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THE p-NILPOTENCY OF FINITE GROUPS WITH SOME WEAKLY PRONORMAL SUBGROUPS

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Abstract. For a finite group G and a fixed Sylow p-subgroup P of G, Ballester-Bolinches and Guo proved in 2000 that G is p-nilpotent if every element of $P \cap G'$ with order p lies in the center of $N_G(P)$ and when $p = 2$, either every element of $P \cap G'$ with order 4 lies in the center of $N_G(P)$ or P is quaternion-free and $N_G(P)$ is 2-nilpotent. Asaad introduced weakly pronormal subgroup of G in 2014 and proved that G is p-nilpotent if every element of P with order p is weakly pronormal in G and when $p = 2$, every element of P with order 4 is also weakly pronormal in G. These results generalized famous Itô's Lemma. We are motivated to generalize Ballester-Bolinches and Guo's Theorem and Asaad's Theorem. It is proved that if p is the smallest prime dividing the order of a group G and P, a Sylow p-subgroup of G, then G is p-nilpotent if G is S_4-free and every subgroup of order p in $P \cap P^x \cap G^{\mathfrak{N}_\mathfrak{p}}$ is weakly pronormal in $N_G(P)$ for all $x \in G \setminus N_G(P)$, and when $p = 2$, P is quaternion-free, where $G^{\mathfrak{N}_\mathfrak{p}}$ is the p-nilpotent residual of G.

Keywords: weakly pronormal subgroup; normalizer; minimal subgroup; formation; p-nilpotency

MSC 2020: 20D10, 20D20

1. Introduction

All considered groups are finite groups. Recall that a formation $\mathfrak{F}$ is a class of groups which is closed under taking epimorphic images and such that every group G has a smallest normal subgroup with quotient in $\mathfrak{F}$. This subgroup is called the $\mathfrak{F}$-residual of G and it is denoted by $G^{\mathfrak{F}}$. Throughout this paper, $\mathfrak{N}_\mathfrak{p}$ and $\mathfrak{N}$ will

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denote the classes of p-nilpotent groups and nilpotent groups, respectively. A 2-group is called quaternion-free if it has no section isomorphic to the quaternion group of order 8. The exponent of a group G will be denoted by $\exp(G)$. If G is a p-group, then $\Omega_n(G) = \langle x \in G\colon\ x^{p^n} = 1\rangle$.

A subgroup H of a group G is called pronormal in G if for each $g \in G$, the subgroups H and H^g are conjugate in $\langle H, H^g\rangle$. The pronormality is one of the most significant properties pertaining to subgroups of groups and has been studied widely, for example, see [1], [3], [5], [6], [9], [19], [20], [21]. Recently, Asaad introduced the following concept:

Definition 1.1 ([1], Definition 1.1)**.** A subgroup H of a group G is called weakly pronormal in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is pronormal in G.

He gave examples to show that the above definition is a generalization of pronormality and proved the following theorem:

Theorem 1.2 ([1], Lemma 2.4)**.** *Let p be a prime dividing the order of a group G and let P be a Sylow p-subgroup of G. If every element of P with order p is weakly pronormal in G and when $p = 2$, every element of P with order 4 is also weakly pronormal in G, then G is p-nilpotent.*

The above result generalized the following famous lemma for p-nilpotence given by Itô.

Itô's Lemma ([17])**.** *Let p be a prime dividing the order of a group G. If every element of G of order p lies in $Z(G)$ and when $p = 2$, every element of G of order 4 also lies in $Z(G)$, then G is p-nilpotent.*

In 2000, Ballester-Bolinches and Guo obtained the following nice result, which extends Itô's result in a different way:

Theorem 1.3 ([7], Theorems 1 and 2)**.** *Let p be a prime dividing the order of a group G and let P be a Sylow p-subgroup of G. If every element of $P \cap G'$ with order p lies in the center of $N_G(P)$ and when $p = 2$, either every element of $P \cap G'$ with order 4 lies in the center of $N_G(P)$ or P is quaternion-free and $N_G(P)$ is 2-nilpotent, then G is p-nilpotent, where G' is the commutator subgroup of G.*

These results have been generalized in several papers such as [8], [13], [14], [15], [16], [18], [23]. In this paper we continue on this topic and prove the following main theorem:

Main Theorem. *Let G be an S_4-free group. Also let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every minimal subgroup of $P \cap P^x \cap G^{\mathfrak{N}_p}$ is weakly pronormal in $N_G(P)$ for all $x \in G \setminus N_G(P)$ and when $p = 2$, P is quaternion-free, then G is p-nilpotent, where $G^{\mathfrak{N}_p}$ is the p-nilpotent residual of G.*

Remark.

(1) Since $P \cap P^x \cap G^{\mathfrak{N}_p} \leqslant P \cap G'$, the conditions of Main Theorem are weaker than those of Theorems 1.2 and 1.3.

(2) It can be easily seen that the hypothesis that G is S_4-free and P is quaternion-free in the Main Theorem cannot be removed. For example, let $G = S_4$, the symmetric group of degree 4, and P a Sylow 2-subgroup of G. Then P is a dihedral group of order 8 and $N_G(P) = P$. We have that every minimal subgroup of $P \cap P^x \cap G^{\mathfrak{N}_p}$ is weakly pronormal in $N_G(P)$, but G is not 2-nilpotent. If we set $G = SL(2,3)$, then the Sylow 2-subgroup P of G is the quaternion group of order 8 and $G^{\mathfrak{N}_p} = P$. Thus, every minimal subgroup of $P \cap P^x \cap G^{\mathfrak{N}_p}$ is normal in $N_G(P) = G$ and therefore it is weakly pronormal in $N_G(P)$. However, G is not 2-nilpotent.

2. Preliminaries

In this section we show some lemmas, which are required in the proofs of our main results.

Lemma 2.1 ([1], Lemma 2.2). *Let H be a weakly pronormal subgroup of a group G. Then the following statements are true:*

(a) *H is weakly pronormal subgroup in K for every subgroup K of G with $H \leqslant K$.*

(b) *Let N be a normal subgroup of G. Then HN/N is weakly pronormal in G/N if $N \leqslant H$ or $(|H|, |N|) = 1$.*

Lemma 2.2. *Let P be a p-subgroup of a group G and N a normal p'-subgroup of G for a prime p. If every minimal subgroup of P is weakly pronormal in $N_G(P)$, then every minimal subgroup of PN/N is weakly pronormal in $N_{G/N}(PN/N)$.*

Proof. Suppose that AN/N is a minimal subgroup of PN/N with $A \leqslant P$. By hypotheses, there exists a subgroup K of $N_G(P)$ such that $N_G(P) = AK$ and $A \cap K$ is pronormal in $N_G(P)$. It is clear that $N_G(P)N/N = AKN/N = AN/N \cdot KN/N$. The minimality of AN/N implies that $AN/N \cap KN/N$ is trivial or equal to AN/N. We only need to consider the latter case. Clearly, $A \leqslant KN$ and hence $N_G(P)N/N =$

KN/N. If $A \nleq K$, then $A \cap K = 1$. It follows that $|P| \nmid |KN/N| = |N_G(P)N/N|$, this contradiction forces that $A \leqslant K$ and so A is pronormal in $N_G(P)$. Let $g = hn \in N_G(P)N$ with $h \in N_G(P)$ and $n \in N$. Since A is pronormal in $N_G(P)$, we have that $A^h = A^x$ with $x \in J = \langle A, A^h\rangle$. Thus, $(AN)^g = (AN)^{hn} = (AN)^{xn}$ and $xn \in JN = \langle AN, (AN)^h\rangle = \langle AN, (AN)^g\rangle$. Consequently, AN is pronormal in $N_G(P)N$ and it follows directly that AN/N is pronormal in $N_{G/N}(PN/N) = N_G(P)N/N$. Hence, AN/N is weakly pronormal in $N_{G/N}(PN/N)$. $\square$

Lemma 2.3 ([10], Lemma 6.3)**.** *If a subgroup H of a group G is both pronormal and subnormal in G, then H is normal in G.*

Lemma 2.4. *Let P be a Sylow p-subgroup of a group G and H a normal subgroup of G. If N is a normal p'-subgroup of G, then for any $x \in G \setminus N_G(P)$ there exists some $n \in N$ such that $HN \cap PN \cap P^xN = (H \cap P \cap P^{xn})N$.*

P r o o f. From Sylow's Theorem and $H \trianglelefteq G$ we have $HN \cap PN = (HN \cap P)N = (H \cap P)N$. So $HN \cap PN \cap P^xN = (H \cap P \cap P^xN)N$. Take $P_0 = H \cap P \cap P^xN$. Then P_0 is contained in a Sylow p-subgroup of P^xN. Thus, by Sylow's Theorem again there exists an element n in N such that $P_0 \leqslant P^{xn}$. It follows that $P_0 = H \cap P \cap P^xN \geqslant H \cap P \cap P^{xn} \geqslant P_0$ and hence $P_0 = H \cap P \cap P^{xn}$. This implies that $HN \cap PN \cap P^xN = (H \cap P \cap P^{xn})N$. $\square$

Lemma 2.5 ([11], Theorem 2.8)**.** *If a solvable group G has a Sylow 2-subgroup P which is quaternion-free, then $P \cap Z(G) \cap G^{\mathfrak{N}} = 1$.*

Lemma 2.6. *Let H be a subgroup of a group G. Then $H^{\mathfrak{N}_\mathfrak{p}} \leqslant G^{\mathfrak{N}_\mathfrak{p}}$.*

P r o o f. Since $HG^{\mathfrak{N}_\mathfrak{p}}/G^{\mathfrak{N}_\mathfrak{p}} \leqslant G/G^{\mathfrak{N}_\mathfrak{p}}$, we have that $H/(H \cap G^{\mathfrak{N}_\mathfrak{p}})$ is p-nilpotent and so $H^{\mathfrak{N}_\mathfrak{p}} \leqslant H \cap G^{\mathfrak{N}_\mathfrak{p}}$, as desired. $\square$

Lemma 2.7 ([2], Lemma 2)**.** *Let $\mathfrak{F}$ be a saturated formation. Assume that G is a non-$\mathfrak{F}$-group and there exists a maximal subgroup M of G such that $M \in \mathfrak{F}$ and $G = F(G)M$, where $F(G)$ is the Fitting subgroup of G. Then*

(i) *$G^{\mathfrak{F}}/(G^{\mathfrak{F}})'$ is a chief factor of G,*
(ii) *$G^{\mathfrak{F}}$ is a p-group for a prime p,*
(iii) *$G^{\mathfrak{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$,*
(iv) *$G^{\mathfrak{F}}$ is either an elementary abelian group or $(G^{\mathfrak{F}})' = Z(G^{\mathfrak{F}}) = \Phi(G^{\mathfrak{F}})$ is an elementary abelian group.*

3. Main results

The proof of Main Theorem can be obtained from the following results.

Theorem 3.1. *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If every minimal subgroup of $P \cap P^x \cap G^{\mathfrak{N}_\mathfrak{p}}$ is weakly pronormal in G and when $p = 2$, either every cyclic subgroup of $P \cap P^x \cap G^{\mathfrak{N}_\mathfrak{p}}$ with order 4 is weakly pronormal in G or P is quaternion-free, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. Then G is not p-nilpotent. Noticing that all its Sylow p-subgroups are conjugate in G, we see that the hypotheses of our theorem are subgroup-closure by Lemma 2.1. Consequently, G is a minimal non-p-nilpotent group (that is, every proper subgroup of the group is p-nilpotent but itself is not p-nilpotent). Now, by the result of Itô, see [22], Theorem 10.3.3, G must be a minimal non-nilpotent group. By the result of Schmidt, see [22], Theorem 9.1.9 and Exercise 9.1.11, we know that G is of order $p^a q^b$, where q is a prime which is different from p, P is normal in G and any Sylow q-subgroup Q of G is cyclic. Moreover, $P = G^{\mathfrak{N}_\mathfrak{p}}$ and P is of exponent p when p is odd and of exponent at most 4 when $p = 2$. Let A be a minimal subgroup of P. Then by our hypotheses, there exists a subgroup K of G such that $G = AK$ and $A \cap K$ is pronormal in G. If $A \cap K = 1$, then K is a maximal subgroup of G with index p. Since p is the smallest prime dividing the order of G, we see that K is a normal subgroup of G and therefore the Sylow q-subgroup of K is normal in G since K is p-nilpotent. This leads to the nilpotence of G, a contradiction. Hence $A \leqslant K$, which means that A is pronormal in G and so is normal in G by Lemma 2.3. Therefore every minimal subgroup of P is in the center of G. If p is odd, then G is p-nilpotent by Itô's result, which is a contradiction. If $p = 2$ and every cyclic subgroup $B = \langle b \rangle$ of $P \cap P^x \cap G^{\mathfrak{N}_\mathfrak{p}}$ with order 4 is weakly pronormal in G, then from our hypotheses there exists a subgroup K of G such that $G = BK$ and $B \cap K$ is pronormal in G. If $[G : K] = 4$, then $K\langle b^2 \rangle$ is a subgroup of G with index 2 and therefore $K\langle b^2 \rangle$ is normal in G. This implies that the Sylow q-subgroup of $K\langle b^2 \rangle$ is normal in G and therefore G is nilpotent, which is a contradiction. If $[G : K] = 2$, then K itself is a normal subgroup of G with index 2. We still get a contradiction. It follows that $G = K$ and so B is normal in G by the pronormality of B in G and Lemma 2.3. If $P = B$, then it is clear that G is p-nilpotent, a contradiction. Thus $B \neq P$. Since the exponent of P is at most 4, we have $P \leqslant C_G(Q)$ and therefore $G = P \times Q$, another contradiction. If $p = 2$ and P is quaternion-free, then from Lemma 2.5 we have $\Omega_1(P) \leqslant P \cap G^{\mathfrak{N}_p} \cap Z(G) = 1$, a contradiction. By all these contradictions, we show that the theorem is true. $\square$

If p is an arbitrary prime, the corresponding result is as follows:

Theorem 3.2. *Let G be an S_4-free group. Also let p be a prime dividing the order of G and let P be a Sylow p-subgroup of G. If every minimal subgroup of $P \cap P^x \cap G^{\mathfrak{N}_\mathfrak{p}}$ is weakly pronormal in P and $N_G(P)$ is p-nilpotent for all $x \in G \setminus N_G(P)$ and when $p=2$, P is quaternion-free, then G is p-nilpotent.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order. We split the proof into the following steps:

(1) Suppose M is a subgroup of G such that $P \leqslant M < G$. Then M is p-nilpotent.

It is clear that $N_M(P)$ is p-nilpotent. Let $x \in M \setminus N_M(P)$. Then from Lemma 2.6 $P \cap P^x \cap M^{\mathfrak{N}_\mathfrak{p}} \leqslant P \cap P^x \cap G^{\mathfrak{N}_\mathfrak{p}}$. It follows that every minimal subgroup of $P \cap P^x \cap M^{\mathfrak{N}_\mathfrak{p}}$ is weakly pronormal in P by Lemma 2.1. Now we can see that M satisfies the hypotheses of our theorem. By the choice of G, M is p-nilpotent.

(2) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then we may choose a minimal normal subgroup N of G such that N is contained in $O_{p'}(G)$. Set $\overline{G} = G/N$. Clearly, $N_{\overline{G}}(\overline{P}) = N_G(P)N/N$ is p-nilpotent. For any $xN \in \overline{G} \setminus N_{\overline{G}}(\overline{P})$, by Lemma 2.4, we have $\overline{P} \cap \overline{P}^{xN} \cap \overline{G}^{\mathfrak{N}_\mathfrak{p}} = (P \cap P^{xn} \cap G^{\mathfrak{N}_\mathfrak{p}})N/N$ for some $n \in N$ as $\overline{G}^{\mathfrak{N}_\mathfrak{p}} = G^{\mathfrak{N}_\mathfrak{p}}N/N$. Furthermore, every minimal subgroup of $\overline{P} \cap \overline{P}^{xN} \cap \overline{G}^{\mathfrak{N}_\mathfrak{p}}$ is weakly pronormal in $\overline{P}$ by Lemma 2.1. If $p=2$ and P is quaternion-free, then of course $\overline{P}$ is quaternion-free. Therefore $\overline{G}$ satisfies the hypotheses of our theorem. The minimality of G means that $\overline{G}$ is p-nilpotent and so is G, a contradiction.

(3) $O_p(G) \neq 1$.

Because G is not p-nilpotent, by Frobenius's theorem, see [22], Theorem 10.3.2, there exists a subgroup H of P such that $N_G(H)$ is not p-nilpotent. Since $N_G(P)$ is p-nilpotent, we may choose a subgroup H of P such that $N_G(H)$ is not p-nilpotent but $N_G(K)$ is p-nilpotent for every subgroup K of P with $H < K \leqslant P$. If $N_G(H) < G$, then $H < P_1 \leqslant P$ for some $P_1 \in \mathrm{Syl}_p(N_G(H))$. Set $F = N_G(H)$. By the choice of H, we know that $N_G(P_1)$ is p-nilpotent and therefore $N_F(P_1)$ is p-nilpotent. Let $x \in F \setminus N_F(P_1)$. Since $P_1 = P \cap F$, we can see that $x \in G \setminus N_G(P)$. Again, $P_1 \cap P_1^x \cap F^{\mathfrak{N}_\mathfrak{p}} \leqslant P \cap P^x \cap G^{\mathfrak{N}_\mathfrak{p}}$, so every minimal subgroup of $P_1 \cap P_1^x \cap F^{\mathfrak{N}_\mathfrak{p}}$ is weakly pronormal in P_1 by Lemma 2.1. Moreover, P_1 is quaternion-free. Now, by the minimality of G, we have that $F = N_G(H)$ is p-nilpotent, this contradiction forces that $N_G(H) = G$, as desired.

(4) $G/O_p(G)$ is p-nilpotent and $C_G(O_p(G)) \leqslant O_p(G)$.

From the proof of Statement (3) we know that $N_G(K)$ is p-nilpotent for every subgroup K of P with $O_p(G) < K \leqslant P$. Hence, by Frobenius theorem again, we see that $G/O_p(G)$ is p-nilpotent and so G is p-solvable. Consequently, we obtain that $C_G(O_p(G)) \leqslant O_p(G)$ by Statement (2) and [22], Theorem 9.3.1.

(5) $G = PQ$, where Q is an elementary abelian Sylow q-subgroup of G for a prime $q \neq p$. Moreover, P is a maximal subgroup of G and $QO_p(G)/O_p(G)$ is a minimal normal subgroup of $G/O_p(G)$.

Since G is p-solvable, there exists a Sylow q-subgroup Q of G such that $PQ = QP$ for any prime $q \neq p$ by [12], Theorem 6.3.5. If $PQ < G$, then PQ is p-nilpotent by Statement (1). It follows that $Q \leqslant C_G(O_p(G))$, and Statement (4) provides a contradiction. Thus $G = PQ$ and so G is solvable. Now let $N/O_p(G)$ be a minimal normal subgroup of $G/O_p(G)$ contained in $O_{pp'}(G)/O_p(G)$. Then $N = O_p(G)(N \cap Q)$. If $N \cap Q < Q$, then $PN < G$ and hence PN is p-nilpotent by Statement (1) again. This implies that $1 < N \cap Q \leqslant C_G(O_P(G)) \leqslant O_p(G)$, a contradiction. Hence $N = O_{pp'}(G)$ and so $QO_p(G)/O_p(G)$ is an elementary abelian q-group complementing $P/O_p(G)$. This yields that P is a maximal subgroup of G.

(6) $|P : O_p(G)| = p$.

It is clear that $O_p(G) < P$. Let P_1 be a maximal subgroup of P containing $O_p(G)$ and let $G_1 = P_1 O_{pp'}(G)$. Then P_1 is a Sylow p-subgroup of G_1. The maximality of P means that either $N_G(P_1) = P$ or $N_G(P_1) = G$. If the former holds, then $N_{G_1}(P_1)$ is p-nilpotent. In view of Lemma 2.6, we have $P_1 \cap P_1^x \cap G_1^{\mathfrak{N}_\mathfrak{p}} \leqslant P \cap P^x \cap G^{\mathfrak{N}_\mathfrak{p}}$ for every $x \in G_1 \setminus N_{G_1}(P_1)$. It follows from Lemma 2.1 that G_1 satisfies the hypotheses of the theorem. Thereby G_1 is p-nilpotent by the choice of G and $Q \leqslant C_G(O_p(G)) \leqslant O_p(G)$, this contradiction means $P_1 = O_p(G)$, as desired.

(7) $G = G^{\mathfrak{N}_\mathfrak{p}} L$, where $L = \langle a \rangle \ltimes Q$ is a non-abelian split extension of Q by a cyclic p-subgroup $\langle a \rangle$, $a^p \in Z(L)$ and the action of a (by conjugate) on Q is irreducible.

Since $G/O_p(G)$ is p-nilpotent, $G^{\mathfrak{N}_p} \leqslant O_p(G)$. We can see that $K = G^{\mathfrak{N}_p}Q$ is normal in G since $G/G^{\mathfrak{N}_p}$ is p-nilpotent. Let P_1 be a maximal subgroup of P containing $G^{\mathfrak{N}_\mathfrak{p}}$ and let $G_1 = P_1K = P_1Q$. Then P_1 is a Sylow p-subgroup of G_1. The maximality of P means that either $N_G(P_1) = G$ or $N_G(P_1) = P$. If the latter holds, then $N_{G_1}(P_1)$ is p-nilpotent. By Lemma 2.6, we have $P_1 \cap P_1^x \cap {G_1}^{\mathfrak{N}_\mathfrak{p}} \leqslant P \cap P^x \cap G^{\mathfrak{N}_\mathfrak{p}}$ for every $x \in G_1 \setminus N_{G_1}(P_1)$. It follows from Lemma 2.1 that G_1 satisfies the hypotheses of the theorem. Hence G_1 is p-nilpotent by the choice of G. Therefore $K = G^{\mathfrak{N}_p} \times Q$ and so $Q \trianglelefteq G$, a contradiction. Thus, P_1 is normal in G and so $P_1 = O_p(G)$, this implies that $P/G^{\mathfrak{N}_p}$ is cyclic. On the other hand, by the Frattini argument we have $G = G^{\mathfrak{N}_p}N_G(Q)$. Thus, we may assume that $G = G^{\mathfrak{N}_\mathfrak{p}} L$, where $L = \langle a \rangle \ltimes Q$ is a non-abelian split extension of Q by a cyclic p-subgroup $\langle a \rangle$. By Statement (6) and $O_p(G) \cap N_G(Q) \trianglelefteq N_G(Q)$, we see that $a^p \in Z(L)$. Also since P is a maximal subgroup of G, we know that $G^{\mathfrak{N}_\mathfrak{p}}Q/G^{\mathfrak{N}_\mathfrak{p}}$ is minimal normal in $G/G^{\mathfrak{N}_\mathfrak{p}}$ and consequently the action of a (by conjugate) on Q is irreducible.

(8) $\exp(G^{\mathfrak{N}_\mathfrak{p}}) \neq p$.

Otherwise, $\exp(G^{\mathfrak{N}_\mathfrak{p}}) = p$. If every minimal subgroup of $G^{\mathfrak{N}_\mathfrak{p}}$ is pronormal in P, then from Lemma 2.3 we have $G^{\mathfrak{N}_\mathfrak{p}} \leqslant Z(P)$. Then, by using the Frattini argument, we obtain that $G = C_G(G^{\mathfrak{N}_p})N_G(P)$ and so $G = C_G(G^{\mathfrak{N}_p})$ as $N_G(P) = P$. Hence $G^{\mathfrak{N}_p}$ normalizes Q and therefore Q is normal in G, a contradiction. Let A_1 be a minimal subgroup of $G^{\mathfrak{N}_p}$ and not pronormal in P. Then, by our hypotheses, there exists a subgroup K_1 of P such that $P = A_1K_1$ and $A_1 \cap K_1 = 1$. In general, we may find minimal subgroups $A_1, A_2, \ldots, A_s$ of $G^{\mathfrak{N}_p}$ and also subgroups $K_1, K_2, \ldots, K_s$ of P such that $P = A_iK_i$, $A_i \cap K_i = 1$ for $i = 1, 2, \ldots, s$ and every minimal subgroup of $G^{\mathfrak{N}_p} \cap K_1 \cap \ldots \cap K_s$ is pronormal in P. Furthermore, we may assume that $A_i \leqslant K_1 \cap \ldots \cap K_{i-1}$ $(i = 2, 3, \ldots, s)$ and therefore we can see $K_1 \cap \ldots \cap K_{i-1} = A_i(K_1 \cap \ldots \cap K_i)$. It is easy to see that $G^{\mathfrak{N}_p} \cap K_i$ is normal in P and $(G^{\mathfrak{N}_p} \cap K_i)\langle a\rangle$ is a complement of A_i in P, so we may replace K_i by $(G^{\mathfrak{N}_p} \cap K_i)\langle a\rangle$ and further assume that $\langle a\rangle \leqslant K_i$ for each i. Since $P = G^{\mathfrak{N}_p}\langle a\rangle$, we see that $K_1 \cap \ldots \cap K_s = (G^{\mathfrak{N}_p} \cap K_1 \cap \ldots \cap K_s)\langle a\rangle$. According to our choice, every minimal subgroup B of $G^{\mathfrak{N}_p} \cap K_1 \cap \ldots \cap K_s$ is pronormal in P, thus $B \trianglelefteq P$ by Lemma 2.3 and therefore $G^{\mathfrak{N}_p} \cap K_1 \cap \ldots \cap K_s \leqslant Z(P)$. It follows that $K_1 \cap \ldots \cap K_s$ is abelian. If p is an odd prime, then from [12], Theorem 6.5.2 we get that $K_1 \cap \ldots \cap K_s \leqslant O_p(G)$ and so $P = G^{\mathfrak{N}_p}(K_1 \cap \ldots \cap K_s) \leqslant O_p(G)$, a contradiction. Hence we may assume that $p = 2$. We proceed now to consider the following two cases:

Case 1: $|\langle a\rangle| = 2^n$, where $n > 1$.

Since $K_1 \cap \ldots \cap K_s$ is an abelian normal subgroup of P and $a \in K_1 \cap \ldots \cap K_s$, we have $\Phi(K_1 \cap \ldots \cap K_s) = \Phi(G^{\mathfrak{N}_p} \cap K_1 \cap \ldots \cap K_s)\Phi(\langle a\rangle) = \langle a^2\rangle$ and so $\Omega_1(\langle a^2\rangle) \leqslant Z(P)$. On the other hand, $\Omega_1(\langle a^2\rangle) \leqslant Z(L)$ by Statement (7) and $\Omega_1(\langle a^2\rangle) \leqslant Z(G)$. By Lemma 2.5, we see that $\Omega_1(\langle a^2\rangle) \cap G^{\mathfrak{N}_p} = 1$. Set $\overline{G} = G/N$ and $N = \Omega_1(\langle a^2\rangle)$. It is clear that $N_{\overline{G}}(\overline{P}) = N_G(P)/N = P/N$ is p-nilpotent. Since every minimal subgroup of $G^{\mathfrak{N}_p}$ is weakly pronormal in P, we can obtain that every minimal subgroup of $\overline{G}^{\mathfrak{N}_p} = G^{\mathfrak{N}_p}N/N$ is weakly pronormal in $\overline{P}$ by Lemma 2.1 and therefore $\overline{G}$ satisfies our hypotheses. The choice of G implies that G/N is p-nilpotent and so G is also p-nilpotent, a contradiction.

Case 2: $|\langle a\rangle| = 2$.

By Statement (7), we see that a is an automorphism of Q with order 2 and Q is a cyclic group of q satisfying $b^a = b^{-1}$, where $Q = \langle b\rangle$. In this case, $G^{\mathfrak{N}_p}$ is a minimal normal subgroup of G. In fact, let N be a minimal normal subgroup of G contained in $G^{\mathfrak{N}_p}$. Take $H = NL$. Since $N\langle a\rangle$ is maximal but not normal in H, we see that $N_H(N\langle a\rangle) = N\langle a\rangle$. Noticing that $N\langle a\rangle \cap H^{\mathfrak{N}_p} \leqslant N$, every minimal subgroup of $N\langle a\rangle \cap H^{\mathfrak{N}_p}$ is weakly pronormal in $N\langle a\rangle$ by Lemma 2.1. If further $H < G$, then the choice of G implies that H is 2-nilpotent. Consequently, $NQ = N \times Q$ and therefore $1 \neq N \cap Z(P) \leqslant Z(G)$. Lemma 2.5 provides a contradiction. Hence $G^{\mathfrak{N}_p}$ is a minimal normal subgroup of G. Since $G^{\mathfrak{N}_p} \cap N_G(Q) \trianglelefteq N_G(Q)$, we know that $G^{\mathfrak{N}_p} \cap N_G(Q) = 1$ and so b acts fixed-point-freely on $G^{\mathfrak{N}_p}$. We may assume that $N_1 = \{1, c_1, c_2, \ldots, c_q\}$ is a subgroup of $G^{\mathfrak{N}_p}$ with $c_1 \in Z(P)$ and $b = (c_1, c_2, \ldots, c_q)$ is a permutation of the set $\{c_1, c_2, \ldots, c_q\}$. Noticing that $b^a = b^{-1}$ and $(c_1)^{a^{-1}ba} = (c_1)^{b^{-1}}$, $(c_2)^a = c_q$. If we use $(b^i)^a = b^{-i}$ and consider $(c_1)^{a^{-1}b^ia} = (c_1)^{b^{-1}}$, we see that $(c_{i+1})^a = c_{q-i+1}$ for $i = 1, 2, \ldots, (q+1)/2$. Hence, N_1 is normalized by both $G^{\mathfrak{N}_\mathfrak{p}}$ and L and so N_1 is normal in G. The minimal normality of $G^{\mathfrak{N}_\mathfrak{p}}$ implies that $G^{\mathfrak{N}_\mathfrak{p}} = N_1$, thus we have $Z(P) = \{1, c_1\}$. Since $G^{\mathfrak{N}_\mathfrak{p}} \cap K_1 \cap \ldots \cap K_s$ is centralized by both $G^{\mathfrak{N}_\mathfrak{p}}$ and $\langle a\rangle$, we have $1 < G^{\mathfrak{N}_\mathfrak{p}} \cap K_1 \cap \ldots \cap K_s \leqslant Z(P)$. In view of P not being abelian, we get $\Phi(P) = P' = Z(P)$. Thus, P is an extra-special 2-group. From [22], Theorem 5.3.8, there exists some positive integer n such that $|P| = 2^{2n+1}$, therefore $|G^{\mathfrak{N}_\mathfrak{p}}| = 2^{2n}$. However, $2^{2n}-1 = (2^n+1)(2^n-1)$ and $q = 2^{2n}-1$, hence $n = 1$, $q = 3$ and $|P| = 2^3$. Now it is easy to see that $G \cong S_4$, which is a contradiction to our hypotheses on G.

(9) $p = 2$ and $\exp(G^{\mathfrak{N}_\mathfrak{p}})$=4.

In view of Lemma 2.7, it will suffice to show that there exists a p-nilpotent maximal subgroup M of G such that $G = G^{\mathfrak{N}_\mathfrak{p}}M$. In fact, let M be a maximal subgroup of G containing L. Then $M = L(M \cap G^{\mathfrak{N}_\mathfrak{p}})$ and $G = G^{\mathfrak{N}_\mathfrak{p}}M$. We can see that $M \cap G^{\mathfrak{N}_\mathfrak{p}} \trianglelefteq G$ and therefore $M = (\langle a\rangle(M \cap G^{\mathfrak{N}_\mathfrak{p}}))Q$. Let $P_1 = \langle a\rangle(M \cap G^{\mathfrak{N}_\mathfrak{p}})$ and M_1 a maximal subgroup of M containing P_1. Then $M_1 = P_1(M_1 \cap Q)$ and $G^{\mathfrak{N}_\mathfrak{p}}M_1 < G$. By Statement (1) we see that $G^{\mathfrak{N}_\mathfrak{p}}M_1$ is p-nilpotent. Thus $M_1 \cap Q \leqslant C_G(O_p(G)) \leqslant O_p(G)$. It follows from Statement (4) that $M_1 \cap Q = 1$ and so P_1 is maximal in M. In this case, if $P_1 \trianglelefteq M$, then $\langle a\rangle = P_1 \cap L \trianglelefteq L$, which is contrary to Statement (7). Hence, $N_M(P_1) = P_1$ and M satisfies the hypotheses of our theorem. The choice of G implies that M is p-nilpotent, as desired.

(10) The final contradiction.

By Statement (9) and Lemma 2.7, $Z(G^{\mathfrak{N}_2}) = \Phi(G^{\mathfrak{N}_2})$ is an elementary abelian 2-group. For any minimal subgroup A_1 of $\Phi(G^{\mathfrak{N}_2})$, since A_1 is weakly pronormal in P, we have that $A_1 \trianglelefteq P$ by Lemma 2.3 and therefore $\Phi(G^{\mathfrak{N}_2}) \leqslant Z(P)$. By the Frattini argument we further obtain $G = N_G(\Phi(G^{\mathfrak{N}_2})) = C_G(\Phi(G^{\mathfrak{N}_2}))N_G(P)$. As $N_G(P) = P$ and $P \leqslant C_G(\Phi(G^{\mathfrak{N}_2}))$, we get $\Phi(G^{\mathfrak{N}_2}) \leqslant Z(G)$. Hence we can take an element x in $\Phi(G^{\mathfrak{N}_2})$ such that x is of order 2 and $x \in Z(G)$, which is a contradiction to Lemma 2.5. This completes our proof. $\square$

P r o o f of the Main Theorem. Combining Theorem 3.1 and Theorem 3.2, we obtain that our theorem holds. $\square$

In the following, we shall extend the Main Theorem to formations.

Theorem 3.3. *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups and G a group such that G is S_4-free. Also let N be a normal subgroup of G such that $G/N \in \mathfrak{F}$. If for every prime p dividing the order of N and for every Sylow p-subgroup P of N, every minimal subgroup of $P \cap P^x \cap G^{\mathfrak{N}_\mathfrak{p}}$ is weakly* *pronormal in $N_G(P)$ for all $x \in G \setminus N_G(P)$ and when $p = 2$, P is quaternion-free, then $G \in \mathfrak{F}$.*

P r o o f. Suppose that the theorem is false and let G be a counterexample of minimal order. By Lemma 2.1 and our main theorem, we know that N is a Sylow tower group of supersolvable type. Thus, if p is the largest prime dividing the order of N and P is a Sylow p-subgroup of N, then P must be normal in G and $G/P/N/P \cong G/N \in \mathfrak{F}$. It is clear that G/P satisfies the hypotheses of our theorem for its normal subgroup N/P by Lemma 2.4 and Lemma 2.1. Then the minimality of G implies that $G/P \in \mathfrak{F}$ and therefore $G^{\mathfrak{F}} \leqslant P \cap G^{\mathfrak{N}}$. Furthermore, we claim that $G^{\mathfrak{F}} \leqslant P \cap G^{\mathfrak{N}_p}$. Let P^* be a Sylow p-subgroup of G. As $G/G^{\mathfrak{N}_p}$ is p-nilpotent, we can see that $P^*G^{\mathfrak{N}_p} \cap O^p(G)G^{\mathfrak{N}_p} = G^{\mathfrak{N}_p}$ and so $P^* \cap O^p(G) \leqslant G^{\mathfrak{N}_p}$, which means that $P^* \cap O^p(G) = P \cap G^{\mathfrak{N}_p}$. A similar argument shows that $P^* \cap O^p(G) = P \cap G^{\mathfrak{N}}$ and this proves our claim. By [4], Theorem 3.5, there exists a maximal subgroup M of G such that $G = MF'(G)$, where $F'(G) = \mathrm{Soc}(G \bmod \Phi(G))$ and $G/M_G \notin \mathfrak{F}$. Then $G = MG^{\mathfrak{F}}$ and so $G = MF(G)$, where $F(G)$ is the Fitting subgroup of G. It is now clear that M satisfies the hypotheses of our theorem for its normal subgroup $M \cap P$. Hence, by the minimality of G, we have $M \in \mathfrak{F}$.

If $G^{\mathfrak{F}}$ is an elementary abelian group, then from Lemma 2.7 $G^{\mathfrak{F}}$ is a minimal normal subgroup of G. Let A be a minimal subgroup of $G^{\mathfrak{F}}$. Then A is weakly pronormal in $N_G(P) = G$ by our hypotheses. Thus, there exists a subgroup K of G such that $G = AK$ and $A \cap K$ is pronormal in G. If $A \leqslant K$, then from Lemma 2.3 and the subnormality of A in G we have that A is normal in G and therefore $G^{\mathfrak{F}} = A$, which implies that $G \in \mathfrak{F}$, a contradiction. Hence, we may assume that $A \cap K = 1$. It is clear that $K \cap G^{\mathfrak{F}}$ is normal in G. The minimal normality of $G^{\mathfrak{F}}$ means that $K \cap G^{\mathfrak{F}} = 1$ and A is normal in G. It follows that $G^{\mathfrak{F}} = A$ is cyclic of order p, a contradiction.

We now suppose that $G^{\mathfrak{F}}$ is not an elementary abelian group. Then $(G^{\mathfrak{F}})' = Z(G^{\mathfrak{F}}) = \Phi(G^{\mathfrak{F}})$ is an elementary abelian group by Lemma 2.7. Suppose that there exists a minimal subgroup B of $G^{\mathfrak{F}}$ such that B is not pronormal in G. Then G has a subgroup K satisfying $G = BK$ and $B \cap K = 1$. Clearly, $\Phi(G^{\mathfrak{F}}) \leqslant K$. We can see that $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}}) \cap K/\Phi(G^{\mathfrak{F}}) \trianglelefteq G/\Phi(G^{\mathfrak{F}})$ and so $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}}) \cap K/\Phi(G^{\mathfrak{F}}) = 1$ by Lemma 2.7. It follows that $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}}) = B\Phi(G^{\mathfrak{F}})/\Phi(G^{\mathfrak{F}})$ and therefore $G^{\mathfrak{F}} = B$, the choice of B provides a contradiction. Hence, every minimal subgroup of $G^{\mathfrak{F}}$ is pronormal in G. Suppose that $\exp(G^{\mathfrak{F}}) = p$. Then from [10], Lemma 6.3 we can obtain that every minimal subgroup $A\Phi(G^{\mathfrak{F}})/\Phi(G^{\mathfrak{F}})$ of $G/\Phi(G^{\mathfrak{F}})$ is pronormal in $G/\Phi(G^{\mathfrak{F}})$ and so is weakly pronormal in $G/\Phi(G^{\mathfrak{F}})$, where $A \leqslant G^{\mathfrak{F}} \setminus \Phi(G^{\mathfrak{F}})$. This implies that $G/\Phi(G^{\mathfrak{F}})$ satisfies the hypotheses of our theorem for its normal subgroup $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$, and $G/\Phi(G^{\mathfrak{F}}) \in \mathfrak{F}$ by the choice of G. Thus $G \in \mathfrak{F}$ because $\mathfrak{F}$

is a saturated formation, a contradiction. Hence $p = 2$ and $G^{\mathfrak{F}}$ has exponent 4 by Lemma 2.7. Let Q be a Sylow q-subgroup of M with $q \neq 2$. By Theorem 3.1, $G^{\mathfrak{F}}Q$ is 2-nilpotent and $O^2(M)$ is normalized by $G^{\mathfrak{F}}$. This means $O^2(M) \trianglelefteq G$. Now, we have $G^{\mathfrak{F}} \leqslant O^2(M) \leqslant M$ since $G/O^2(M)$ is a 2-group, a contradiction. Thus, our proof is completed. $\square$

As an immediate consequence of Theorem 3.3, we have:

Corollary 3.4. *Let G be a group such that G is S_4-free. If for every prime p dividing the order of G and for every Sylow p-subgroup P of G, every minimal subgroup of $P \cap P^x \cap G^{\mathfrak{N}_\mathfrak{p}}$ is weakly pronormal in $N_G(P)$ for all $x \in G \setminus N_G(P)$ and when $p = 2$, P is quaternion-free, then G is supersolvable.*

References

[1] *M. Asaad*: On weakly pronormal subgroups of finite groups. J. Group Theory *17* (2014), 407–418. zbl MR doi

[2] *M. Asaad, A. Ballester-Bolinches, M. C. Pedraza Aguilera*: A note on minimal subgroups of finite groups. Commun. Algebra *24* (1996), 2771–2776. zbl MR doi

[3] *M. Asaad, M. Ramadan*: On the intersection of maximal subgroups of a finite group. Arch. Math. *61* (1993), 206–214. zbl MR doi

[4] *A. Ballester-Bolinches*: $\mathfrak{H}$-normalizers and local definitions of saturated formations of finite groups. Isr. J. Math. *67* (1989), 312–326. zbl MR doi

[5] *A. Ballester-Bolinches, J. C. Beidleman, A. D. Feldman, M. F. Ragland*: On generalised pronormal subgroups of finite groups. Glasg. Math. J. *56* (2014), 691–703. zbl MR doi

[6] *A. Ballester-Bolinches, J. C. Beidleman, A. D. Feldman, M. F. Ragland*: Finite groups in which pronormality and $\mathfrak{F}$-pronormality coincide. J. Group Theory *19* (2016), 323–329. zbl MR doi

[7] *A. Ballester-Bolinches, X. Guo*: Some results on p-nilpotence and solubility of finite groups. J. Algebra *228* (2000), 491–496. zbl MR doi

[8] *A. Ballester-Bolinches, X. Guo, Y. Li, N. Su*: On finite p-nilpotent groups. Monatsh. Math. *181* (2016), 63–70. zbl MR doi

[9] *B. Brewster, A. Martínez-Pastor, M. D. Pérez-Ramos*: Pronormal subgroups of a direct product of groups. J. Algebra *321* (2009), 1734–1745. zbl MR doi

[10] *K. Doerk, T. Hawkes*: Finite Soluble Groups. De Gruyter Expositions in Mathematics 4, Walter de Gruyter, Berlin, 1992. zbl MR doi

[11] *L. Dornhoff*: M-groups and 2-groups. Math. Z. *100* (1967), 226–256. zbl MR doi

[12] *D. Gorenstein*: Finite Groups. Harper's Series in Modern Mathematics, Harper & Row Publishers, New York, 1968. zbl MR

[13] *X. Y. Guo, K. P. Shum*: The influence of minimal subgroups of focal subgroups on the structure of finite groups. J. Pure Appl. Algebra *169* (2002), 43–50. zbl MR doi

[14] *X. Guo, K. P. Shum*: On p-nilpotency and minimal subgroups of finte groups. Sci. China, Ser. A *46* (2003), 176–186. zbl MR doi

[15] *X. Guo, K. P. Shum*: Permutability of minimal subgroups and p-nilpotentcy of finite groups. Isr. J. Math. *136* (2003), 145–155. zbl MR doi

[16] *X. Guo, K. P. Shum*: p-nilpotence of finite groups and minimal subgroups. J. Algebra *270* (2003), 459–470. zbl MR doi

[17] *N. Itô*: Über eine zur Frattini-Gruppe duale Bildung. Nagoya Math. J. *9* (1955), 123–127. (In German.) zbl MR doi
[18] *Y. Li, N. Su, Y. Wang*: A generalization of Burnside's p-nilpotency criterion. J. Group Theory *20* (2017), 185–192. zbl MR doi
[19] *I. A. Malinowska*: Finite groups all of whose small subgroups are pronormal. Acta Math. Hung. *147* (2015), 324–337. zbl MR doi
[20] *G. Navarro*: Pronormal subgroups and zeros of characters. Proc. Am. Math. Soc. *142* (2014), 3003–3005. zbl MR doi
[21] *T. A. Peng*: Finite groups with pro-normal subgroups. Proc. Am. Math. Soc. *20* (1969), 232–234. zbl MR doi
[22] *D. J. S. Robinson*: A Course in the Theory of Groups. Graduate Texts in Mathematics 80, Springer, New York, 1982. zbl MR doi
[23] *J. Shi, W. Shi, C. Zhang*: A note on p-nilpotence and solvability of finite groups. J. Algebra *321* (2009), 1555–1560. zbl MR doi

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