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SOLUTIONS TO CONJECTURES ON A NONLINEAR RECURSIVE EQUATION

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Abstract. We obtain solutions to some conjectures about the nonlinear difference equation

$$x_{n+1} = \alpha + \beta x_{n-1} \mathrm{e}^{-x_n}, \quad n = 0, 1, \ldots, \ \alpha, \beta > 0.$$

More precisely, we get not only a condition under which the equilibrium point of the above equation is globally asymptotically stable but also a condition under which the above equation has a unique positive cycle of prime period two. We also prove some further results.

Keywords: recursive equation; nonlinear difference equation; equilibrium point; stability

MSC 2020: 39A10, 39A21, 11B39

1. Introduction

In [1], the authors consider the nonlinear difference equation

$$x_{n+1} = \alpha + \beta x_{n-1} \mathrm{e}^{-x_n}, \quad n = 0, 1, \ldots, \tag{1.1}$$

with initial values x_{-1} and x_0. Because of the biological point of view, assume that $\alpha, \beta > 0$. Then we know that equation (1.1) has exactly one equilibrium point $\bar{x}$, and furthermore $\bar{x} > \alpha$. In [1], they obtained the following results:

Theorem 1.1. *A point $\bar{x}$ is locally asymptotically stable if*

$$\beta < \frac{-\alpha + \sqrt{\alpha^2 + 4\alpha}}{\alpha + \sqrt{\alpha^2 + 4\alpha}} \mathrm{e}^{(\alpha + \sqrt{\alpha^2 + 4\alpha})/2}$$

and is unstable (and in fact is a saddle point) if

$$\beta > \frac{-\alpha + \sqrt{\alpha^2 + 4\alpha}}{\alpha + \sqrt{\alpha^2 + 4\alpha}} \mathrm{e}^{(\alpha + \sqrt{\alpha^2 + 4\alpha})/2}.$$

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Theorem 1.2. *The following statements are true:*

(i) *Every positive solution of equation* (1.1) *is bounded if* $\beta < \mathrm{e}^{\alpha}$.

(ii) *Equation* (1.1) *has positive unbounded solutions if* $\beta > \mathrm{e}^{\alpha}$.

Theorem 1.3. *The equilibrium* $\bar{x}$ *of equation* (1.1) *is globally asymptotically stable if*

$$\beta \leqslant \frac{-\alpha + \sqrt{\alpha^2 + 4}}{2}\mathrm{e}^{\alpha}.$$

After the above results, the following conjectures and open problem have been stated in [1]:

Conjecture 1.4. *The equilibrium* $\bar{x}$ *of equation* (1.1) *is globally asymptotically stable if*

$$\frac{-\alpha + \sqrt{\alpha^2 + 4}}{2}\mathrm{e}^{\alpha} < \beta < \frac{-\alpha + \sqrt{\alpha^2 + 4\alpha}}{\alpha + \sqrt{\alpha^2 + 4\alpha}}\mathrm{e}^{(\alpha + \sqrt{\alpha^2 + 4\alpha})/2}. \tag{1.2}$$

Conjecture 1.5. *Equation* (1.1) *has a unique positive cycle of prime period two if*

$$\beta > \frac{-\alpha + \sqrt{\alpha^2 + 4\alpha}}{\alpha + \sqrt{\alpha^2 + 4\alpha}}\mathrm{e}^{(\alpha + \sqrt{\alpha^2 + 4\alpha})/2}. \tag{1.3}$$

Moreover, if $\beta < \mathrm{e}^{\alpha}$, *then every positive solution* $\{x_n\}_{n=-1}^{\infty}$ *of equation* (1.1), *which does not converge to* $\bar{x}$, *converges to the unique period* 2 *cycle.*

Open Problem. *Determine the boundedness character of the positive solution of equation* (1.1) *when* $\beta = \mathrm{e}^{\alpha}$.

We should note that Conjecture 1.5 has been partially solved by Fotiades and Papaschinopoulos in [2], Proposition 2.2 under the condition $\alpha\beta > 2(\beta - 1)$.

In the present paper, we prove that Conjectures 1.4 and 1.5 are always true. Hence we do not need the extra condition $\alpha\beta > 2(\beta - 1)$, which improves Proposition 2.2 in [2]. We also show that when $\beta = \mathrm{e}^{\alpha}$, equation (1.1) has no (bounded) periodic solution (or, period 2 solution), which partially solves the above open problem stated in [1]. At the end of the paper, we prove that the same situation is also valid for the case of $\beta > \mathrm{e}^{\alpha}$.

2. Auxiliary results

We first get the next result.

Lemma 2.1. *Positive solutions* $\{x_{2n}\}$ *and* $\{x_{2n+1}\}$ *of equation* (1.1) *are eventually monotonic, i.e., either non-decreasing or non-increasing for every sufficiently large* n.

Proof. From (1.1), we get

$$x_{2n+3} - x_{2n+1} = \beta \frac{e^{x_{2n}} x_{2n+1} - e^{x_{2n+2}} x_{2n-1}}{e^{x_{2n}} e^{x_{2n+2}}}$$

and

$$x_{2n+4} - x_{2n+2} = \beta \frac{e^{x_{2n+1}} x_{2n+2} - e^{x_{2n+3}} x_{2n}}{e^{x_{2n+1}} e^{x_{2n+3}}}.$$

Using these equations, we now consider the following possible cases:

(a) Let $x_{-1} \geqslant x_1$ and $x_0 \leqslant x_2$. Then we obtain that

$$x_3 - x_1 = \beta \frac{e^{x_0} x_1 - e^{x_2} x_{-1}}{e^{x_0} e^{x_2}} \leqslant \beta \frac{(e^{x_0} - e^{x_2}) x_{-1}}{e^{x_0} e^{x_2}} \leqslant 0 \ \Rightarrow \ x_3 \leqslant x_1,$$

and

$$x_4 - x_2 = \beta \frac{e^{x_1} x_2 - e^{x_3} x_0}{e^{x_1} e^{x_3}} \geqslant \beta \frac{e^{x_1} (x_2 - x_0)}{e^{x_1} e^{x_3}} \geqslant 0 \ \Rightarrow \ x_4 \geqslant x_2.$$

Applying the same procedure, we see that

$$x_{-1} \geqslant x_1 \geqslant x_3 \geqslant \ldots \quad \text{and} \quad x_0 \leqslant x_2 \leqslant x_4 \leqslant \ldots,$$

whence the result.

(b) Let $x_{-1} \leqslant x_1$ and $x_0 \geqslant x_2$. The proof follows easily as in (a). In this case, we get

$$x_{-1} \leqslant x_1 \leqslant x_3 \leqslant \ldots \quad \text{and} \quad x_0 \geqslant x_2 \geqslant x_4 \geqslant \ldots.$$

(c) Let $x_{-1} > x_1$ and $x_0 > x_2$. Now, define

$$N := \min\{n \in \mathbb{N} \colon \ x_n \leqslant x_{n+2}\}. \tag{2.1}$$

Then we have three possible cases:

▷ If there is no finite natural number N as in (2.1), then the proof is done.

▷ Assume that N is an odd integer. Then, from (2.1), it is clear that

$$\begin{aligned} &x_{-1} > x_1 > x_3 > \ldots > x_{N-2} > x_N, \\ &x_N \leqslant x_{N+2}, \\ &x_0 > x_2 > x_4 > \ldots > x_{N-1} > x_{N+1}. \end{aligned}$$

Hence, we may write that

$$x_{N+3} - x_{N+1} = \beta \frac{e^{x_N} x_{N+1} - e^{x_{N+2}} x_{N-1}}{e^{x_N} e^{x_{N+2}}} \leqslant \beta \frac{e^{x_{N+2}} (x_{N+1} - x_{N-1})}{e^{x_N} e^{x_{N+2}}} < 0,$$

which means

$$x_{N+1} > x_{N+3}. \tag{2.2}$$

Now using (2.3) and the fact that N is odd, we observe that

$$x_{N+4} - x_{N+2} = \beta \frac{e^{x_{N+1}} x_{N+2} - e^{x_{N+3}} x_N}{e^{x_{N+1}} e^{x_{N+3}}} \geqslant \beta \frac{x_N (e^{x_{N+1}} - e^{x_{N+3}})}{e^{x_{N+1}} e^{x_{N+3}}} > 0,$$

which gives

$$x_{N+2} < x_{N+4}. \tag{2.3}$$

Therefore, as in the cases (a) and (b), we conclude from (2.2) and (2.3) that

$$\begin{gathered} x_N \leqslant x_{N+2} < x_{N+4} < x_{N+6} < \ldots, \\ x_0 > x_2 > x_4 > \ldots > x_{N+1} > x_{N+3} > x_{N+5} > \ldots, \end{gathered}$$

whenever N is odd.

▷ If N is an even integer, then we arrive at the following situation:

$$\begin{gathered} x_N \leqslant x_{N+2} < x_{N+4} < x_{N+6} < \ldots, \\ x_{-1} > x_1 > x_3 > \ldots > x_{N+1} > x_{N+3} > x_{N+5} > \ldots. \end{gathered}$$

(d) Finally, let $x_{-1} < x_1$ and $x_0 < x_2$. This is the symmetric case of (c). □

Due to Lemma 2.1, one can say that $\{x_n\}$ is a positive unbounded solution of equation (1.1) if and only if

$$\text{either} \quad \lim_{n\to\infty} x_{2n} = \infty \quad \text{or} \quad \lim_{n\to\infty} x_{2n+1} = \infty,$$

which is equivalent to

$$\text{either} \quad \lim_{n\to\infty} x_{2n+1} = \alpha \quad \text{or} \quad \lim_{n\to\infty} x_{2n} = \alpha.$$

Hence, if $\{x_n\}$ is a positive bounded solution of equation (1.1), there exist $x, y \in (\alpha, \infty)$ such that

$$\lim_{n\to\infty} x_{2n} = x \quad \text{and} \quad \lim_{n\to\infty} x_{2n+1} = y. \tag{2.4}$$

Then we get the next lemma.

Lemma 2.2. *Let* $\{x_n\}$ *be a positive bounded solution of equation* (1.1), *and let* $x, y \in (\alpha, \infty)$ *as in* (2.4). *Then, the following statements hold:*

(i) $x = \bar{x} \Leftrightarrow y = \bar{x}$,

(ii) $x < \bar{x} \Leftrightarrow y > \bar{x}$,

(iii) $x > \bar{x} \Leftrightarrow y < \bar{x}$.

Proof. We observe from (1.1) and (2.4) that $x = \alpha + \beta x \mathrm{e}^{-y}$ and $y = \alpha + \beta y \mathrm{e}^{-x}$. Hence we get

$$y = \ln \frac{\beta x}{x - \alpha}. \tag{2.5}$$

Since $\ln(\beta\bar{x}/(\bar{x} - \alpha)) = \bar{x}$ and $\bar{x}$ is unique, the statement (i) follows from (2.5), immediately.

Since the function $f(x) = \ln(\beta x/(x - \alpha))$ is strictly decreasing on (α, ∞), we can say that $\alpha < x < \bar{x}$ if and only if $f(x) > f(\bar{x}) = \bar{x}$, which gives $y > \bar{x}$. Hence, the proof of (ii) is done.

Similarly, replacing x by y, we see that $\alpha < y < \bar{x}$ if and only if $f(y) > f(\bar{x}) = \bar{x}$, which implies $x > \bar{x}$. $\square$

Observe that $\alpha\beta > 2(\beta - 1)$ for any $\alpha \geqslant 2$.

Now define the following functions on the interval $(0, 2)$:

$$f(\alpha) = \frac{2}{2 - \alpha}, \quad g(\alpha) = \frac{-\alpha + \sqrt{\alpha^2 + 4\alpha}}{\alpha + \sqrt{\alpha^2 + 4\alpha}} \mathrm{e}^{(\alpha + \sqrt{\alpha^2 + 4\alpha})/2},$$
$$h(\alpha) = \frac{-\alpha + \sqrt{\alpha^2 + 4}}{2} \mathrm{e}^{\alpha}.$$

From the definitions of f and g we first obtain that

$$f\Big(\frac{1}{2}\Big) = \frac{4}{3} \approx 1.3333 \quad \text{and} \quad g\Big(\frac{1}{2}\Big) = \frac{\mathrm{e}}{2} \approx 1.3591,$$
$$f(1) = 2 \quad \text{and} \quad g(1) = \frac{3 - \sqrt{5}}{2} \mathrm{e}^{(1+\sqrt{5})/2} \approx 1.9263.$$

Since $f(\frac{1}{2}) < g(\frac{1}{2})$ and $f(1) > g(1)$, the equation $f(\alpha) = g(\alpha)$ has at least one solution in the interval $(\frac{1}{2}, 1)$. On the other hand, we also get the following facts for every $\alpha \in (0, 2)$:

$$f(0^+) = g(0^+) = h(0^+) = 1,$$
$$f(2^-) = \infty > g(2^-) = \frac{1}{2 + \sqrt{3}} \mathrm{e}^{1+\sqrt{3}} > h(2^-) = \big(-1 + \sqrt{2}\big)\mathrm{e}^2,$$
$$f'(\alpha) > 0, \quad g'(\alpha) > 0, \quad h'(\alpha) > 0, \quad f''(\alpha) > 0, \quad g''(\alpha) > 0, \quad h''(\alpha) > 0,$$
$$h(\alpha) < f(\alpha) \quad \text{and} \quad h(\alpha) < g(\alpha).$$

The above observations imply that the functions f and g are strictly increasing and convex on $(0, 2)$. Hence, the equation $f(\alpha) = g(\alpha)$ has exactly one solution in the interval $(0, 2)$, say α_0. It follows from a numerical approximation that

$$\alpha_0 \approx 0.81464. \tag{2.6}$$

Thus, we get the next lemma.

Lemma 2.3.

(i) *Condition* (1.2) *implies* $\beta > 1$.

(ii) *If* $\alpha > \alpha_0$, *then condition* (1.2) *implies*

$$\alpha\beta > 2(\beta - 1). \tag{2.7}$$

Now using α_0 in (2.6) we consider the following functions defined on the interval $[0, \alpha_0]$:

$$u(t) = -2\mathrm{e}^t(t-1)(t-2), \quad v(t) = t^2 + 2t - 4.$$

Then we get

$$u'(t) = -2\mathrm{e}^t(t^2 - t - 1), \quad u''(t) = -2\mathrm{e}^t(t^2 + t - 2), \quad v'(t) = 2t + 2, \quad v''(t) = 2$$

and

$$u(0) = v(0) = -4, \quad u'(0) = v'(0) = 2.$$

On the other hand, defining the function $\varphi := u''$ on the interval $[0, \alpha_0]$, we see that

$$\varphi'(t) = -2\mathrm{e}^t(t^2 + 3t - 1) \quad \text{and} \quad \varphi''(t) = -2\mathrm{e}^t(t^2 + 5t + 2).$$

Hence, φ has exactly one critical point t_0 in $[0, \alpha_0]$, where $t_0 = \frac{1}{2}(-3+\sqrt{13}) \approx 0.3028$. Then we observe that φ has a maximum at the critical point t_0. Also, since $\varphi(0) = 4 > \varphi(\alpha_0) \approx 2.3565$, we obtain that the function $\varphi = u''$ has the minimum value at the point $t = \alpha_0$. Then we get, for every $t \in [0, \alpha_0]$, that

$$u''(t) \geqslant u''(\alpha_0) \approx 2.3565 > v''(t).$$

So, integrating twice the last inequality on the interval $[0, \alpha_0]$, we immediately see that

$$u(\alpha) > v(\alpha) \quad \text{for every } \alpha \in (0, \alpha_0]. \tag{2.8}$$

3. Proofs of Conjectures 1.4 and 1.5

Now we are ready to give our main result, which answers Conjecture 1.4.

Theorem 3.1. *The equilibrium solution* $\bar{x}$ *of equation* (1.1) *is globally asymptotically stable if* (1.2) *holds.*

Proof. Assume that (1.2) holds. Due to Theorems 1.1 and 1.3 it is enough to show that $\bar{x}$ is a global attractor of equation (1.1). Now, using the fact that

$$\frac{-\alpha+\sqrt{\alpha^2+4\alpha}}{\alpha+\sqrt{\alpha^2+4\alpha}}\mathrm{e}^{(\alpha+\sqrt{\alpha^2+4\alpha})/2}<\mathrm{e}^{\alpha},$$

we get $\beta<\mathrm{e}^{\alpha}$ and hence from (1.2) and Theorem 1.2 (i) we observe that every solution of equation (1.1) is bounded. Now let $\{x_n\}$ be any bounded solution of equation (1.1). Then Lemma 2.1 implies that there exist $x, y \in (\alpha,\infty)$ such that (2.4) holds. Hence we get from equation (1.1) that

$$x=\alpha+\beta x\mathrm{e}^{-y} \quad \text{and} \quad y=\alpha+\beta y\mathrm{e}^{-x}. \tag{3.1}$$

If we show that $x=y$, the proof is completed. Observe that, by Lemma 2.2 (i), the case of $x=\bar{x}$ is clear. To get a contradiction assume now that $x\neq\bar{x}$, which is equivalent to $x\neq y$. From the equation (3.1), since

$$x=\frac{\alpha}{1-\beta\mathrm{e}^{-y}} \quad \text{and} \quad y=\frac{\alpha}{1-\beta\mathrm{e}^{-x}},$$

we get

$$x=\alpha+\beta x\mathrm{e}^{-\alpha/(1-\beta\mathrm{e}^{-x})} \quad \text{and} \quad y=\alpha+\beta y\mathrm{e}^{-\alpha/(1-\beta\mathrm{e}^{-y})}.$$

Now consider the function:

$$F(z)=\alpha+\beta z\mathrm{e}^{-\alpha/(1-\beta\mathrm{e}^{-z})}-z, \quad z\in(\alpha,\infty). \tag{3.2}$$

We first claim that

$$F'(\bar{x})<0. \tag{3.3}$$

Indeed, since

$$F'(z)=\beta\mathrm{e}^{-\alpha/(1-\beta\mathrm{e}^{-z})}+\frac{\alpha\beta^2z\mathrm{e}^{-z}}{(1-\beta\mathrm{e}^{-z})^2}\mathrm{e}^{-\alpha/(1-\beta\mathrm{e}^{-z})}-1$$

and

$$\bar{x} = \frac{\alpha}{1 - \beta e^{-\bar{x}}}, \tag{3.4}$$

we may write that

$$F'(\bar{x}) = \beta e^{-\bar{x}} + \frac{\beta^2}{\alpha}\bar{x}^3 e^{-2\bar{x}} - 1 = \frac{\beta^2}{\alpha}\bar{x}^3 e^{-2\bar{x}} - \frac{\alpha}{\bar{x}}.$$

Hence, our claim is true if and only if

$$\beta \bar{x}^2 e^{-\bar{x}} < \alpha. \tag{3.5}$$

As in the proof of Proposition 2.1 in [2], define the function h as follows:

$$h(x) = \frac{x - \alpha}{x} e^{x}, \quad x > \alpha.$$

Then we know that the function h is strictly increasing on (α, ∞). Also, we get from (3.4) that

$$\beta = \frac{\bar{x} - \alpha}{\bar{x}} e^{\bar{x}}, \tag{3.6}$$

which implies

$$h(\bar{x}) = \beta.$$

From the hypothesis (1.2), we observe that

$$h(\bar{x}) = \beta < \frac{-\alpha + \sqrt{\alpha^2 + 4\alpha}}{\alpha + \sqrt{\alpha^2 + 4\alpha}} e^{(\alpha + \sqrt{\alpha^2 + 4\alpha})/2} = h\left(\frac{\alpha + \sqrt{\alpha^2 + 4\alpha}}{2}\right),$$

which gives

$$\bar{x} < \frac{\alpha + \sqrt{\alpha^2 + 4\alpha}}{2}.$$

From the last inequality, we immediately see that

$$\bar{x} - \alpha < \frac{-\alpha + \sqrt{\alpha^2 + 4\alpha}}{2},$$

which yields

$$\bar{x}(\bar{x} - \alpha) < \alpha. \tag{3.7}$$

Then, using (3.6) and (3.7), we get that (3.5) holds, which corrects our claim (3.3). Now, from the definition of the function F in (3.2), we observe that x, y and $\bar{x}$ are solutions of the equation $F(z) = 0$. Assuming $x < \bar{x}$, we immediately get from Lemma 2.2 (ii) that $y > \bar{x}$. Now, since

$$F(x) = F(\bar{x}) = 0, \quad F(\alpha) > 0 \quad \text{and} \quad F'(\bar{x}) < 0,$$

the graph of F in the interval $(\alpha, \bar{x})$ becomes at least once concave up and once concave down, respectively (or has more fluctuations). Similarly, since

$$F(y) = F(\bar{x}) = 0, \quad F'(\bar{x}) < 0 \quad \text{and} \quad \lim_{z\to\infty} F(z) = -\infty,$$

the graph of F in the interval $(\bar{x}, \infty)$ becomes at least once concave up and once concave down, respectively (or has more fluctuations). Therefore, the graph of F in the interval (α, ∞) becomes at least once concave up, concave down, concave up and concave down, respectively (or has more fluctuations), which implies that the equation $F''(z) = 0$ has at least three solutions in (α, ∞).

Now we complete the proof by investigating two main parts for $\alpha_0 \approx 0.81464$ given by (2.6):

Case (I): Let $\alpha > \alpha_0$. Then from Lemma 2.3 we get the condition (2.7). Now using the same idea as in the proof of Proposition 2.2 in [2], we see that $F''(z) = 0$ has exactly one solution in (α, ∞) under the condition (2.7), which gives a contradiction.

Case (II): Let $0 < \alpha \leqslant \alpha_0$. By an argument similar to the case (I), if condition (2.7) holds, the proof follows immediately. Otherwise, we get

$$\frac{2}{2-\alpha} \leqslant \beta < \frac{-\alpha + \sqrt{\alpha^2 + 4\alpha}}{\alpha + \sqrt{\alpha^2 + 4\alpha}} \mathrm{e}^{(\alpha + \sqrt{\alpha^2 + 4\alpha})/2}. \tag{3.8}$$

We know from the proof of Proposition 2.2 in [2] that

$$F''(z) = -\frac{\alpha\beta^2 \mathrm{e}^{-3z} \mathrm{e}^{-\alpha/(1-\beta \mathrm{e}^{-z})}}{(1-\beta \mathrm{e}^{-z})^4} \cdot G(z) \quad \text{for } z > \alpha, \tag{3.9}$$

where

$$G(z) = \mathrm{e}^{2z}(z-2) - \beta \mathrm{e}^{z}(\alpha z - 4) - \beta^2 (z+2). \tag{3.10}$$

We can define the above function G on $[\alpha, \infty)$. Then we get, for $z \geqslant \alpha$, that

$$\begin{aligned} G'(z) &= \mathrm{e}^{2z}(2z-3) - \beta \mathrm{e}^{z}(\alpha z + \alpha - 4) - \beta^2 \\ G''(z) &= \mathrm{e}^{z}\{4z\mathrm{e}^{z}(z-1) - \mathrm{e}^{z}(\alpha z + 2\alpha - 4)\}. \end{aligned}$$

We now consider the function H on $[\alpha, \infty)$ defined by

$$H(z) = 4\mathrm{e}^z(z-1) - \beta(\alpha z + 2\alpha - 4). \tag{3.11}$$

Then we see that

$$H'(z) = 4z\mathrm{e}^z - \alpha\beta.$$

The function H' is strictly increasing in $[\alpha, \infty)$. Since $H'(\alpha) = 4\alpha\mathrm{e}^\alpha - \alpha\beta > 0$ and $\lim_{z\to\infty} H'(z) = \infty$, the function H is strictly increasing in $[\alpha, \infty)$. Now we observe from the inequality (2.8) that

$$\frac{4\mathrm{e}^\alpha(\alpha-1)}{\alpha^2+2\alpha-4} < \frac{2}{2-\alpha}$$

for every $\alpha \in (0, \alpha_0]$. Using the last inequality and also considering (3.8) we see that

$$\beta > \frac{4\mathrm{e}^\alpha(\alpha-1)}{\alpha^2+2\alpha-4},$$

which yields

$$H(\alpha) = 4\mathrm{e}^\alpha(\alpha-1) - \beta(\alpha^2+2\alpha-4) > 0 \quad \text{for every } \alpha \in (0, \alpha_0].$$

Hence, $H(z) > 0$ for every $z \in [\alpha, \infty)$. Since $G''(z) = \mathrm{e}^z H(z)$, we have $G''(z) > 0$ for $z \in [\alpha, \infty)$. Then the function G' is strictly increasing in $[\alpha, \infty)$. In this case, we obtain that $G'(z) = 0$ has at most one root in (α, ∞). Therefore, $F''(z) = 0$ has at most two solutions in (α, ∞). This is a contradiction. $\square$

Remark 3.2. Assume that

$$\beta = \frac{-\alpha + \sqrt{\alpha^2+4\alpha}}{\alpha + \sqrt{\alpha^2+4\alpha}}\mathrm{e}^{(\alpha+\sqrt{\alpha^2+4\alpha})/2} \tag{3.12}$$

holds. Then we claim that equation (1.1) has no periodic solution, and hence every solution converges to the equilibrium point $\bar{x}$. Indeed otherwise, $F''(z) = 0$ would have at least 3 solutions in (α, ∞) since $F(\alpha) > 0$, $\lim_{z\to\infty} F(z) = -\infty$, $F(\bar{x}) = F'(\bar{x}) = 0$ due to (3.12). However, as in the cases (I) and (II) in the proof of Theorem 3.1, we see that $F''(z) = 0$ has at most 2 solutions in (α, ∞), which is a contradiction.

Now we prove Conjecture 1.5, which also improves Proposition 2.2 in [2] since we do not need the extra condition (2.7).

Theorem 3.3. *Every positive solution of equation* (1.1) *except for the equilibrium solution converges to the unique period* 2 *cycle if*

$$\frac{-\alpha+\sqrt{\alpha^2+4\alpha}}{\alpha+\sqrt{\alpha^2+4\alpha}}\mathrm{e}^{(\alpha+\sqrt{\alpha^2+4\alpha})/2}<\beta<\mathrm{e}^{\alpha} \tag{3.13}$$

holds.

Proof. Let (x_n) be a positive solution which does not converge to $\bar{x}$. Then we know from Proposition 2.1 in [2] that if (3.13) holds, then equation (1.1) has a positive periodic solution of prime period two. In this case, (x_n) converges to this periodic solution. Hence, we complete the proof if we show that this periodic solution is unique. Indeed, if equation (1.1) had at least two positive periodic solutions of prime period two, say (x_n) and (x'_n), then there would exist $x, y, x', y' \in (\alpha,\infty)$ such that

$$\lim_{n\to\infty} x_{2n}=x \quad \text{and} \quad \lim_{n\to\infty} x_{2n+1}=y,$$
$$\lim_{n\to\infty} x'_{2n}=x' \quad \text{and} \quad \lim_{n\to\infty} x'_{2n+1}=y'.$$

Assuming $x<x'<\bar{x}$, we get $\bar{x}<y'<y$. Now consider the function

$$F(z)=\alpha+\beta z\mathrm{e}^{-\alpha/(1-\beta\mathrm{e}^{-z})}-z, \quad z\in(\alpha,\infty).$$

Since

$$F(x)=F(x')=F(\bar{x})=0, \quad F(\alpha)>0 \quad \text{and} \quad F'(\bar{x})>0,$$

the graph of F in the interval $(\alpha,\bar{x})$ becomes at least once concave up, concave down, concave up, respectively (or has more fluctuations). Similarly, since

$$F(y)=F(y')=F(\bar{x})=0, \quad F'(\bar{x})>0 \quad \text{and} \quad \lim_{z\to\infty} F(z)=-\infty,$$

the graph of F in the interval $(\bar{x},\infty)$ becomes at least once concave down, concave up and concave down, respectively (or has more fluctuations). Therefore, the equation $F''(z)=0$ has at least 5 solutions in (α,∞). However, using an idea similar to in Theorem 3.1, we observe that since $H'(\alpha)>0$ and $\lim_{z\to\infty} H'(z)=\infty$, the function H is strictly increasing in $[\alpha,\infty)$. Hence, $H(z)=0$ has at most one solution in (α,∞), which implies $G'(z)=0$ has at most 2 solutions in (α,∞). But in this case we obtain that $G(z)=0$ has at most 3 solutions in (α,∞). These observations yield that, in all possible cases, $F''(z)=0$ has at most 3 solutions in (α,∞). This contradiction proves that equation (1.1) has always a unique positive periodic solution of prime period two if (3.13) holds. Hence, in this case, the condition $\alpha\beta>2(\beta-1)$ in Proposition 2.2 in [2] is redundant. $\square$

4. Some further results

Now we study the case of $\beta = \mathrm{e}^{\alpha}$, which partially answers the open problem stated in [1].

Theorem 4.1. *If* $\beta = \mathrm{e}^{\alpha}$, *then equation* (1.1) *has no (bounded) periodic solution.*

P r o o f. Let $\beta = \mathrm{e}^{\alpha}$. To get a contradiction assume that there exists a (bounded) periodic solution of equation (1.1) except for the equilibrium solution. In this case, by the monotonicity, the prime period must be 2. Then there exist $x, y \in (\alpha, \infty)$ such that

$$\lim x_{2n} = x \quad \text{and} \quad \lim x_{2n+1} = y.$$

Hence, the function

$$F(z) = \alpha + z\mathrm{e}^{\alpha(1-1/(1-\mathrm{e}^{\alpha-z}))} - z \quad \text{for } z > \alpha,$$

has at least one root in $(\alpha, \bar{x})$ and one root in $(\bar{x}, \infty)$. We first observe that

$$F(\bar{x}) = 0, \quad F'(\bar{x}) > 0, \quad F(\alpha^{+}) = 0, \quad F'(\alpha^{+}) = -1, \quad \lim_{z\to\infty} F(z) = \alpha > 0.$$

Therefore, $F''(z) = 0$ has at least 4 solutions in (α, ∞). On the other hand, putting $\beta = \mathrm{e}^{\alpha}$ in (3.9), (3.10) and (3.11), we obtain that

$$F''(z) = -\frac{\alpha\mathrm{e}^{2\alpha-3z}\mathrm{e}^{-\alpha/(1-\mathrm{e}^{\alpha-z})}}{(1-\mathrm{e}^{\alpha-z})^{4}} \cdot G(z) \quad \text{for } z > \alpha,$$

where

$$G(z) := \mathrm{e}^{2z}(z-2) - \mathrm{e}^{\alpha+z}(\alpha z - 4) - \mathrm{e}^{2\alpha}(z+2).$$

We again consider the function G on $[\alpha, \infty)$. Then, for $z \geqslant \alpha$,

$$G'(z) = \mathrm{e}^{2z}(2z-3) - \mathrm{e}^{\alpha+z}(\alpha z + \alpha - 4) - \mathrm{e}^{2\alpha}, \quad G''(z) = \mathrm{e}^{z}H(z),$$

where

$$H(z) := 4\mathrm{e}^{z}(z-1) - \mathrm{e}^{\alpha}(\alpha z + 2\alpha - 4).$$

Then we see that

$$H'(z) = 4z\mathrm{e}^{z} - \alpha\mathrm{e}^{\alpha}.$$

The function H' is strictly increasing in $[\alpha, \infty)$. Since $H'(\alpha) = 3\alpha\mathrm{e}^{\alpha} > 0$ and $\lim\limits_{z\to\infty} H'(z) = \infty$, the function H is strictly increasing in $[\alpha, \infty)$. Observe that

$$H(\alpha) = \alpha\mathrm{e}^{\alpha}(2-\alpha).$$

We have two possible cases:

(a) The case of $\alpha \geqslant 2$. In this case, condition (2.7) holds. Then, from Proposition 2.2 in [2], we see that $F''(z) = 0$ has exactly one solution in (α, ∞) under the condition (2.7), which gives a contradiction.

(b) The case of $0 < \alpha < 2$. In this case we get $H(\alpha) > 0$. Hence $H(z) > 0$ for every $z \in [\alpha, \infty)$. Since $G''(z) = \mathrm{e}^z H(z)$, we have $G''(z) > 0$ for $z \in [\alpha, \infty)$. Then the function G' is strictly increasing in $[\alpha, \infty)$. So, we obtain that $G'(z) = 0$ has at most one root in (α, ∞). Also, since $G(\alpha) = -\alpha^2 \mathrm{e}^{2\alpha} < 0$, we get that $F''(z) = 0$ has exactly one solution in (α, ∞). This is a contradiction. $\square$

The following result shows that a similar situation in Theorem 4.1 is also valid for $\beta > \mathrm{e}^{\alpha}$.

Theorem 4.2. *If $\beta > \mathrm{e}^{\alpha}$, then equation* (1.1) *has no (bounded) periodic solution.*

Proof. Let $\beta > \mathrm{e}^{\alpha}$. As in the proof of Theorem 4.1, to get a contradiction assume that there exists a (bounded) periodic solution of equation (1.1) except for the equilibrium solution. We again consider the following function:

$$F(z) = \alpha + \beta z \mathrm{e}^{-\alpha/(1-\beta \mathrm{e}^{-z})} - z \quad \text{for } z > \alpha \text{ and } z \neq \ln \beta.$$

We know that

$$F(\bar{x}) = 0, \quad F'(\bar{x}) > 0, \quad F(\ln \beta^-) = +\infty, \quad F(\ln \beta^+) = \alpha - \ln \beta < 0$$
$$\lim_{z \to \infty} F(z) = +\infty.$$

It is easy to check that $\bar{x} > \ln \beta$. We first observe that if

$$z \in (\alpha, \ln \beta) \;\Rightarrow\; 1 - \beta \mathrm{e}^{-\alpha} < 1 - \beta \mathrm{e}^{-z} < 0 \;\Rightarrow\; -\frac{\alpha}{1 - \beta \mathrm{e}^{-z}} > 0$$
$$\Rightarrow\; \mathrm{e}^{-\alpha/(1-\beta \mathrm{e}^{-z})} > 1 \;\Rightarrow\; \beta \mathrm{e}^{-\alpha/(1-\beta \mathrm{e}^{-z})} > \beta > 1.$$

Since

$$F'(z) = \beta \mathrm{e}^{-\alpha/(1-\beta \mathrm{e}^{-z})} + \frac{\alpha \beta^2 z \mathrm{e}^{-z}}{(1 - \beta \mathrm{e}^{-z})^2} \mathrm{e}^{-\alpha/(1-\beta \mathrm{e}^{-z})} - 1,$$

we see that $F'(z) > 0$ for every $\alpha < z < \ln \beta$. Now using the fact that $F(\alpha) > 0$, the equation $F(z) = 0$ has no solution in the interval $(\alpha, \ln \beta)$ since F is strictly increasing in $(\alpha, \ln \beta)$. If equation (1.1) has a periodic solution, then we immediately see that $F(z) = 0$ has at least one root in $(\ln \beta, \bar{x})$ and one root in $(\bar{x}, \infty)$. This means that $F''(z) = 0$ has at least 3 solutions in $(\ln \beta, \infty)$. Now, for $z \geqslant \ln \beta$, we consider the functions G and H defined in (3.10) and (3.11), respectively. Then, we know that

$$F''(z) = -\frac{\alpha\beta^2 \mathrm{e}^{-3z}\mathrm{e}^{-\alpha/(1-\beta\mathrm{e}^{-z})}}{(1-\beta\mathrm{e}^{-z})^4}\cdot G(z), \quad G''(z) = \mathrm{e}^z H(z).$$

Then using the cases (I) and (II) in the proof of Theorem 3.1, one can say that $F''(z) = 0$ has at most 2 solutions in $(\ln\beta, \infty)$. This contradiction completes the proof. □

Finally, combining all results we get the following characterization for positive solutions of equation (1.1).

Corollary 4.3. *Every positive solution of equation* (1.1) *except for the equilibrium solution converges to the unique period* 2 *cycle if and only if* (3.13) *holds.*

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