_{E,0}) = 0$.

Hence we complete the proof. $\square$

Now we prove Theorem 1.5.

P r o o f of Theorem 1.5. Suppose that H is not a Poincaré set. Then there exists a set $S \subseteq \mathbb{N}$ with $\bar{d}(S) > 0$ such that

$$(S - S) \cap H = \emptyset.$$

Define

$$A_S = \left\{ x = \sum_{n=1}^{\infty} \frac{x_n}{b^n} \colon\ x_n = 0 \text{ if } n \notin S \text{ and } x_n \in \{0, 1, 2, \dots, b-1\} \text{ otherwise} \right\},$$

it is easy to see that the set A_S hits exactly

$$b^{\sum_{k=1}^{n} \chi_S(k)}$$

b-adic intervals (half-open intervals) of generation n. Then

$$\log_b \widetilde{N}(A_S, b^{-n}) = \sum_{k=1}^{n} \chi_S(k),$$

where $\widetilde{N}(A_S, b^{-n})$ means the minimal number of b-adic intervals of length b^{-n} needed to cover A_S. Thus

$$\frac{\log \widetilde{N}(A_S, b^{-n})}{\log b^n} = \frac{1}{n} \sum_{k=1}^{n} \chi_S(k),$$

which implies

$$\overline{\dim}_{\mathcal{M}}(A_S) = \bar{d}(S) > 0,$$

where $\overline{\dim}_{\mathcal{M}}$ denotes the upper Minkowski dimension (see [2]). Next we prove $A_S \subseteq X_{H,t}^b$. Let $x \in A_S$. For any $n \geqslant 1$, we consider the following two cases: if $n \notin S$, then $x_n = 0$ and thus $x_n x_{n+h} = 0$ for all $h \in H$, if $n \in S$, since H is not a Poincaré set for all $h \in H$, we have $(S+h) \cap S = \emptyset$. So $n + h \notin S$ and $x_{n+h} = 0$, which implies that $x_n x_{n+h} = 0$. Then we have $A_S \subseteq X_{H,t}^b$.

Similarly, we can easily to check that $X_{H,t}^b$ is compact and T_b-invariant. By Lemma 2.3, we have

$$\dim_{\mathcal{H}}(X_{H,t}^b) = \overline{\dim}_{\mathcal{M}}(X_{H,t}^b) \geqslant \overline{\dim}_{\mathcal{M}}(A_S) > 0.$$

This is a contradiction. □

We have the following useful and interesting corollaries.

Corollary 3.1. *Let $b = 2^l p_1^{l_1} p_2^{l_2} \dots p_m^{l_m}$, where p_i and p_j are distinct odd primes for any $i \neq j$ and $l_1, \dots, l_m$ are all positive integers. Suppose H is a Poincaré set. Then $X_{H,t}^b = \emptyset$ if one of the followings happens:*

(i) *there exists p_i such that $(t/p_i) = -1$, where (t/p_i) means the Legendre symbol,*
(ii) *$l = 2$ and $t \equiv 3 \pmod 4$,*
(iii) *$l > 2$ and $t \equiv 3, 5, 7 \pmod 8$,*
(iv) *there exist p_i and an odd prime denoted by a_0 with $a_0 < l_i$ such that $(p_i^{l_i}, t) = p_i^{a_0}$,*
(v) *there exists an odd number denoted by a_1 with $a_1 < l$ such that $(2^l, t) = 2^{a_1}$.*

P r o o f. To prove it, it suffices to use a lemma in number theory, see [7].

Lemma 3.2. *Let p be a prime with $p \nmid n$. Suppose that the Quadratic Residues Equation*

$$x^2 \equiv n \pmod{p^l}, \quad l > 0,$$

has $1+(n/p)$ solutions when $p > 2$. When $p = 2$, there are the following three types:

(i) *if $l = 1$, it has only one solution,*

(ii) *if $l = 2$, it has respectively two or no solutions when $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$,*

(iii) *if $l > 2$, it has respectively four or no solutions when $n \equiv 1 \pmod 8$ or $n \not\equiv 1 \pmod 8$.*

By Lemma 3.2, we can immediately achieve Corollary 3.1. □

Corollary 3.3. *Suppose H is a Poincaré set. Then $\dim_{\mathcal{H}}(X^b_{H,t}) = 0$ if one of the following happens:*

(i) *$t = 0$, $b = p_1p_2\dots p_m$, where p_i and p_j are distinct primes for any $i \neq j$,*

(ii) *$t = 1$, $b = 2$,*

(iii) *$t = p_1p_2\dots p_m$, $b = 2p_1p_2\dots p_m$, where p_i and p_j are distinct odd primes for any $i \neq j$.*

P r o o f. (i) By the Chinese Remainder Theorem, we know that the equation $a^2 \equiv 0 \pmod b$ has only one solution. Then by Theorem 1.5 we have $\dim_{\mathcal{H}}(X^b_{H,t}) = 0$. (ii) Since the equation $y^2 \equiv 1 \pmod 2$ has only one solution, we have $\dim_{\mathcal{H}}(X^b_{H,t}) = 0$ by Theorem 1.5. (iii) Combining the Chinese Remainder Theorem with Corollary 3.1, we have that the equation $y^2 \equiv t \pmod b$ has only one solution. Then by Theorem 1.5 again, we have $\dim_{\mathcal{H}}(X^b_{H,t}) = 0$. □

We denote $\Sigma_b = \{0, 1, \dots, b-1\}$ and $\Sigma_b^n = \{\sigma = \sigma_1\sigma_2\dots\sigma_n \colon \sigma_i \in \Sigma_b, 1 \leqslant i \leqslant n\}$. For any set $X \subseteq [0,1]$, we define

$$\begin{aligned} N_n(X) = \#\Big\{\sigma \in \Sigma_b^n \colon \Big[a_\sigma, a_\sigma + \frac{1}{b^n}\Big] \cap X \neq \emptyset, \text{ where} \\ a_\sigma = \frac{\sigma_1}{b} + \frac{\sigma_2}{b^2} + \ldots + \frac{\sigma_n}{b^n}, \text{ if } \sigma = \sigma_1\sigma_2\dots\sigma_n, \ \sigma_i \in \Sigma_b, 1 \leqslant i \leqslant n\Big\}. \end{aligned}$$

It is straightforward to verify that

$$\overline{\dim}_{\mathcal{M}}(X) = \limsup_{n\to\infty} \frac{\log N_n(X)}{n \log b}.$$

We conclude this section by a proposition for which we calculate the dimension of another more general set. Let $H=\{1,3,5,\dots\}$ and $b\geqslant 2$. We define

$$\widetilde{X}^b_{H,0} := \biggl\{x=\sum_{n=1}^{\infty}\frac{x_n}{b^n}\colon\ x_n\in\{0,1,\dots,b-1\} \text{ with } x_nx_{n+h}\equiv 0 \pmod b$$

$$\text{for all } n\geqslant 1,\ h\in H \text{ and } \gcd(x_n,b)=1 \text{ for all } x_n>0\biggr\}.$$

Proposition 3.4. *With the notation above,* $\dim_{\mathcal M}(\widetilde{X}^b_{H,0}) = \dim_{\mathcal H}(\widetilde{X}^b_{H,0}) = \frac12\log_b(\varphi(b)+1)$, *where* $\varphi(k)$ *is the Euler function of* k.

Proof. It is easy to see that $\widetilde{X}^b_{H,0}$ is compact and $T_b(\widetilde{X}^b_{H,0})=\widetilde{X}^b_{H,0}$, and then by Lemma 2.3, we have $\dim_{\mathcal M}(\widetilde{X}^b_{H,0})=\dim_{\mathcal H}(\widetilde{X}^b_{H,0})$. Define

$$g(n)=\begin{cases}(\varphi(b)+1)^{(n-1)/2} & \text{if } n \text{ is odd},\\ (\varphi(b)+1)^{n/2-1} & \text{if } n \text{ is even}.\end{cases}$$

We decompose the points in $\widetilde{X}^b_{H,0}$ into two cases:

(i) Fix $x_1\in\{1,2,\dots,b-1\}$, then $\gcd(x_1,b)=1$. So we have $x_{2k}=0$. Since $\gcd(x_{2k+1},b)=1$ for all $x_{2k+1}>0$, we know the number of distinct combining forms from x_1 to x_n is $g(n)$.

(ii) If $x_1=0$, then the number of distinct combining forms from x_1 to x_n is $N_{n-1}(\widetilde{X}^b_{H,0})$.

By (i) and (ii), we have $N_n(\widetilde{X}^b_{H,0})=N_{n-1}(\widetilde{X}^b_{H,0})+\varphi(b)g(n)$ for $n\geqslant 2$ and $N_1(\widetilde{X}^b_{H,0})=\varphi(b)+1$. If $n=2k$, we have $N_n(\widetilde{X}_{H,0})=2(\varphi(b)+1)^{n/2}-1$. Otherwise, we have $N_n(\widetilde{X}^b_{H,0})=(\varphi(b)+1)^{(n-1)/2}(\varphi(b)+2)-1$. Then

$$\dim_{\mathcal M}(\widetilde{X}^b_{H,0})=\lim_{n\to\infty}\frac{\log N_n(\widetilde{X}^b_{H,0})}{n\log b}=\frac12\log_b(\varphi(b)+1).$$

$\square$

4. Proof of Theorem 1.6

In this section we begin by proving the following general result and as an immediate corollary, we will prove Theorem 1.6.

Proposition 4.1. *Let* M_1 *be a nonempty subset of* $I\cap\mathbb{Z}_b$, *where* I *is an ideal of* K ($K=\mathbb{Z}_b$ *or* $\mathbb{Z}$). *Let* M_2 *be a nonempty subset of* $\mathbb{Z}_b$ *and* $\#M_1=m_1<\#M_2=m_2$. *If* $\dim_{\mathcal H}(X^b_{H,I})\leqslant\log_b m_1$, *then* H *is a Poincaré set.*

Proof. Suppose that H is not a Poincaré set. Then there exists $S \subseteq \mathbb{N}$ with $\bar{d}(S) > 0$ such that

$$(S - S) \cap H = \emptyset.$$

We define

$$A_{M_1,M_2}=\left\{x=\sum_{n=1}^{\infty} \frac{x_n}{b^n}\colon\ x_n\in\{0,1,\ldots,b-1\},\ x_n\in M_1 \text{ if } n\notin S \text{ and } x_n\in M_2 \text{ otherwise}\right\}.$$

Let $x \in A_{M_1,M_2}$ with $x = \sum\limits_{n=1}^{\infty} x_n/b^n$. If $n \in S$, then $n + h \notin S$ and thus $x_{n+h} \in I$. Since I is an ideal of K, $x_n x_{n+h} \in I$ for all $h \in H$. If $n \notin S$, then $x_n \in I$ and thus $x_n x_{n+h} \in I$ for all $h \in H$. Then $x_n x_{n+h} \in I$ for all $n \geqslant 1$, $h \in H$. Thus $x \in X^b_{H,I}$. Therefore $A_{M_1,M_2} \subseteq X^b_{H,I}$. Since $X^b_{H,I}$ is compact and T_b-invariant, it follows that

$$\dim_{\mathcal{H}}(X^b_{H,I}) = \overline{\dim}_{\mathcal{M}}(X^b_{H,I}) \geqslant \overline{\dim}_{\mathcal{M}}(A_{M_1,M_2})$$

by Lemma 2.3.

Moreover, since the set A_{M_1,M_2} hits exactly $m_2^{\sum_{k=1}^n \chi_S(k)} m_1^{n-\sum_{k=1}^n \chi_S(k)}$ b-adic intervals of generation n, we have

$$\overline{\dim}_{\mathcal{M}}(A_{M_1,M_2}) = \limsup_{n\to\infty} \frac{\log\left(m_2^{\sum_{k=1}^n \chi_S(k)} m_1^{n-\sum_{k=1}^n \chi_S(k)}\right)}{n\log b} = \log_b m_1 + \bar{d}(S)\log\frac{m_2}{m_1}.$$

From $\bar{d}(S) > 0$ and $\log m_2/m_1 > 0$, we know that $\dim_{\mathcal{H}}(X^b_{H,I}) \geqslant \overline{\dim}_{\mathcal{M}}(A_{M_1,M_2}) > \log_b m_1$. Then we get a contradiction. Thus H is a Poincaré set. $\square$

By Theorem 1.6 we immediately have the following two corollaries.

Corollary 4.2. *Let $K = \mathbb{Z}$, $I = k\mathbb{Z}$, where $k \in \mathbb{N}$ with $k \geqslant 2$. Then H is a Poincaré set if one of the following holds:*

(i) *$b \geqslant k$, $b = kj + i$ for $0 \leqslant i < k$, and $\dim_{\mathcal{H}}(X^b_{H,I}) \leqslant \log_b(j + 1)$,*

(ii) *$b < k$, and $\dim_{\mathcal{H}}(X^b_{H,I}) = 0$.*

Corollary 4.3. *Let $K = \mathbb{Z}$, and $I = \{0\}$. Then we have*

$$X^b_{H,I} = \left\{x = \sum_{n=1}^{\infty} \frac{x_n}{b^n}\colon\ x_n \in \{0, 1, \ldots, b-1\},\ x_n x_{n+h} = 0 \text{ for all } n \geqslant 1,\ h \in H\right\}.$$

Moreover, if $\dim_{\mathcal{H}}(X^b_{H,I}) = 0$, then H is a Poincaré set.

5. More examples

Through the following first example, we illustrate that our theory provides a new method to determine whether some sets are Poincaré sets or not. In the next example, the Hausdorff dimension of a more general and interesting class of sets defined by digit restrictions is computed.

Example 5.1. For any positive integers k, $b \geqslant 2$, let $H = \mathbb{N} \setminus k\mathbb{N}$ and let

$$X'^{b}_{H,0} := \Big\{x = \sum_{n=1}^{\infty} \frac{x_n}{b^n} \colon x_n \in \{0, 1, \dots, b-1\},\ x_n x_{n+h} = 0 \text{ for all } n \geqslant 1,\ h \in H\Big\}.$$

Then we have $\dim_{\mathcal{H}}(X'^{b}_{H,0}) = \dim_{\mathcal{M}}(X'^{b}_{H,0}) = 1/k$ and H is not a Poincaré set.

P r o o f. Define

$$f(n) = \begin{cases} b^{n/k-1}, & n \equiv 0 \pmod{k}, \\ b^{(n-1)/k}, & n \equiv 1 \pmod{k}, \\ b^{(n-2)/k}, & n \equiv 2 \pmod{k}, \\ \vdots & \\ b^{(n-(k-1))/k}, & n \equiv (k-1) \pmod{k}. \end{cases}$$

We decompose the points in $X'^{b}_{H,0}$ into two cases:

(i) Fix $x_1 \in \{1, 2, \dots, b-1\}$, then we have $x_m = 0$ if $m \not\equiv 1 \pmod{k}$ and $x_m \in \{0, 1, 2, \dots, b-1\}$ if $m \equiv 1 \pmod{k}$. So the number of distinct combining forms from x_1 to x_n is $f(n)$.

(ii) If $x_1 = 0$, then the number of distinct combining forms from x_1 to x_n is $N_{n-1}(X'^{b}_{H,0})$.

By (i) and (ii), we have $N_n(X'^{b}_{H,0}) = N_{n-1}(X'^{b}_{H,0}) + (b-1)f(n)$ $(n \geqslant 2)$ and $N_1(X'^{b}_{H,0}) = b$. Therefore, by a simple computation, we have

$$\dim_{\mathcal{M}}(X'^{b}_{H}) = \lim_{n\to\infty} \frac{\log N_n(X'^{b}_{H,0})}{n \log b} = \frac{1}{k}.$$

Then we have $\dim_{\mathcal{H}}(X'^{b}_{H,0}) = \dim_{\mathcal{M}}(X'^{b}_{H,0}) = 1/k$. Moreover, let b be a prime, then we know $y^2 \equiv 0 \pmod{b}$ has only one solution and $\dim_{\mathcal{H}}(X^{b}_{H,0}) \geqslant \dim_{\mathcal{H}}(X'^{b}_{H,0}) \geqslant 1/k$. Then, by Proposition 1.4, H is not a Poincaré set. $\square$

Remark 5.2. By the definition of Poincaré sets, we clearly have that any subset of a non Poincaré set is also a non Poincaré one. Then we have that $k\mathbb{N} - 1$, $k\mathbb{N} - 2, \dots, k\mathbb{N} - (k-1)$ are all not Poincaré sets.

Example 5.3. Let $H = 2\mathbb{N}$, $b = 4$, $K = \mathbb{Z}$ and $I = 2\mathbb{N}$. Then

$$X^b_{H,I} = \left\{ x = \sum_{n=1}^{\infty} \frac{x_n}{4^n} \colon x_n \in \{0,1,2,3\},\ x_n x_{n+h} \equiv 0 \pmod 2 \text{ for all } n \geqslant 1,\ h \in 2\mathbb{N} \right\}.$$

We have $\dim_{\mathcal{H}}(X^b_{H,I}) = \dim_{\mathcal{M}}(X^b_{H,I}) = \frac{1}{2}$.

Proof. We decompose the points in $X^b_{H,I}$ into two cases:

(i) Fix $x_1 \equiv 0 \pmod 2$, then the number of distinct combining forms from x_1 to x_n is $N_{n-1}(X^b_{H,I})$.

(ii) Fix $x_1 \equiv 1 \pmod 2$, then we have $x_{2k+1} \equiv 0 \pmod 2$ for all $k \in \mathbb{N}$. In this case, the number of distinct combining forms from x_1 to x_n is denoted by T_n. If $x_2 \equiv 1 \pmod 2$, then we have $x_{2k} \equiv 0 \pmod 2$ for all $k \in \mathbb{N}$, and the number of distinct combining forms from x_1 to x_n is 2^{n-1}. If $x_2 \equiv 0 \pmod 2$, then the number of distinct combining forms from x_1 to x_n is $2^2 T_{n-2}$.

By (i) and (ii), we have $N_n(X^b_{H,I}) = 2N_{n-1}(X^b_{H,I}) + 2T_n$ and $T_n = 2^{n-1} + 2^2 T_{n-2}$, where $N_1(X^b_{H,I}) = 4$, $T_1 = 1$ and $T_2 = 4$.

Then, by a simple computation, we have

$$2^{n-1}(n+2) + 2^{n+1} \leqslant N_n(X^b_{H,I}) \leqslant n2^{n-1}(n+2) + 2^{n+1}.$$

Thus

$$\dim_{\mathcal{M}}(X^b_{H,I}) = \lim_{n\to\infty} \frac{\log N_n(X^b_{H,I})}{n \log 4} = \frac{1}{2}.$$

Then we have

$$\dim_{\mathcal{H}}(X^b_{H,I}) = \dim_{\mathcal{M}}(X^b_{H,I}) = \frac{1}{2}.$$

$\square$

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