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CHARACTERIZATIONS OF PARTIAL ISOMETRIES AND TWO SPECIAL KINDS OF EP ELEMENTS

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Abstract. We give some sufficient and necessary conditions for an element in a ring to be an EP element, partial isometry, normal EP element and strongly EP element by using solutions of certain equations.

Keywords: EP element; partial isometry; normal EP element; strongly EP element; solutions of equation

MSC 2010: 15A09, 16U99, 16W10

1. Introduction

Let R be an associative ring with 1 and let $a \in R$. The element a is said to be group invertible if there exists $a^{\#} \in R$ such that

$$aa^{\#}a = a, \quad a^{\#}aa^{\#} = a^{\#}, \quad aa^{\#} = a^{\#}a.$$

The element $a^{\#}$ is called a group inverse of a, which is uniquely determined by the above equations, see [3]. We denote the set of all group invertible elements of R by $R^{\#}$.

An involution in R is an anti-isomorphism $*\colon R \to R$, $a \mapsto a^*$ of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.$$

If $a^*a = aa^*$, then the element a is called normal.

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An element $a^\dagger$ is called the Moore-Penrose inverse (or MP-inverse) of a, see [14], if

$$aa^\dagger a = a, \quad a^\dagger aa^\dagger = a^\dagger, \quad (aa^\dagger)^* = aa^\dagger, \quad (a^\dagger a)^* = a^\dagger a.$$

If $a^\dagger$ exists, then it is unique, see [6], [7], [8]. Denote by $R^\dagger$ the set of all MP-invertible elements of R. An element a is called a partial isometry if $aa^*a = a$, that is, $a \in R^\dagger$ and $a^* = a^\dagger$. An element $a \in R^\# \cap R^\dagger$ satisfying $a^\# = a^\dagger$ is said to be EP. We denote the set of all EP elements of R by $R^{\rm EP}$. Note that if $a \in R^\dagger$ is normal, then $a \in R^{\rm EP}$, see [9]. An element a is called a normal EP element if $a \in R^\dagger$ is normal. We denote the set of all normal EP elements of R by $R^{\rm NEP}$. Obviously, if $a \in R^{\rm EP}$ is normal, then $a \in R^{\rm NEP}$. If $a \in R^{\rm EP}$ is a partial isometry, we say a is a strongly EP element. Denote by $R^{\rm SEP}$ the set of all strongly EP elements of R.

In [1], Baksalary, Styan and Trenkler explored various classes of matrices, such as partial isometries and EP elements, by using the representation of complex matrices and the matrix rank described in [7]. Recent researches on partial isometries in rings with involution have produced some interesting findings, see [10], [11]. At the same time, various characterizations of EP elements in rings with involution were investigated in [2], [5], [12], [13]. In general, EP elements are considered in the contexts of semigroups, rings and C^*-algebras.

Motivated by the above results, this work is intended to provide some equivalent conditions for an element to be an EP element and partial isometry in rings with involution by using solutions of some equations. Normal EP elements and strongly EP elements, two special classes of EP elements, are also investigated. Let $\chi_a = \{a, a^\#, a^\dagger, a^*, (a^\#)^*, (a^\dagger)^*\}$. We show that $a \in R^{\rm EP}$ if and only if the equation $axa^\# + axa^* = xaa^\dagger + a^*ax$ has at least one solution in χ_a. Replacing the above equation by $xa^*a = xaa^*$, we obtain $a \in R^{\rm NEP}$. We also prove that if the equation $x = xaa^*$ or the equation $x = xa^*a$ has at least one solution in χ_a, then a is a partial isometry. Finally, we describe an element a to be a strongly EP element by discussing the solutions of equations $x = axa^*$, $xa^\dagger a = xaa^*$, $axa^* = xa^\dagger a$ and $a^*xa = xaa^\dagger$ in χ_a.

2. Results

Lemma 2.1. *Let $a \in R^\# \cap R^\dagger$. Then the following conditions are satisfied:*
(1) $a^*R = a^*a^2R = a^*aa^\#R = (a^\#)^*R$;
(2) $Ra = Ra^\# = Raa^*a^\# = Ra^*a = Ra^*a^*a = Ra^\dagger a^*a$;
(3) $(a^\#)^*aa^\dagger R = (a^\#)^*a^\#a^\dagger R = (a^\#)^*a^\#a^*R$;
(4) $a^\#R = aR$ *and* $Ra^* = Ra^\dagger$.

P r o o f. We only give the proof of the item (1), the rest of them are left to the reader to be proven by similar techniques.

$$\begin{aligned}a^*R &= (aa^\dagger a)^*R = a^*aa^\dagger R = a^*a^2a^\# a^\dagger R \subseteq a^*a^2R = a^*aa^\# a^2R \subseteq a^*aa^\# R \subseteq a^*R \\ &= (a^2a^\#)^*R = (a^\#)^*a^*a^*R \subseteq (a^\#)^*R = ((a^\#)^2a)^*R \subseteq a^*R.\end{aligned}$$

□

Lemma 2.2 ([15], Theorem 3.9). *Let $a \in R^\#$. Then $a \in R^{\mathrm{EP}}$ if and only if one of the following conditions holds:*

$$(1)\ a^*R \subseteq aR; \quad (2)\ aR \subseteq a^*R; \quad (3)\ Ra \subseteq Ra^*; \quad (4)\ Ra^* \subseteq Ra.$$

The following lemma follows from Lemmas 2.1 and 2.2.

Lemma 2.3. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{\mathrm{EP}}$ if and only if one of the following conditions holds:*

$$\begin{array}{llll}(1)\ a^*R \subseteq a^\# R; & (2)\ Ra^\# \subseteq Ra^\dagger; & (3)\ Ra^\# \subseteq Ra^*; & (4)\ Ra \subseteq Ra^\dagger; \\ (5)\ Ra^\dagger \subseteq Ra; & (6)\ a^\dagger R \subseteq aR; & (7)\ Ra^\dagger \subseteq Ra; & (8)\ aR \subseteq a^\dagger R.\end{array}$$

Let $a \in R^\# \cap R^\dagger$ and $\chi_a = \{a, a^\#, a^\dagger, a^*, (a^\#)^*, (a^\dagger)^*\}$.
We first consider the equation

$$axa^\# + axa^* = xaa^\dagger + a^*ax. \tag{2.1}$$

By discussing the solutions of the equation (2.1) in χ_a, we give a novel characterization of EP elements.

Theorem 2.4. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{\mathrm{EP}}$ if and only if the equation* (2.1) *has at least one solution in χ_a.*

P r o o f. $\Rightarrow$ The conclusion is proved by writing $x = a^\dagger$.
$\Leftarrow$ (1) If $x = a$, then $a^2a^\# + a^2a^* = a^2a^\dagger + a^*a^2$. That is,

$$a + a^2a^* = a^2a^\dagger + a^*a^2.$$

It follows from Lemma 2.1 that

$$a^*R = a^*a^2R = (a + a^2a^* - a^2a^\dagger)R \subseteq aR.$$

Hence $a \in R^{\mathrm{EP}}$.

(2) If $x = a^{\#}$, then $aa^{\#}a^{\#} + aa^{\#}a^* = a^{\#}aa^{\dagger} + a^*aa^{\#}$. Indeed that

$$a^{\#} + a^{\#}aa^* = a^{\#}aa^{\dagger} + a^*aa^{\#}.$$

By Lemma 2.1, we have

$$a^*R = a^*aa^{\#}R = (a^{\#} + a^{\#}aa^* - a^{\#}aa^{\dagger})R \subseteq a^{\#}R = aR.$$

It follows from Lemma 2.2 that $a \in R^{\mathrm{EP}}$.

(3) If $x = a^{\dagger}$, then $aa^{\dagger}a^{\#} + aa^{\dagger}a^* = a^{\dagger}aa^{\dagger} + a^*aa^{\dagger}$. That is,

$$a^{\#} + aa^{\dagger}a^* = a^{\dagger} + a^*.$$

We thus get

$$Ra^{\#} = R(a^{\dagger} + a^* - aa^{\dagger}a^*) \subseteq Ra^{\dagger} + Ra^* = Ra^{\dagger}$$

by Lemma 2.1. The fact that $a \in R^{\mathrm{EP}}$ follows from Lemma 2.3.

(4) If $x = a^*$, then $aa^*a^{\#} + aa^*a^* = a^*aa^{\dagger} + a^*aa^*$. It is immediate that

$$aa^*a^{\#} + aa^*a^* = a^* + a^*aa^*.$$

Lemma 2.1 now leads to

$$Ra^{\#} = Raa^*a^{\#} = R(a^* + a^*aa^* - aa^*a^*) \subseteq Ra^*.$$

Therefore $a \in R^{\mathrm{EP}}$ by Lemma 2.3.

(5) If $x = (a^{\#})^*$, then $a(a^{\#})^*a^{\#} + a(a^{\#})^*a^* = (a^{\#})^*aa^{\dagger} + a^*a(a^{\#})^*$. It is easy to see that

$$a(a^{\#})^*a^{\#} + a(a^{\#})^*a^* = (a^{\#})^* + a^*a(a^{\#})^*.$$

Applying involution to the above equality, we deduce that

$$(a^{\#})^*a^{\#}a^* + aa^{\#}a^* = a^{\#} + a^{\#}a^*a.$$

We conclude from Lemma 2.1 that

$$\begin{aligned} a^*R &= (a^{\#})^*R = (aa^{\dagger}a^{\#})^*R = (a^{\#})^*aa^{\dagger}R = (a^{\#})^*a^{\#}a^{\dagger}R \\ &= (a^{\#})^*a^{\#}a^*R = (a^{\#} + a^{\#}a^*a - aa^{\#}a^*)R \subseteq aR. \end{aligned}$$

Hence, $a \in R^{\mathrm{EP}}$.

(6) If $x = (a^{\dagger})^*$, then $a(a^{\dagger})^*a^{\#} + a(a^{\dagger})^*a^* = (a^{\dagger})^*aa^{\dagger} + a^*a(a^{\dagger})^*$. Taking involution of the above equality, we obtain that

$$(a^{\#})^*a^{\dagger}a^* + aa^{\dagger}a^* = aa^{\dagger}a^{\dagger} + a^{\dagger}a^*a.$$

By Lemma 2.1, we get

$$Ra = Ra^{\dagger}a^*a = R((a^{\#})^*a^{\dagger}a^* + aa^{\dagger}a^* - aa^{\dagger}a^{\dagger}) \subseteq Ra^* + Ra^{\dagger} = Ra^{\dagger}.$$

Therefore, $a \in R^{\mathrm{EP}}$. □

Many achievements have been made in partial isometry, see [4], [10], [11]. Motivated by some known results, we proceed with this study. Specifically, we establish the relation between partial isometry and the solutions of equation in χ_a. Let R be a ring and $w \in R$. The element w is called a *semi-idempotent*, if $w - w^2 \in J(R)$, where $J(R)$ is the Jacobson radical of R. As usual, denote by $E(R)$ the set of all idempotents of R.

Theorem 2.5. *Let $a \in R^\dagger$. Then the following conditions are equivalent:*
(1) *a is a partial isometry;*
(2) *$aa^* \in E(R)$;*
(3) *$a^*a \in E(R)$.*

P r o o f. (1) $\Rightarrow$ (2) The equality $a^* = a^\dagger$ implies $aa^* = aa^\dagger \in E(R)$.

(2) $\Rightarrow$ (3) Since $aa^* = aa^*aa^*$, pre-multiplying by $a^\dagger$, we know that $a^* = a^*aa^*$. Post-multiplying by a, we get the desired result.

(3) $\Rightarrow$ (1) From the assumption $a^*a = a^*aa^*a$, we deduce that

$$a^* = a^*aa^\dagger = a^*aa^*aa^\dagger = a^*aa^*.$$

It is clear that $a^\dagger aa^* = a^* = a^*aa^*$. Post-multiplying by $(a^\dagger)^*$, we get $a^\dagger a = a^*a$. So

$$a^\dagger = a^\dagger aa^\dagger = a^*aa^\dagger = a^*.$$

□

Theorem 2.6. *Let $a \in R^\dagger$. Then a is a partial isometry if and only if the following two conditions hold:*
(1) *aa^* is a semi-idempotent;*
(2) *$a^\dagger - a^* \in E(R)$.*

P r o o f. The equality $a^* = a^\dagger$ gives

$$aa^* - aa^*aa^* = 0 \in J(R) \quad \text{and} \quad a^\dagger - a^* = 0 \in E(R).$$

Conversely, write $x = aa^* - aa^*aa^* \in J(R)$. Then $a^\dagger aa^* - a^\dagger aa^*aa^* = a^\dagger x$, namely $a^* - a^*aa^* = a^\dagger x$. Pre-multiplying by $(a^\dagger)^*$, we obtain that

$$aa^\dagger - aa^* = (a^\dagger)^*a^\dagger x.$$

Pre-multiplying by $a^\dagger$, we get

$$a^\dagger - a^* = a^\dagger(a^\dagger)^*a^\dagger x.$$

On the other hand, $a^\dagger - a^* \in E(R)$. It is easy to see that $a^\dagger - a^* \in E(R) \cap J(R)$. This gives $a^\dagger(a^\dagger)^*a^\dagger x = 0$, because $E(R) \cap J(R) = \{0\}$. Consequently, $a^\dagger = a^*$. □

Theorem 2.7. *Let $a \in R^{\dagger}$. Then a is a partial isometry if and only if $aa^{\dagger} - aa^* \in E(R)$.*

Proof. $\Rightarrow$ From the assumption, we know that $aa^* \in E(R)$. It follows that

$$(aa^{\dagger} - aa^*)(aa^{\dagger} - aa^*) = aa^{\dagger} - aa^* - aa^* + aa^* = aa^{\dagger} - aa^*.$$

$\Leftarrow$ We first consider the equality

$$aa^{\dagger} - aa^* = (aa^{\dagger} - aa^*)^2 = aa^{\dagger} - 2aa^* + aa^*aa^*.$$

From this equality, we have $aa^* = aa^*aa^*$. That is, $aa^* \in E(R)$. By Theorem 2.5, a is a partial isometry. □

Theorem 2.8. *Let $a \in R^{\#} \cap R^{\dagger}$. Then a is a partial isometry if and only if the equation $x = xaa^*$ has at least one solution in χ_a.*

Proof. $\Rightarrow$ Since $a^* = a^{\dagger}$, the conclusion is obtained by taking $x = a^{\dagger}$.

$\Leftarrow$ (1) If $x = a$, then $a = a^2a^*$. This clearly forces $Ra = Ra^2a^* \subseteq Ra^*$. It follows from Lemma 2.2 that $a \in R^{\mathrm{EP}}$. Thus,

$$aa^{\dagger} = a^{\dagger}a = a^{\dagger}a^2a^* = aa^*.$$

Pre-multiplying by $a^{\dagger}$, we get

$$a^{\dagger} = a^{\dagger}aa^* = a^*.$$

(2) If $x = a^{\#}$, then $a^{\#} = a^{\#}aa^*$. Now, by Lemma 2.1, we have $Ra = Ra^{\#} = Ra^{\#}aa^* = Ra^*$. From Lemma 2.2, we observe $a \in R^{\mathrm{EP}}$. Therefore,

$$a^{\dagger} = a^{\#} = a^{\#}aa^* = a^{\dagger}aa^* = a^*.$$

(3) If $x = a^{\dagger}$, then $a^{\dagger} = a^{\dagger}aa^* = a^*$.

(4) If $x = a^*$, then $a^* = a^*aa^*$. By Theorem 2.5, we know $a^{\dagger} = a^*$.

(5) If $x = (a^{\#})^*$, then $(a^{\#})^* = (a^{\#})^*aa^*$. Thus, $a^{\dagger} = a^*$ by [10], Theorem 2.2.

(6) If $x = (a^{\dagger})^*$, then $(a^{\dagger})^* = (a^{\dagger})^*aa^*$. Applying involution to the above equality, we deduce that $a^{\dagger} = aa^*a^{\dagger}$. It follows that $a^{\dagger}R = aa^*a^{\dagger}R \subseteq aR$ and $a \in R^{\mathrm{EP}}$ by Lemma 2.3. Moreover,

$$a^{\dagger}a = aa^*a^{\dagger}a = aa^*aa^{\dagger} = aa^*.$$

That is, $aa^{\dagger} = aa^*$, which gives $a^{\dagger} = a^*$. □

Theorem 2.9. *Let $a \in R^{\#} \cap R^{\dagger}$. Then a is a partial isometry if and only if the equation $x = xa^*a$ has at least one solution in χ_a.*

Proof. $\Rightarrow$ Taking $x = a$, we complete the proof.

$\Leftarrow$ (1) If $x = a$, then $a = aa^*a$. By the definition of partial isometry, we see $a^{\dagger} = a^*$.

(2) If $x = a^{\#}$, then $a^{\#} = a^{\#}a^*a$. Therefore, $a^{\dagger} = a^*$ follows from [10], Theorem 2.2.

(3) If $x = a^{\dagger}$, then $a^{\dagger} = a^{\dagger}a^*a$. Observe that $Ra^{\dagger} = Ra^{\dagger}a^*a \subseteq Ra$. So, $a \in R^{\mathrm{EP}}$. It is straightforward that

$$a^{\dagger}a = aa^{\dagger} = aa^{\dagger}a^*a = a^{\dagger}aa^*a = a^*a.$$

Hence, $a^{\dagger} = a^*$.

(4) If $x = a^*$, then $a^* = a^*a^*a$. According to the above equality, we conclude that $Ra^* = Ra^*a^*a \subseteq Ra$ and $a = a^*a^2$, hence that $a \in R^{\mathrm{EP}}$ and finally

$$a^{\dagger}a = aa^{\dagger} = a^*a^2a^{\dagger} = a^*a.$$

Consequently, $a^{\dagger} = a^*$.

(5) If $x = (a^{\#})^*$, then $(a^{\#})^* = (a^{\#})^*a^*a$. Applying involution to $(a^{\#})^* = (a^{\#})^*a^*a$, we have $a^{\#} = a^*aa^{\#}$. It is understood that $a^{\#}R = a^*aa^{\#}R \subseteq a^*R$. This means that $a \in R^{\mathrm{EP}}$. We thus get

$$a^{\dagger} = a^{\#} = a^*aa^{\#} = a^*aa^{\dagger} = a^*.$$

(6) If $x = (a^{\dagger})^*$, then

$$(a^{\dagger})^* = (a^{\dagger})^*a^*a = aa^{\dagger}a = a.$$

Taking involution of the above equality, we obtain $a^{\dagger} = a^*$. □

Normal EP elements, a special kind of EP elements, are very important for the development of matrices and operators on Hilbert spaces. Here some new conclusions about them are proposed.

Lemma 2.10. *Let $a \in R^{\dagger}$. Then $(aa^*)^{\#} = (a^{\dagger})^*a^{\dagger} = (aa^*)^{\dagger}$.*

Proof. It is obvious. □

Lemma 2.11. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{\mathrm{NEP}}$ if and only if $(a^{\dagger})^*a^{\dagger} = a^{\dagger}(a^{\dagger})^*$.*

P r o o f. From the hypothesis $(a^\dagger)^* a^\dagger = a^\dagger (a^\dagger)^*$, post-multiplying by a, we see that $(a^\dagger)^* = a^\dagger (a^\dagger)^* a$. Applying involution to the above equality, we deduce that $a^\dagger = a^* a^\dagger (a^\dagger)^*$. It is understood that

$$Ra^\dagger = Ra^* a^\dagger (a^\dagger)^* \subseteq R(a^\dagger)^* = Ra.$$

Therefore, $a \in R^{\rm EP}$ by Lemma 2.3. Post-multiplying $a^\dagger = a^* a^\dagger (a^\dagger)^*$ by a^*, we have

$$a^\dagger a^* = a^* a^\dagger (a^\dagger)^* a^* = a^* a^\dagger (aa^\dagger)^* = a^* a^\dagger.$$

Pre-multiplying by a, we get $a^* = aa^* a^\dagger$. Post-multiplying by a, we have

$$a^* a = aa^* a^\dagger a = aa^* a^\# a = aa^* aa^\# = aa^* aa^\dagger = aa^*.$$

The converse can easily be verified by $a^\# = a^\dagger$ and the double commutativity of the group inverse. □

Theorem 2.12. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{\rm NEP}$ if and only if the equation $xa^* a = xaa^*$ has at least one solution in χ_a.*

P r o o f. Using the assumption $a^* a = aa^*$, we assert that $xa^* a = xaa^*$ for any $x \in \chi_a$. The converse is obvious by [9], Theorem 2.2 (v), (ii), (xi), (vi), (iv), (x). □

Strongly EP elements are a special kind of EP elements. At the end of this article, through the research on solutions of some equations in χ_a, we present some necessary and sufficient conditions for an element a of a ring with involution to be a strongly EP element.

Theorem 2.13. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{\rm SEP}$ if and only if the equation $x = axa^*$ has at least one solution in χ_a.*

P r o o f. Writing $x = a^\dagger$, we complete the proof. In fact,

$$aa^\dagger a^* = a^\dagger aa^* = a^* = a^\dagger.$$

The converse follows from [10], Theorem 2.3 (xx), (xviii), (v). □

Theorem 2.14. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{\rm SEP}$ if and only if the equation $xa^\dagger a = xaa^*$ has at least one solution in χ_a.*

P r o o f. $\Rightarrow$ The conclusion holds if we take $x = a^\dagger$.

$\Leftarrow$ (1) If $x = a$, then $a = aa^\dagger a = a^2a^*$. It follows that $a \in R^{\rm SEP}$ by [10], Theorem 2.3 (xx).

(2) If $x = a^\#$, then $a^\# a^\dagger a = a^\# aa^*$. Then $a \in R^{\rm SEP}$ follows from [10], Theorem 2.3 (v).

(3) If $x = a^\dagger$, then $a^\dagger a^\dagger a = a^\dagger aa^* = a^*$. It is clear that $Ra^* = Ra^\dagger a^\dagger a \subseteq Ra$, which yields $a \in R^{\rm EP}$. The above equality implies

$$a^* = a^\dagger a^\dagger a = a^\# a^\# a = a^\# = a^\dagger.$$

(4) If $x = a^*$, then $a^* a^\dagger a = a^* aa^*$. Applying involution to $a^* a^\dagger a = a^* aa^*$, we get $a^\dagger a^2 = aa^* a$. It is immediate that

$$aR = aa^* aR = a^\dagger a^2 R \subseteq a^\dagger R,$$

which gives $a \in R^{\rm EP}$. We have $aa^* \in E(R)$, because $a = a^\dagger a^2 = aa^* a$. Hence $a^\dagger = a^*$.

(5) If $x = (a^\#)^*$, then $(a^\#)^* a^\dagger a = (a^\#)^* aa^*$. Taking involution of the above equality, we deduce that $a^\dagger aa^\# = aa^* a^\#$. It is easy to see that

$$a^\dagger R = a^\dagger aa^\# aa^\dagger R \subseteq a^\dagger aa^\# R = aa^* a^\# R \subseteq aR,$$

which leads to $a \in R^{\rm EP}$. On the other hand,

$$a^\dagger = a^\dagger aa^\dagger = a^\dagger aa^\# = aa^* a^\#.$$

Post-multiplying by a, we verify that

$$a^\dagger a = aa^* a^\# a = aa^* a^\dagger a = aa^*.$$

This clearly forces $aa^\dagger = aa^*$. That is, $a^\dagger = a^*$.

(6) If $x = (a^\dagger)^*$, then $(a^\dagger)^* a^\dagger a = (a^\dagger)^* aa^*$. Thus, $a \in R^{\rm SEP}$ by [10], Theorem 2.3 (xvi). □

Theorem 2.15. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{\rm SEP}$ if and only if the equation $axa^* = xa^\dagger a$ has at least one solution in χ_a.*

P r o o f. $\Rightarrow$ The result holds if we take $x = a$. In fact,

$$a^2a^* = a^2a^\dagger = a^2a^\# = aa^\# a = aa^\dagger a.$$

$\Leftarrow$ (1) If $x = a$, then $a^2a^* = aa^\dagger a = a$. We thus get $Ra = a^2a^* \subseteq Ra^*$, which shows that $a \in R^{\mathrm{EP}}$. It follows that

$$aa^\dagger = aa^\# = a^\# a = a^\# a^2 a^* = aa^*.$$

Therefore, $a^\dagger = a^*$.

(2) If $x = a^\#$, then $aa^\# a^* = a^\# a^\dagger a$. Observe that

$$Ra = Raa^\dagger a = Ra^2a^\# a^\dagger a \subseteq Ra^\# a^\dagger a = Raa^\# a^* \subseteq Ra^*,$$

which yields $a \in R^{\mathrm{EP}}$. According to the above, we have

$$aa^* = aa^\# aa^* = aa^\# a^\dagger a = aa^\# a^\# a = aa^\# = aa^\dagger.$$

Hence $a^\dagger = a^*$.

(3) If $x = a^\dagger$, then $aa^\dagger a^* = a^\dagger a^\dagger a$. Taking involution of the equality, we know $a^2a^\dagger = a^\dagger a(a^\dagger)^*$. It is evident that

$$aR = a^2a^\# R = a^2a^\dagger aa^\# R \subseteq a^2a^\dagger R = a^\dagger a(a^\dagger)^* R \subseteq a^\dagger R,$$

which proves that $a \in R^{\mathrm{EP}}$. It remains to show that $a^\dagger = a^*$. We need only to prove that $aa^* = aa^\dagger$. In fact,

$$aa^* = a^2a^\# a^* = a^2a^\dagger a^* = aa^\dagger a^\dagger a = aa^\# a^\# a = aa^\# = aa^\dagger.$$

(4) If $x = a^*$, then

$$aa^*a^* = a^*a^\dagger a = a^*(a^\dagger a)^* = (a^\dagger a^2)^*.$$

Applying involution to the above equality, we assert $a^\dagger a^2 = a^2a^*$. It is easy to check that

$$Ra = Ra^\# a^2 = Ra^\# aa^\dagger a^2 \subseteq Ra^\dagger a^2 = Ra^2a^* \subseteq Ra^*,$$

which implies $a \in R^{\mathrm{EP}}$. Moreover,

$$aa^\dagger = aa^\# = a^\# a = (a^\#)^2a^2 = a^\# a^\dagger a^2 = a^\# a^2a^* = aa^*.$$

This means that $a^\dagger = a^*$.

(5) If $x = (a^\#)^*$, then $a(a^\#)^*a^* = (a^\#)^*a^\dagger a$. Taking involution of the equality, we deduce $aa^\# a^* = a^\dagger aa^\#$. Pre-multiplying by a, we see that

$$aa^* = a^2a^\# a^* = aa^\dagger aa^\# = aa^\#.$$

It is straightforward that

$$Ra^{\#} = Ra^{\#}aa^{\#} \subseteq Raa^{\#} = Raa^* \subseteq Ra^*,$$

which gives $a \in R^{\mathrm{EP}}$. Furthermore, $aa^* = aa^{\#} = aa^{\dagger}$. Consequently, $a^{\dagger} = a^*$.

(6) If $x = (a^{\dagger})^*$, then

$$a(a^{\dagger})^* a^* = (a^{\dagger})^* a^{\dagger} a = (a^{\dagger} a a^{\dagger})^* = (a^{\dagger})^*.$$

On the other hand, $a(a^{\dagger})^* a^* = a^2 a^{\dagger}$. That is, $a^2 a^{\dagger} = (a^{\dagger})^*$. Pre-multiplying by a^*, we find out that $a^* a^2 a^{\dagger} = a^*(a^{\dagger})^* = a^{\dagger} a$. It is evident that

$$Ra = Raa^{\dagger}a \subseteq Ra^{\dagger}a = Ra^* a^2 a^{\dagger} \subseteq Ra^{\dagger},$$

which yields $a \in R^{\mathrm{EP}}$. Next, we only need to show that $aa^* = aa^{\dagger}$. In fact,

$$aa^{\dagger} = a^* a^2 a^{\dagger} = a^* a^2 a^{\#} = a^* a.$$

□

Theorem 2.16. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{\mathrm{SEP}}$ if and only if the equation $a^* x a = x a a^{\dagger}$ has at least one solution in χ_a.*

P r o o f. $\Rightarrow$ Taking $x = a$, we complete the proof. In fact,

$$a^* a^2 = a^{\#} a^2 = a^2 a^{\#} = a^2 a^{\dagger}.$$

$\Leftarrow$ (1) If $x = a$, then $a^* a^2 = a^2 a^{\dagger}$. It is clear that

$$Ra^{\dagger} = Ra^{\dagger}aa^{\dagger} \subseteq Raa^{\dagger} = Ra^{\#}a^2 a^{\dagger} \subseteq Ra^2 a^{\dagger} = Ra^* a^2 \subseteq Ra,$$

which shows that $a \in R^{\mathrm{EP}}$. To the dual with $a^{\dagger} = a^*$, we note that

$$a^* a = a^* a^2 a^{\#} = a^2 a^{\dagger} a^{\#} = a^2 (a^{\#})^2 = a^{\#} a = a^{\dagger} a.$$

(2) If $x = a^{\#}$, then $a^* a^{\#} a = a^{\#} a a^{\dagger}$. It is obvious that

$$\begin{aligned} Ra &= Raa^{\#}a = Ra^2(a^{\#})^2 a \subseteq Ra(a^{\#})^2 a = Raa^{\dagger}a(a^{\#})^2 a \\ &= R(a^{\dagger})^* a^* a^{\#} a \subseteq Ra^* a^{\#} a = Ra^{\#} a a^{\dagger} \subseteq Ra^{\dagger}, \end{aligned}$$

which implies $a \in R^{\mathrm{EP}}$. We also have

$$a^*a = a^*a^\# a^2 = a^\# aa^\dagger a = a^\# a = a^\dagger a.$$

Consequently, $a^* = a^\dagger$.

(3) If $x = a^\dagger$, then $a^*a^\dagger a = a^\dagger aa^\dagger = a^\dagger$. It follows that $Ra^\dagger = Ra^*a^\dagger a \subseteq Ra$, which gives $a \in R^{\mathrm{EP}}$. Furthermore,

$$a^\dagger a = a^*a^\dagger aa = a^*a^\# a^2 = a^*a.$$

Hence $a^\dagger = a^*$.

(4) If $x = a^*$, then

$$a^*a^*a = a^*aa^\dagger = a^*(aa^\dagger)^* = (aa^\dagger a)^* = a^*.$$

It is obvious that $a \in R^{\mathrm{SEP}}$ by [10], Theorem 2.3 (xix).

(5) If $x = (a^\#)^*$, then

$$a^*(a^\#)^*a = (a^\#)^*aa^\dagger = (a^\#)^*(aa^\dagger)^* = (aa^\dagger a^\#)^*.$$

It follows from [10], Theorem 2.3 (vi) that $a \in R^{\mathrm{SEP}}$.

(6) If $x = (a^\dagger)^*$, then $a^*(a^\dagger)^*a = (a^\dagger)^*aa^\dagger$. On the other hand, $a^\dagger a^2 = a^*(a^\dagger)^*a$. Thus, $a^\dagger a^2 = (a^\dagger)^*aa^\dagger$. Pre-multiplying by a^*, we get

$$a^*a^\dagger a^2 = a^*(a^\dagger)^*aa^\dagger = a^\dagger a^2 a^\dagger.$$

It is evident that

$$\begin{aligned} Ra^\dagger &= Ra^\dagger aa^\dagger \subseteq Raa^\dagger = Ra^\# a^2 a^\dagger \subseteq Ra^2 a^\dagger \\ &= Raa^\dagger a^2 a^\dagger \subseteq Ra^\dagger a^2 a^\dagger = Ra^*a^\dagger a^2 \subseteq Ra, \end{aligned}$$

which proves that $a \in R^{\mathrm{EP}}$. Moreover,

$$a^*a = a^*a^\# a^2 = a^*a^\dagger a^2 = a^\dagger a^2 a^\dagger = a^\# a^2 a^\# = a^\# a = a^\dagger a.$$

This means $a^* = a^\dagger$. □

Applying involution to the equations in Theorems 2.15 and 2.16, we have the following corollaries.

Corollary 2.17. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{\mathrm{SEP}}$ if and only if the equation $axa^* = a^\dagger ax$ has at least one solution in χ_a.*

Corollary 2.18. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{\mathrm{SEP}}$ if and only if the equation $a^*xa = aa^{\dagger}x$ has at least one solution in χ_a.*

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