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BOUNDARY VALUE PROBLEMS FOR THE SCHRÖDINGER EQUATION INVOLVING THE HENSTOCK-KURZWEIL INTEGRAL

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Abstract. In the present paper, we investigate the existence of solutions to boundary value problems for the one-dimensional Schrödinger equation $-y''+qy=f$, where q and f are Henstock-Kurzweil integrable functions on $[a,b]$. Results presented in this article are generalizations of the classical results for the Lebesgue integral.

Keywords: Henstock-Kurzweil integral; Schrödinger operator; ACG$_*$-function; bounded variation function

MSC 2010: 26A39, 34B24, 26A45

1. Introduction

Let q be a real valued function defined on $[a,b]$ and let L be the one-dimensional Schrödinger operator defined by $Ly=-y''+qy$. In [5] an existence and uniqueness theorem is given for initial value problems with the differential equation $Ly=f$, where q and f are Henstock-Kurzweil integrable functions on $[a,b]$. In the present paper we use this theorem in order to give a solution to the boundary value problem

$$\begin{cases} Ly=f, \\ m_1y(a)+n_1y'(a)+p_1y(b)+q_1y'(b)=h_1, \\ m_2y(a)+n_2y'(a)+p_2y(b)+q_2y'(b)=h_2, \end{cases} \tag{1.1}$$

where $m_i,n_i,p_i,q_i,h_i\in\mathbb{C}$, $i=1,2$ and q, f are Henstock-Kurzweil integrable functions on $[a,b]$.

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2. Preliminaries

We say that a function $f\colon [a,b] \to \mathbb{C}$ is Henstock-Kurzweil integrable (shortly, HK-integrable), if there exists a number $A \in \mathbb{C}$ such that for each $\varepsilon > 0$, there exists a function $\gamma_\varepsilon\colon [a,b] \to (0,\infty)$ (named a gauge) for which

$$\left|\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - A\right| < \varepsilon$$

for any partition $P = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ such that $t_i \in [x_{i-1}, x_i]$ and $[x_{i-1}, x_i] \subseteq [t_i - \gamma_\varepsilon(t_i), t_i + \gamma_\varepsilon(t_i)]$ for all $i = 1, 2, \ldots, n$. The number A is the integral of f on $[a,b]$ and is denoted by $\int_a^b f = A$. We denote by $\mathrm{HK}([a,b])$ the set of Henstock-Kurzweil integrable functions on $[a,b]$. This set is a linear space over the field $\mathbb{C}$, furthermore $\mathrm{HK}([a,b])$ is a semi-normed space with the Alexiewicz semi-norm defined as

$$\|f\|_{[a,b]} = \sup_{[c,d]\subseteq[a,b]} \left|\int_c^d f(t)\,\mathrm{d}t\right|.$$

The variation of $\varphi\colon [a,b] \to \mathbb{C}$ is defined by

$$V_{[a,b]}\varphi = \sup\left\{\sum_{i=1}^{n} |\varphi(x_i) - \varphi(x_{i-1})|\colon\ a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\right\}.$$

The function φ is of bounded variation on $[a,b]$ if $V_{[a,b]}\varphi < \infty$. The space of all functions of bounded variation on $[a,b]$ is denoted by $\mathrm{BV}([a,b])$.

Theorem 2.1 (Multiplier Theorem, [1], Theorem 10.12). *If $f \in \mathrm{HK}([a,b])$ and $g \in \mathrm{BV}([a,b])$ then the product fg belongs to $\mathrm{HK}([a,b])$ and*

$$\int_a^b fg = F(b)g(b) - \int_a^b F\,\mathrm{d}g,$$

where F is the indefinite integral $F(x) = \int_a^x f$ of f on $[a,b]$, and the latter integral is a Riemann-Stieltjes one.

Next, a type of Hölder inequality for HK-integrable functions is given.

Theorem 2.2 ([7], Lemma 24). *If $f \in \mathrm{HK}([a,b])$ and $g \in \mathrm{BV}([a,b])$, then*

$$\left|\int_a^b fg\right| \leqslant \inf_{t\in[a,b]} |g(t)| \left|\int_a^b f(t)\,\mathrm{d}t\right| + \|f\|_{[a,b]} V_{[a,b]} g.$$

A function $F\colon [a,b] \to \mathbb{C}$ is absolutely continuous (respectively, absolutely continuous in the restricted sense) on a set $E \subseteq [a,b]$, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{i=1}^{s} |F(d_i) - F(c_i)| < \varepsilon, \quad \text{respectively,} \quad \sum_{i=1}^{s} \sup\{|F(x) - F(y)|\colon\ x, y \in [c_i, d_i]\} < \varepsilon$$

whenever $\{[c_i, d_i]\}_{i=1}^{s}$ is a collection of non-overlapping intervals with endpoints in E and such that $\sum\limits_{i=1}^{s} (d_i - c_i) < \delta$. The space of absolutely continuous functions on E is denoted by $\mathrm{AC}(E)$ and the space of absolutely continuous functions in the restricted sense on E is denoted by $\mathrm{AC}_*(E)$.

The function F is generalized absolutely continuous in the restricted sense on $[a,b]$ ($F \in \mathrm{ACG}_*([a,b])$), if F is continuous on $[a,b]$ and there exists a countable collection $(E_n)_{n=1}^{\infty}$ of subsets of $[a,b]$ such that $[a,b] = \bigcup\limits_{i=1}^{\infty} E_n$ and $F \in \mathrm{AC}_*(E_n)$ for all $n \in \mathbb{N}$. This concept leads to a very strong version of the Fundamental Theorem of Calculus:

Theorem 2.3 (Fundamental Theorem of Calculus, [3]). *Let $f, F\colon [a,b] \to \mathbb{C}$ be functions and let $c \in [a,b]$.*

(1) *If $f \in \mathrm{HK}([a,b])$ and $F(x) = \int_c^x f$ for all $x \in [a,b]$, then $F \in \mathrm{ACG}_*([a,b])$ and $F' = f$ almost everywhere on $[a,b]$. In particular, if f is continuous at $x \in [a,b]$, then $F'(x) = f(x)$.*

(2) *If $F \in \mathrm{ACG}_*([a,b])$ and $F' = f$ almost everywhere on $[a,b]$, then $f \in \mathrm{HK}([a,b])$ and $F(x) = \int_c^x f + F(c)$ for all $x \in [a,b]$.*

(3) *$F \in \mathrm{ACG}_*([a,b])$ if and only if F' exists almost everywhere on $[a,b]$ and $\int_c^x F' = F(x) - F(c)$ for all $x \in [a,b]$.*

The following result gives a formula of integration by parts for functions in $\mathrm{ACG}_*([a,b])$.

Corollary 2.4 (Integration by parts). *If $u \in \mathrm{ACG}_*([a,b])$ and $v \in \mathrm{AC}([a,b])$ then $u'v \in \mathrm{HK}([a,b])$, $uv' \in L([a,b])$ and*

$$\int_a^b u'(t)v(t)\,\mathrm{d}t = u(b)v(b) - u(a)v(a) - \int_a^b u(t)v'(t)\,\mathrm{d}t.$$

Proof. By Theorem 2.3, u' exists almost everywhere on $[a,b]$, $u' \in \mathrm{HK}([a,b])$ and $\int_a^x u' = u(x) - u(a)$ for all $x \in [a,b]$. Then by [3], Theorem 12.8, $u'v \in \mathrm{HK}([a,b])$

and

$$\begin{aligned}\int_a^b u'(t)v(t)\,\mathrm{d}t &= \int_a^b u'(t)\,\mathrm{d}t\,v(b) - \int_a^b \Big(\int_a^s u'(t)\,\mathrm{d}t\Big) v'(s)\,\mathrm{d}s \\ &= (u(b)-u(a))v(b) - \int_a^b (u(s)-u(a))v'(s)\,\mathrm{d}s \\ &= u(b)v(b) - u(a)v(b) - \int_a^b u(s)v'(s)\,\mathrm{d}s + u(a)(v(b)-v(a)) \\ &= u(b)v(b) - u(a)v(a) - \int_a^b u(s)v'(s)\,\mathrm{d}s.\end{aligned}$$

□

Proposition 2.5. *If $f, g \in \mathrm{ACG}_*([a,b])$ then $fg \in \mathrm{ACG}_*([a,b])$.*

Proof. Take $M > 0$ such that $|f(x)| \leqslant M$ and $|g(x)| \leqslant M$ for all $x \in [a,b]$. Let (E_n), (G_n) be such that $[a,b] = \bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} G_n$ and for which $f \in \mathrm{AC}_*(E_n)$ and $g \in \mathrm{AC}_*(G_n)$ for all $n \in \mathbb{N}$. Define $\mathcal{V} = \{E_n \cap G_m\colon\ n, m \in \mathbb{N}$ and $E_n \cap G_m \neq \emptyset\}$. Then $[a,b] = \bigcup_{V\in\mathcal{V}} V$ and $f, g \in \mathrm{AC}_*(V)$ for all $V \in \mathcal{V}$. Let $\varepsilon > 0$. There exist $\delta_f > 0$ and $\delta_g > 0$ such that

$$\sum_{i=1}^s \sup\{|f(x)-f(y)|\colon\ x,y \in [c_i,d_i]\} < \varepsilon$$

and

$$\sum_{i=1}^s \sup\{|g(x)-g(y)|\colon\ x,y \in [c_i^*,d_i^*]\} < \varepsilon,$$

whenever $\{[c_i,d_i]\}_{i=1}^s$ and $\{[c_i^*,d_i^*]\}_{i=1}^s$ are collections of non-overlapping intervals that have endpoints in V and satisfy

$$\sum_{i=1}^s (d_i - c_i) < \delta_f \quad\text{and}\quad \sum_{i=1}^s (d_i^* - c_i^*) < \delta_g.$$

Let $\delta = \min\{\delta_f, \delta_g\}$, if $\sum\limits_{i=1}^s (d_i - c_i) < \delta$, then

$$\begin{aligned}&\sum_{i=1}^s \sup_{x,y\in[c_i,d_i]} |f(x)g(x) - f(y)g(y)| \\ &\qquad\leqslant M\Big[\sum_{i=1}^s \sup_{x,y\in[c_i,d_i]} |f(x)-f(y)| + \sum_{i=1}^s \sup_{x,y\in[c_i,d_i]} |g(x)-g(y)|\Big] < 2M\varepsilon.\end{aligned}$$

□

To finish this section we enunciate a well known result about integral transforms, see for example [4].

Theorem 2.6. *If G is a continuous complex function defined on $[a,b]\times[a,b]$ then Ψ, defined on $L^2([a,b])$ as*

$$\Psi(f)(x)=\int_a^b G(x,t)f(t)\,\mathrm{d}t,$$

satisfies the inclusion $\Psi(L^2([a,b]))\subseteq C([a,b])$ and

$$\Psi\colon\ (L^2([a,b]),\|\cdot\|_2)\to(L^2([a,b]),\|\cdot\|_2)$$

is a compact linear operator. Moreover, if $\Psi\neq 0$, Ψ is symmetric, and $\Psi(L^2([a,b]))$ is a dense subspace of $L^2([a,b])$, then there exist two sequences (λ_n) in $\mathbb{R}\setminus\{0\}$ and (ϕ_n) in $\Psi(L^2([a,b]))\setminus\{0\}$ such that
(1) $\|\phi_n\|_2=1$, $\Psi(\phi_n)=\lambda_n\phi_n$ *and* $|\lambda_{n+1}|\leqslant|\lambda_n|$ *for all* $n\in\mathbb{N}$,
(2) $\lim\limits_{n\to\infty}|\lambda_n|=0$,
(3) $\{\phi_1,\phi_2,\ldots\}$ *is a complete orthonormal system in* $L^2([a,b])$.

3. The existence and uniqueness theorem

The Wronskian of $u_1,u_2\in C^1([a,b])$ at $x\in[a,b]$ is given by

$$W_x(u_1,u_2)=u_1(x)u_2'(x)-u_1'(x)u_2(x).$$

It is well known that if $W_c(u_1,u_2)\neq 0$ for some $c\in[a,b]$, then u_1, u_2 are linearly independent. We consider

$$\mathcal{A}=\{y\in\mathrm{AC}([a,b])\colon\ y'\in\mathrm{ACG}_*([a,b])\}.$$

This set is a linear space over $\mathbb{C}$. Note that if $y\in\mathcal{A}$, then y' exists and is continuous on $[a,b]$, $|y'|$ is integrable, y is of bounded variation, y'' exists almost everywhere on $[a,b]$, and

$$\int_a^x y''=y'(x)-y'(a) \tag{3.1}$$

for all $x\in[a,b]$, where the integral is the HK-integral. The equality in (3.1) is important in order to analyse the second order differential equation $-y''+qy=f$ (see [5], Lemma 3.1). Now, we define the linear space

$$\mathcal{A}_*=\{y\in\mathcal{A}\colon\ Ly=0\text{ a.e. on }[a,b]\}$$

over the field $\mathbb{C}$. By [5], Theorem 3.2, we have that if $y_1, y_2 \in \mathcal{A}_*$ are linearly independent then $W_x(y_1, y_2) \neq 0$ for all $x \in [a,b]$. Moreover, the Wronskian of two elements $y_1, y_2 \in \mathcal{A}_*$ is a constant function on $[a,b]$, indeed; by Proposition 2.5, $y_1 y_2' - y_2 y_1' \in \mathrm{ACG}_*([a,b])$ and since $(y_1 y_2' - y_2 y_1')' = y_1 y_2'' - y_2 y_1'' = y_1 q y_2 - y_2 q y_1 = 0$ a.e. on $[a,b]$, it follows by Theorem 2.3, case (2) that

$$\begin{aligned} y_1(x)y_2'(x) - y_2(x)y_1'(x) &= \int_a^x (y_1 y_2' - y_2 y_1')'(t)\,\mathrm{d}t + y_1(a)y_2'(a) - y_2(a)y_1'(a) \\ &= y_1(a)y_2'(a) - y_2(a)y_1'(a) \end{aligned}$$

for all $x \in [a,b]$.

Proposition 3.1. $\dim \mathcal{A}_* = 2$.

P r o o f. For some fixed $c \in [a,b]$ we can find, by [5], Theorem 3.2, $y_1, y_2 \in \mathcal{A}$ such that $Ly_i = 0$ a.e. on $[a,b]$ for $i = 1,2$, $y_1(c) = 1$, $y_1'(c) = 0$, $y_2(c) = 0$ and $y_2'(c) = 1$. Therefore, $y_1, y_2 \in \mathcal{A}_*$ and $W_c(y_1, y_2) = 1$, so y_1, y_2 are linearly independent. Now, let $y \in \mathcal{A}_*$ and define w on $[a,b]$ as $w = (y(c)/y_1(c))y_1 + (y'(c)/y_2'(c))y_2 - y$; then $w \in \mathcal{A}$, $w(c) = w'(c) = 0$, and $Lw = 0$ a.e. on $[a,b]$. Consequently, by using again [5], Theorem 3.2 we have that $w = 0$, thus $y = (y(c)/y_1(c))y_1 + (y'(c)/y_2'(c))y_2$, i.e. $\{y_1, y_2\}$ is a basis of $\mathcal{A}_*$. $\square$

The boundary conditions in the problem (1.1) can be written as $Uy = h$, where

$$Uy = \begin{pmatrix} m_1 & n_1 & p_1 & q_1 \\ m_2 & n_2 & p_2 & q_2 \end{pmatrix} \begin{pmatrix} y(a) \\ y'(a) \\ y(b) \\ y'(b) \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}. \tag{3.2}$$

It is clear that U is a linear operator.

Theorem 3.2 (Theorem of the alternative). *Let* $h \in \mathbb{C}^2$ *and* $f \in \mathrm{HK}([a,b])$. *Consider the problems*

$$\text{(A)} \begin{cases} Ly = f \text{ a.e.,} \\ U(y) = h, \end{cases} \qquad \text{(B)} \begin{cases} Ly = 0 \text{ a.e.,} \\ U(y) = 0. \end{cases}$$

Then, either

(1) *the problem* (A) *has a unique solution in* $\mathcal{A}$, *or*

(2) *the problem* (B) *has a nonzero solution in* $\mathcal{A}$.

P r o o f. Let $\{y_1, y_2\}$ be a basis of $\mathcal{A}_*$.

Case I: $\{Uy_1, Uy_2\}$ is a linearly dependent set. Let $\alpha, \beta \in \mathbb{C}$ be such that $(\alpha, \beta) \neq (0,0)$ and $\alpha Uy_1 + \beta Uy_2 = 0$. Then $\alpha y_1 + \beta y_2 \in \mathcal{A}_*$, $\alpha y_1 + \beta y_2 \neq 0$ and $U(\alpha y_1 + \beta y_2) = 0$. Therefore $\alpha y_1 + \beta y_2$ is a nonzero solution of the problem (B). If $y \in \mathcal{A}$ is a solution of the problem (A) and $z = y + \alpha y_1 + \beta y_2$, then $z \in \mathcal{A}$ is also a solution of the problem (A) and $z \neq y$.

Case II: $\{Uy_1, Uy_2\}$ is a linearly independent set. By [5], Theorem 3.2, there exists $\widetilde{y} \in \mathcal{A}$ such that $L\widetilde{y} = f$ a.e. on $[a,b]$. Since $\det(Uy_1, Uy_2) \neq 0$ it follows that there exist $a_1, a_2 \in \mathbb{C}$ such that $h - U\widetilde{y} = a_1 Uy_1 + a_2 Uy_2$. Thus $\widetilde{y} + a_1 y_1 + a_2 y_2$ is a solution of the problem (A). Now, let $y \in \mathcal{A}$ be another solution of the problem (A). Then $\widetilde{y} - y \in \mathcal{A}_*$ and so there exist $\alpha, \beta \in \mathbb{C}$ such that $y - \widetilde{y} = \alpha y_1 + \beta y_2$. This implies that $h - U\widetilde{y} = \alpha Uy_1 + \beta Uy_2$ and hence $\alpha = a_1$ and $\beta = a_2$, from which $y = \widetilde{y} + a_1 y_1 + a_2 y_2$.

Finally, if $z \in \mathcal{A}$ is a solution of the problem (B) then there exist $\lambda, \mu \in \mathbb{C}$ such that $z = \lambda y_1 + \mu y_2$ and $\lambda Uy_1 + \mu Uy_2 = 0$, therefore $\lambda = \mu = 0$, i.e. $z = 0$. □

Remark 3.3. Let $h \in \mathbb{C}^2$ and $f \in \mathrm{HK}([a,b])$. If the problem (B) has only a trivial solution in $\mathcal{A}$ and $\{y_1, y_2\}$ is a basis of $\mathcal{A}_*$ then from Case I of Theorem 3.2 it follows that $\det(Uy_1, Uy_2) \neq 0$. Thus, there exist constants $\alpha, \beta \in \mathbb{C}$ such that $\alpha Uy_1 + \beta Uy_2 = h$. Therefore, if y is a solution of the problem

$$\begin{cases} Ly = f \text{ a.e.,} \\ U(y) = 0, \end{cases}$$

then $y + \alpha y_1 + \beta y_2$ is the unique solution of the problem (A).

Lemma 3.4. *Let $\{y_1, y_2\}$ be a basis of $\mathcal{A}_*$ such that $W(y_1, y_2) = 1$ and let $f \in \mathrm{HK}([a,b])$. If $z\colon [a,b] \to \mathbb{C}$ is defined as*

$$z(x) = y_1(x) \int_a^x y_2(t) f(t)\,\mathrm{d}t - y_2(x) \int_a^x y_1(t) f(t)\,\mathrm{d}t \tag{3.3}$$

then $z \in \mathcal{A}$,

$$z'(x) = y_1'(x) \int_a^x y_2(t) f(t)\,\mathrm{d}t - y_2'(x) \int_a^x y_1(t) f(t)\,\mathrm{d}t \tag{3.4}$$

and $Lz = f$ a.e. on $[a,b]$.

P r o o f. We know that y_1, y_2 are of bounded variation on $[a,b]$. Then by Theorem 2.1, $y_1 f$ and $y_2 f$ are HK-integrable on $[a,b]$ and hence z is well defined. Now, by Theorem 2.3 (1),

$$\int_a^{(\cdot)} y_1(t) f(t)\,\mathrm{d}t, \int_a^{(\cdot)} y_2(t) f(t)\,\mathrm{d}t \in \mathrm{ACG}_*([a,b]), \tag{3.5}$$

thus by Corollary 2.4,

$$\int_a^x (y_2(t)f(t))y_1(t)\,\mathrm{d}t = \Big(\int_a^x y_2(t)f(t)\,\mathrm{d}t\Big)y_1(x) - \int_a^x \Big(\int_a^s y_2(t)f(t)\,\mathrm{d}t\Big)y_1'(s)\,\mathrm{d}s$$

and

$$\int_a^x (y_1(t)f(t))y_2(t)\,\mathrm{d}t = \Big(\int_a^x y_1(t)f(t)\,\mathrm{d}t\Big)y_2(x) - \int_a^x \Big(\int_a^s y_1(t)f(t)\,\mathrm{d}t\Big)y_2'(s)\,\mathrm{d}s.$$

Consequently,

$$z(x) = \int_a^x \Big[y_1'(s)\int_a^s y_2(t)f(t)\,\mathrm{d}t - y_2'(s)\int_a^s y_1(t)f(t)\,\mathrm{d}t\Big]\,\mathrm{d}s \tag{3.6}$$

and so by Theorem 2.3 (1),

$$z'(x) = y_1'(x)\int_a^x y_2(t)f(t)\,\mathrm{d}t - y_2'(x)\int_a^x y_1(t)f(t)\,\mathrm{d}t$$

for all $x \in [a,b]$. Since the integrand in (3.6) is a continuous function, it follows that $z \in \mathrm{AC}([a,b])$. Now, considering the equality in (3.4), we have by (3.5) and Proposition 2.5 that $z' \in \mathrm{ACG}_*([a,b])$. Thus $z \in \mathcal{A}$. Consider $E \subseteq [a,b]$ with $m(E) = 0$ such that for each $x \in [a,b] \setminus E$,

$$\begin{aligned} &-y_1''(x) + q(x)y_1(x) = 0, && -y_2''(x) + q(x)y_2(x) = 0,\\ &\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x y_1(t)f(t)\,\mathrm{d}t = y_1(x)f(x), && \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x y_2(t)f(t)\,\mathrm{d}t = y_2(x)f(x). \end{aligned}$$

Let $x \in [a,b] \setminus E$. Then

$$z''(x) = y_1''(x)\int_a^x y_2(t)f(t)\,\mathrm{d}t - y_2''(x)\int_a^x y_1(t)f(t)\,\mathrm{d}t - W_x(y_1,y_2)f(x),$$

thus

$$\begin{aligned} -z''(x) + q(x)z(x) &= (-y_1''(x) + q(x)y_1(x))\int_a^x y_2(t)f(t)\,\mathrm{d}t\\ &\quad - (-y_2''(x) + q(x)y_2(x))\int_a^x y_1(t)f(t)\,\mathrm{d}t + f(x) = f(x). \end{aligned}$$

Therefore $Lz = f$ a.e. on $[a,b]$. □

Theorem 3.5. *Let $\{y_1, y_2\}$ be a basis of $\mathcal{A}_*$ such that $W(y_1, y_2) = 1$ and let $K\colon [a,b] \times [a,b] \to \mathbb{C}$ be defined as*

$$K(x,t) = \begin{cases} 0, & \text{if } a \leqslant x < t, \\ y_2(t)y_1(x) - y_1(t)y_2(x), & \text{if } t \leqslant x \leqslant b. \end{cases}$$

If the problem (B) *has only a trivial solution and $f \in \mathrm{HK}([a,b])$ then the unique solution $y \in \mathcal{A}$ of the problem*

$$\begin{cases} Ly = f \text{ a.e.}, \\ U(y) = 0, \end{cases} \tag{3.7}$$

is given by

$$y(x) = \int_a^b [K(x,t) + c_1(t)y_1(x) + c_2(t)y_2(x)]f(t)\,\mathrm{d}t,$$

where

$$c_1(t) = \frac{\det\Big((y_1(t)y_2(b) - y_2(t)y_1(b))\left[\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right] + (y_1(t)y_2'(b) - y_2(t)y_1'(b))\left[\begin{smallmatrix} q_1 \\ q_2 \end{smallmatrix}\right], Uy_2\Big)}{\det(Uy_1, Uy_2)} \tag{3.8}$$

and

$$c_2(t) = \frac{\det\Big(Uy_1,\ (y_1(t)y_2(b) - y_2(t)y_1(b))\left[\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right] + (y_1(t)y_2'(b) - y_2(t)y_1'(b))\left[\begin{smallmatrix} q_1 \\ q_2 \end{smallmatrix}\right]\Big)}{\det(Uy_1, Uy_2)} \tag{3.9}$$

for all $t \in [a,b]$.

Proof. Since y_1, y_2 are of bounded variation on $[a,b]$ it follows that $K(x,\cdot)$, c_1 and c_2 are of bounded variation on $[a,b]$ for all $x \in [a,b]$. Thus y is well defined. Let us consider the function z defined in (3.3). Then

$$y(x) = z(x) + y_1(x)\int_a^b c_1(t)f(t)\,\mathrm{d}t + y_2(x)\int_a^b c_2(t)f(t)\,\mathrm{d}t,$$

and so by Lemma 3.4, $y \in \mathcal{A}$ and

$$Ly = Lz + \int_a^b c_1(t)f(t)\,\mathrm{d}t\, Ly_1 + \int_a^b c_2(t)f(t)\,\mathrm{d}t\, Ly_2 = f$$

a.e. on $[a,b]$. On the other hand, observe that

$$UK(\cdot,t) = \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix}\begin{pmatrix} K(b,t) \\ K_1(b,t) \end{pmatrix} = \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix}\begin{pmatrix} y_2(t)y_1(b) - y_1(t)y_2(b) \\ y_2(t)y_1'(b) - y_1(t)y_2'(b) \end{pmatrix}$$

for all $t \in (a,b)$, where K_1 denotes the derivative of K with respect to the first variable. Let A be the matrix whose columns are Uy_1 and Uy_2, and let

$$M = \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} \quad \text{and} \quad B(t) = \begin{pmatrix} -K(b,t) \\ -K_1(b,t) \end{pmatrix}.$$

From (3.8) and (3.9) we have that $(c_1(t), c_2(t))$ is the unique solution of the linear system

$$AX = MB(t)$$

for all $t \in (a,b)$. Thus,

$$c_1(t)Uy_1 + c_2(t)Uy_2 = MB(t) = -UK(\cdot,t)$$

for all $t \in (a,b)$. Then

$$\begin{aligned} Uy &= \int_a^b [c_1(t)Uy_1 + c_2(t)Uy_2 + UK(\cdot,t)]f(t)\,\mathrm{d}t \\ &= \int_a^b [-UK(\cdot,t) + UK(\cdot,t)]f(t)\,\mathrm{d}t = 0. \end{aligned}$$

The uniqueness of the solution is obtained by the theorem of the alternative. □

4. The inverse of the Schrödinger operator

In the rest of this paper we will assume that the problem (B) has only a trivial solution.

Remark 4.1. Consider y_1, y_2, K, c_1 and c_2 as in Theorem 3.5. We set

$$G(x,t) = K(x,t) + c_1(t)y_1(x) + c_2(t)y_2(x)$$

and let

$$\mathcal{D}_* = \{y \in \mathcal{A}\colon\ Uy = 0\}.$$

Then $L\colon \mathcal{D}_* \to \mathrm{HK}([a,b])$ is invertible and its inverse $\Gamma\colon \mathrm{HK}([a,b]) \to \mathcal{D}_*$ is given by

$$\Gamma(f)(x) = \int_a^b G(x,t)f(t)\,\mathrm{d}t.$$

Indeed, if $y \in \mathcal{D}_*$ then $Ly \in \mathrm{HK}([a,b])$ by Theorems 2.1 and 2.3. Thus by Theorem 3.5, $\Gamma(L(y))$ is the unique solution of the problem

$$\begin{cases} Lz = Ly & \text{a.e.,} \\ U(z) = 0, \end{cases}$$

therefore $y = \Gamma(L(y))$. On the other hand, using Theorem 3.5 again, we have that $\Gamma(f) \in \mathcal{D}_*$ and $L(\Gamma(f)) = f$ a.e. on $[a,b]$ for all $f \in \mathrm{HK}([a,b])$.

Theorem 4.2. *If (f_n) is a sequence in* $\mathrm{HK}([a,b])$ *with* $\|f_n\|_{[a,b]} \leqslant 1$ *for all* $n \in \mathbb{N}$, *then there exist a subsequence* (f_{n_k}) *of* (f_n) *and* $g \in C([a,b])$ *such that* $\Gamma(f_{n_k}) \to g$ *uniformly on* $[a,b]$.

P r o o f. Let (f_n) be a sequence in $\mathrm{HK}([a,b])$ such that $\|f_n\|_{[a,b]} \leqslant 1$ for all $n \in \mathbb{N}$. Considering $\mathcal{F} = \{\Gamma(f_n)\colon n \in \mathbb{N}\}$, we prove that $\mathcal{F}$ is equicontinuous. Choose $M_1, M_2 > 0$ such that the variations of c_1, c_2, y_1 and y_2 on $[a,b]$ are bounded by M_1 and the functions y_1, y_2, y_1', y_2' are bounded by M_2. Let $\varepsilon > 0$; since G is continuous on $[a,b] \times [a,b]$ and y_1, y_2 are continuous on $[a,b]$, there exists $\delta_1 > 0$ such that if $x_1, x_2 \in [a,b]$ with $|x_2 - x_1| < \delta_1$ then

$$|G(x_2,a) - G(x_1,a)| < \frac{\varepsilon}{2}$$

and

$$|y_1(x_2) - y_1(x_1)| < \frac{\varepsilon}{16M_1}, \quad |y_2(x_2) - y_2(x_1)| < \frac{\varepsilon}{16M_1}.$$

Let $\delta = \min\{\varepsilon/(8M_2^2), \delta_1\}$. Take $n \in \mathbb{N}$ and $x_1, x_2 \in [a,b]$ with $|x_2 - x_1| < \delta$. Without loss of generality we may suppose that $x_1 < x_2$. By Theorem 2.2,

$$\begin{aligned}
|\Gamma(f_n)(x_2) - \Gamma(f_n)(x_1)| &= \left|\int_a^b [G(x_2,t) - G(x_1,t)] f_n(t)\,\mathrm{d}t\right| \\
&\leqslant \inf_{t\in[a,b]} |G(x_2,t) - G(x_1,t)| \left|\int_a^b f_n\right| + \|f_n\|_{[a,b]} V_{[a,b]}[G(x_2,\cdot) - G(x_1,\cdot)] \\
&\leqslant |G(x_2,a) - G(x_1,a)| + V_{[a,b]}[G(x_2,\cdot) - G(x_1,\cdot)] \\
&< \frac{\varepsilon}{2} + V_{[a,b]}[G(x_2,\cdot) - G(x_1,\cdot)].
\end{aligned}$$

Now, observe that

$$\begin{aligned}
V_{[a,b]}[G(x_2,\cdot) - G(x_1,\cdot)] \leqslant V_{[a,b]}[K(x_2,\cdot) - K(x_1,\cdot)] + |y_1(x_2) \\
- y_1(x_1)| V_{[a,b]} c_1 + |y_2(x_2) - y_2(x_1)| V_{[a,b]} c_2
\end{aligned}$$

and

$$\begin{aligned}
V_{[a,b]}[K(x_2,\cdot) - K(x_1,\cdot)] &= V_{[a,x_1]}[(y_1(x_2) - y_1(x_1))y_2 - (y_2(x_2) - y_2(x_1))y_1] \\
&\quad + V_{[x_1,x_2]}[y_1(x_2)y_2 - y_2(x_2)y_1] \\
&\leqslant |y_1(x_2) - y_1(x_1)| V_{[a,b]} y_2 + |y_2(x_2) - y_2(x_1)| V_{[a,b]} y_1 \\
&\quad + |y_1(x_2)| V_{[x_1,x_2]} y_2 + |y_2(x_2)| V_{[x_1,x_2]} y_1.
\end{aligned}$$

Moreover, since y_1, y_2 are differentiable on $[a,b]$ and y_1', y_2' are bounded by M_2, we have $V_{[x_1,x_2]} y_i \leqslant M_2(x_2 - x_1)$, $i = 1,2$. Thus

$$\begin{aligned}
V_{[a,b]}[G(x_2,\cdot) - G(x_1,\cdot)] &\leqslant |y_1(x_2) - y_1(x_1)| 2M_1 \\
&\quad + |y_2(x_2) - y_2(x_1)| 2M_1 + 2M_2^2(x_2 - x_1) < \frac{\varepsilon}{2}.
\end{aligned}$$

Therefore, $|\Gamma(f_n)(x_2)-\Gamma(f_n)(x_1)| < \varepsilon$. Repeating the same procedure as above we find a constant $M > 0$ such that $|\Gamma(f_n)(x)| \leqslant M$ for all $x \in [a,b]$ and $n \in \mathbb{N}$. Then $\overline{\{\Gamma(f_n)(x)\}}$ is a compact set in $\mathbb{C}$ for all $x \in [a,b]$. Consequently, from Arzelà-Ascoli theorem, $\overline{\mathcal{F}}$ is a compact set in $C([a,b])$, therefore there exists a subsequence (f_{n_k}) of (f_n) and $g \in \overline{\mathcal{F}}$ such that $\Gamma(f_{n_k})$ converges uniformly to g on $[a,b]$. $\square$

Remark 4.3. The operator $\Gamma\colon (\mathrm{HK}([a,b]), \|\cdot\|_{[a,b]}) \to \mathcal{D}_* \subseteq (C([a,b]), \|\cdot\|_\infty)$ is compact.

Let $y \in \mathrm{BV}([a,b])$ and $f \in \mathrm{HK}([a,b])$. We denote the integrals $\int_a^b y(t)\overline{f(t)}\,\mathrm{d}t$ and $\int_a^b f(t)\overline{y(t)}\,\mathrm{d}t$ by $\langle y, f\rangle$ and $\langle f, y\rangle$, respectively. The following properties hold:

(1) $\langle y+u, f\rangle = \langle y, f\rangle + \langle u, f\rangle$ and $\langle y, f+g\rangle = \langle y, f\rangle + \langle y, g\rangle$ for all $g \in \mathrm{HK}([a,b])$ and $u \in \mathrm{BV}([a,b])$.

(2) $\langle \alpha y, f\rangle = \alpha\langle y, f\rangle$ and $\langle y, \alpha f\rangle = \overline{\alpha}\langle y, f\rangle$ for all $\alpha \in \mathbb{C}$.

(3) $\langle y, f\rangle = \overline{\langle f, y\rangle}$.

(4) $|\langle y, f\rangle| \leqslant \|y\|_{\mathrm{BV}}\|f\|_{[a,b]}$, where $\|y\|_{\mathrm{BV}} = |y(a)| + V_{[a,b]}y$. This inequality is true by Theorem 2.2.

Proposition 4.4. *If $y, z \in \mathcal{A}$ then*

$$\langle Ly, z\rangle = W_b(y, \overline{z}) - W_a(y, \overline{z}) + \langle y, Lz\rangle.$$

P r o o f. First note that

$$\langle Ly, z\rangle = -\int_a^b y''(t)\overline{z(t)}\,\mathrm{d}t + \int_a^b q(t)y(t)\overline{z(t)}\,\mathrm{d}t.$$

By Corollary 2.4,

$$\int_a^b y''(t)\overline{z(t)}\,\mathrm{d}t = y'(b)\overline{z(b)} - y'(a)\overline{z(a)} - \int_a^b y'(t)\overline{z'(t)}\,\mathrm{d}t$$

and

$$\int_a^b \overline{z''(t)}y(t)\,\mathrm{d}t = \overline{z'(b)}y(b) - \overline{z'(a)}y(a) - \int_a^b \overline{z'(t)}y'(t)\,\mathrm{d}t.$$

Therefore

$$\begin{aligned}\langle Ly, z\rangle = &-y'(b)\overline{z(b)} + y'(a)\overline{z(a)} + \int_a^b y'(t)\overline{z'(t)}\,\mathrm{d}t + \int_a^b q(t)y(t)\overline{z(t)}\,\mathrm{d}t \\ = &-y'(b)\overline{z(b)} + y'(a)\overline{z(a)} + \overline{z'(b)}y(b) - \overline{z'(a)}y(a) - \int_a^b \overline{z''(t)}y(t)\,\mathrm{d}t \\ &+ \int_a^b q(t)y(t)\overline{z(t)}\,\mathrm{d}t = W_b(y,\overline{z}) - W_a(y,\overline{z}) + \langle y, Lz\rangle.\end{aligned}$$

$\square$

Remark 4.5. Let $y, z \in \mathcal{A}$; if $y(a) = y(b) = y'(a) = y'(b) = 0$ or $z(a) = z(b) = z'(a) = z'(b) = 0$ then

$$\langle Ly, z\rangle = \langle y, Lz\rangle.$$

Let $f \in L^2([a,b])$ then by taking $g = 1 \in L^2([a,b])$ we have $f = fg \in L([a,b])$. Thus,

$$L^2([a,b]) \subseteq \mathrm{HK}([a,b]). \tag{4.1}$$

In the following theorem we use the notation $\overline{D}^{\|\cdot\|}$ to represent the closure of a set D with respect to the norm $\|\cdot\|$.

Theorem 4.6. *The following propositions hold.*

(1) $L^2([a,b])$ *is a dense subspace of* $\mathrm{HK}([a,b])$ *with the Alexiewicz semi-norm.*

(2) $\Gamma(L^2([a,b]))$ *is a dense subspace of* $L^2([a,b])$ *with the semi-norm* $\|\cdot\|_2$.

P r o o f. (1) Consider $S([a,b])$ to be the space of all step functions defined on $[a,b]$. By [6], it follows that

$$\mathrm{HK}([a,b]) = \overline{S([a,b])}^{\|\cdot\|_{[a,b]}} \subseteq \overline{L^2([a,b])}^{\|\cdot\|_{[a,b]}} \subseteq \mathrm{HK}([a,b]).$$

(2) We set $\Delta := \Gamma(L^2([a,b]))$. We show that $\bar{\Delta}^{\|\cdot\|_2} = L^2([a,b])$. Suppose to the contrary that $\bar{\Delta}^{\|\cdot\|_2} \neq L^2([a,b])$, then there exists $k \in L^2([a,b]) \cap \Delta^{\perp}$ such that $k \neq 0$ on a set with positive measure. This implies that $\langle z, k\rangle = 0$ for all $z \in \Delta$. From (4.1) and Lemma 3.4, there exists $h \in \mathcal{A}$ such that $Lh = k$ a.e. on $[a,b]$.

Let $l, l_i\colon C([a,b]) \to \mathbb{C}$, $i = 1,2$, be defined as

$$l(g) = \int_a^b g(t)\overline{h(t)}\,\mathrm{d}t \quad \text{and} \quad l_i(g) = \int_a^b g(t)y_i(t)\,\mathrm{d}t.$$

Since $W(y_1, y_2) = 1$, it follows that l_1, l_2 are linearly independent. Let $g \in \ker(l_1) \cap \ker(l_2)$, then

$$\int_a^b g(t)y_1(t)\,\mathrm{d}t = \int_a^b g(t)y_2(t)\,\mathrm{d}t = 0.$$

Consider $f\colon [a,b] \to \mathbb{C}$ defined as

$$f(x) = y_1(x)\int_a^x y_2(t)g(t)\,\mathrm{d}t - y_2(x)\int_a^x y_1(t)g(t)\,\mathrm{d}t.$$

From Lemma 3.4, we have $f \in \mathcal{A}$,

$$f'(x) = y_1'(x)\int_a^x y_2(t)g(t)\,\mathrm{d}t - y_2'(x)\int_a^x y_1(t)g(t)\,\mathrm{d}t$$

and $L(f) = g$ a.e. on $[a, b]$. Thus, $f(a) = f(b) = f'(a) = f'(b) = 0$ and hence $f \in \mathcal{D}_*$. This implies that $f = \Gamma(g)(\in \Delta)$ and using Remark 4.5 we obtain

$$\int_a^b g(t)\overline{h(t)}\,\mathrm{d}t = \int_a^b L(f)(t)\overline{h(t)}\,\mathrm{d}t = \langle L(f), h\rangle = \langle f, L(h)\rangle = \langle f, k\rangle = 0.$$

Thus $g \in \ker(l)$. Consequently, by [2], Lemma 3.2, $l = \alpha_1 l_1 + \alpha_2 l_2$ for some scalars $\alpha_1, \alpha_2 \in \mathbb{C}$.

Therefore for each $g \in C([a, b])$,

$$\int_a^b g(t)[\overline{h(t)} - \alpha_1 y_1(t) - \alpha_2 y_2(t)]\,\mathrm{d}t = 0.$$

This shows that $\bar{h} = \alpha_1 y_1 + \alpha_2 y_2$ and so $\overline{k} = \overline{L(h)} = L(\bar{h}) = \alpha_1 L(y_1) + \alpha_2 L(y_2) = 0$ a.e. on $[a, b]$, i.e. $k = 0$ a.e. on $[a, b]$, which is a contradiction. $\square$

5. Adding the condition of symmetry to Γ

In this section we show that if Γ is a symmetric operator, i.e. for each $f, g \in \mathrm{HK}([a, b])$, $\langle \Gamma f, g\rangle = \langle f, \Gamma g\rangle$, then the solution of (3.7) can be represented as a series (see Theorem 5.5).

Remark 5.1. If Γ is a symmetric operator then the following propositions hold:

(1) $\sigma_p(\Gamma) \subseteq \mathbb{R}$, where $\sigma_p(\Gamma)$ is the point spectrum of Γ.

(2) Let $\phi_0, \phi_1 \in \mathcal{D}_*$ be such that $\Gamma\phi_0 = \lambda_0\phi_0$ and $\Gamma\phi_1 = \lambda_1\phi_1$. If $\lambda_0 \neq \lambda_1$ then $\langle \phi_0, \phi_1\rangle = 0$.

(3) Let $\lambda \in \mathbb{R}$ with $\lambda \neq 0$ and $\phi \in \mathcal{D}_*$. Then $L\phi = \lambda\phi$ if and only if $\Gamma\phi = \phi/\lambda$.

(4) For each $\lambda \in \sigma_p(\Gamma) \setminus \{0\}$, we have $1 \leqslant \dim\ker(\lambda - \Gamma) \leqslant 2$. In fact, define L_1 as $L_1 y = -y'' + (q - 1/\lambda)y$, then $\ker(\Gamma - \lambda) \subseteq \ker(L - 1/\lambda) \subseteq \{y \in \mathcal{A}\colon L_1 y = 0 \text{ a.e. on } [a, b]\}$, now by replacing the operator L by L_1 in Proposition 3.1, we obtain that $\dim\{y \in \mathcal{A}\colon\ L_1 y = 0 \text{ a.e. on } [a, b]\} = 2$, thus $\dim\ker(\lambda - \Gamma) \leqslant 2$.

Proposition 5.2. *If Γ is a symmetric operator then there exists a sequence (λ_n) such that $\sigma_p(\Gamma) \setminus \{0\} = \{\lambda_n\colon\ n \in \mathbb{N}\}$, and there exists (ϕ_n) in $\mathcal{D}_* \setminus \{0\}$ such that $\{\phi_1, \phi_2, \ldots\}$ is a complete orthonormal system in $L^2([a, b])$ and $\Gamma(\phi_n) = \lambda_n\phi_n$ for all $n \in \mathbb{N}$.*

Proof. Let $\Psi = \Gamma|_{L^2([a,b])}$. Then Ψ is symmetric and, by Remark 4.1, Ψ is injective and so $\Psi \neq 0$. Moreover, by Theorem 4.6, case (2), $\Psi(L^2([a, b]))$ is a dense subspace of $L^2([a, b])$. Therefore, by Theorem 2.6, there exist two sequences (λ_n) in $\mathbb{R} \setminus \{0\}$ and (ϕ_n) in $\Psi(L^2([a, b])) \setminus \{0\}$ such that

(1) $\|\phi_n\|_2 = 1$, $\Psi(\phi_n) = \lambda_n\phi_n$ and $|\lambda_{n+1}| \leqslant |\lambda_n|$ for all $n \in \mathbb{N}$,
(2) $\lim\limits_{n\to\infty} |\lambda_n| = 0$,
(3) $\{\phi_1, \phi_2, \ldots\}$ is a complete orthonormal system in $L^2([a,b])$.

If there exists $\lambda \in \sigma_p(\Gamma) \setminus \{0\}$ for which $\lambda \neq \lambda_n$ for all $n \in \mathbb{N}$, then there exists $\phi \in \mathcal{D}_* \setminus \{0\}$ such that $\Gamma(\phi) = \lambda\phi$ and, by Remark 5.1, case (2), $\langle \phi, \phi_n \rangle = 0$ for all $n \in \mathbb{N}$. This implies that

$$\|\phi\|_2 = \sum_{k=1}^{\infty} |\langle \phi, \phi_n \rangle|^2 = 0$$

and so $\phi = 0$ which is a contradiction. □

Remark 5.3. If $L\colon \mathcal{D}_* \to \mathrm{HK}([a,b])$ is a symmetric operator then $\Gamma\colon \mathrm{HK}([a,b]) \to \mathcal{D}_*$ is symmetric. Indeed, we set $u = \Gamma(f)$ and $v = \Gamma(g)$, this implies that $\langle \Gamma(f), g \rangle = \langle \Gamma(f), L(\Gamma(g)) \rangle = \langle u, Lv \rangle = \langle Lu, v \rangle = \langle L(\Gamma(f)), \Gamma(g) \rangle = \langle f, \Gamma(g) \rangle$.

Proposition 5.4. *Suppose that $L\colon \mathcal{D}_* \to \mathrm{HK}([a,b])$ is a symmetric operator and consider (μ_k), a sequence in $\mathbb{C}$, with $\mu_k \neq \mu_j$ if $k \neq j$, such that $\sigma_p(L) \setminus \{0\} = \{\mu_k\colon k \in \mathbb{N}\}$. Let $P = \{k \in \mathbb{N}\colon \dim\ker(\mu_k - L) = 2\}$. If for every $k \in P$, $\varphi_k, \varphi_k^* \in \ker(\mu_k - L)$ are such that $\{\varphi_k, \varphi_k^*\}$ is an orthonormal set and for each $k \in \mathbb{N} \setminus P$, $\varphi_k \in \ker(\mu_k - L)$ is such that $\|\varphi_k\| = 1$, then*

$$\Omega = \{\varphi_k, \varphi_k^*\colon k \in P\} \cup \{\varphi_k\colon k \in \mathbb{N} \setminus P\}$$

is a complete orthonormal system in $L^2([a,b])$.

Proof. Let (λ_n) and (ϕ_n) be the same as in Proposition 5.2. We set $Q = \{n \in \mathbb{N}\colon \dim\ker(\lambda_n - \Gamma) = 2\}$. Take $n \in Q$ and suppose that for each $m \neq n$, $\phi_m \notin \ker(\lambda_n - \Gamma)$. This implies that for every $g \in \ker(\lambda_n - \Gamma)$ and $m \neq n$, $\langle g, \phi_m \rangle = 0$. Thus by completeness of (ϕ_n), $g = \langle g, \phi_n \rangle \phi_n$ holds for all $g \in \ker(\lambda_n - \Gamma)$, i.e. $\dim\ker(\lambda_n - \Gamma) = 1$, which is a contradiction. Consequently, for each $n \in Q$, there exists a unique $m_n \in \mathbb{N}$ with $m_n \neq n$ such that $\phi_{m_n} \in \ker(\lambda_n - \Gamma)$. We show that

$$\{\langle f, \varphi_k \rangle \varphi_k + \langle f, \varphi_k^* \rangle \varphi_k^*\colon k \in P\} = \{\langle f, \phi_n \rangle \phi_n + \langle f, \phi_{m_n} \rangle \phi_{m_n}\colon n \in Q\}. \tag{5.1}$$

Let us denote by $\mathcal{F}_1$ the first family in (5.1) and by $\mathcal{F}_2$ the second one. Let $n \in Q$, then there exists $k_n \in P$ such that $\lambda_n = 1/\mu_{k_n}$. We set $h_1 = \langle f, \phi_n \rangle \phi_n + \langle f, \phi_{m_n} \rangle \phi_{m_n}$ and $h_2 = \langle f, \varphi_{k_n} \rangle \varphi_{k_n} + \langle f, \varphi_{k_n}^* \rangle \varphi_{k_n}^*$, then $y = h_1 + (f - h_1) = h_2 + (f - h_2)$, $h_1, h_2 \in \ker(\lambda_n - \Gamma)$ and $f - h_1, f - h_2 \in \ker(\lambda_n - \Gamma)^{\perp}$. Therefore, $h_1 = h_2$ and so $h_1 \in \mathcal{F}_1$. In a similar way the opposite inclusion is proved.

On the other hand, it is clear that

$$\{\langle f, \varphi_k \rangle \varphi_k\colon k \in \mathbb{N} \setminus P\} = \{\langle f, \phi_n \rangle \phi_n\colon n \in \mathbb{N} \setminus Q\}. \tag{5.2}$$

Let us denote by $\mathcal{G}_1$ the first family in (5.2) and by $\mathcal{G}_2$ the second one. Then

$$\begin{aligned} f &= \sum_{n\in\mathbb{N}} \langle f, \phi_n\rangle \phi_n = \sum_{n\in Q} \langle f, \phi_n\rangle \phi_n + \sum_{n\in\mathbb{N}\setminus Q} \langle f, \phi_n\rangle \phi_n \\ &= \sum_{F\in\mathcal{F}_2} F + \sum_{G\in\mathcal{G}_2} G = \sum_{F\in\mathcal{F}_1} F + \sum_{G\in\mathcal{G}_1} G \\ &= \sum_{k\in P} (\langle f, \varphi_k\rangle \varphi_k + \langle f, \varphi_k^*\rangle \varphi_k^*) + \sum_{k\in\mathbb{N}\setminus Q} \langle f, \varphi_k\rangle \varphi_k. \end{aligned}$$

$\square$

Theorem 5.5. *Suppose that L is symmetric and take (μ_k) and Ω as in Proposition 5.4. Let (β_n) and (ω_n) be an indexation of (μ_k) and Ω, respectively, such that $L\omega_n = \beta_n\omega_n$ for all $n \in \mathbb{N}$. If there exists a constant $M \geqslant 0$ such that for each $u \in \mathrm{HK}([a,b])$,*

$$\left|\sum_{k=1}^{n} \langle u, \omega_k\rangle \frac{1}{\beta_k}\omega_k(x)\right| \leqslant \|u\|_{[a,b]} M \tag{5.3}$$

for all $x \in [a,b]$ and $n \in \mathbb{N}$, then for every $f \in \mathrm{HK}([a,b])$,

$$\Gamma(f) = \sum_{k=1}^{\infty} \langle f, \omega_k\rangle \frac{1}{\beta_k}\omega_k$$

uniformly on $[a,b]$.

Proof. Let $f \in \mathrm{HK}([a,b])$. Since $L^2([a,b])$ is a dense subspace of $\mathrm{HK}([a,b])$ with the Alexiewicz semi-norm, it follows that there exists a sequence (f_p) in $L^2([a,b])$ such that $\|f_p - f\|_{[a,b]} \to 0$. By Theorem 4.2, there exists a subsequence (f_{p_m}) of (f_p) and $g \in C([a,b])$ such that $\Gamma(f_{p_m})$ converges uniformly to g on $[a,b]$. Moreover, by Remark 4.3, $\Gamma\colon (\mathrm{HK}([a,b]), \|\cdot\|_{[a,b]}) \to (C([a,b]), \|\cdot\|_\infty)$ is bounded and so $\Gamma(f_{p_m}) \to \Gamma(f)$. Therefore $\|g - \Gamma(f)\|_\infty = 0$ and so $g = \Gamma(f)$. Let $\varepsilon > 0$, we consider $m \in \mathbb{N}$ such that

$$|\Gamma(f_{p_m})(x) - \Gamma(f)(x)| < \frac{\varepsilon}{3}$$

for all $x \in [a,b]$ and

$$\|f_{p_m} - f\|_{[a,b]} < \frac{\varepsilon}{3M}.$$

On the other hand, by Proposition 5.4, $\{\omega_1, \omega_2, \ldots\}$ is a complete system in $L^2([a,b])$. Thus, there exists $N \in \mathbb{N}$ such that for every $n \geqslant N$,

$$\left\|f_{p_m} - \sum_{k=1}^{n} \langle f_{p_m}, \omega_k\rangle \omega_k\right\|_2 < \frac{\varepsilon}{3 \max\limits_{x\in[a,b]} \|G(x,\cdot)\|_2}.$$

Therefore, for each $n \geqslant N$ and every $x \in [a,b]$ we have

$$\begin{aligned}\left|\Gamma(f)(x) - \sum_{k=1}^{N}\langle f,\omega_k\rangle\frac{1}{\beta_k}\omega_k(x)\right| &\leqslant |\Gamma(f)(x) - \Gamma(f_{p_m})(x)| \\ &+ \left|\Gamma(f_{p_m})(x) - \sum_{k=1}^{N}\langle f_{p_m},\omega_k\rangle\frac{1}{\beta_k}\omega_k(x)\right| + \left|\sum_{k=1}^{N}\langle f_{p_m} - f,\omega_k\rangle\frac{1}{\beta_k}\omega_k(x)\right| \\ &< \frac{\varepsilon}{3} + \|G(x,\cdot)\|_2\left\|f_{p_m} - \sum_{k=1}^{N}\langle f_{p_m},\omega_k\rangle\omega_k\right\|_2 + \|f_{p_m} - f\|_{[a,b]}M \; < \varepsilon.\end{aligned}$$

□

Theorem 5.6. *Suppose that L is symmetric and take (β_n) and (ω_n) as in Theorem 5.5. If there exists $M > 0$ such that $|\omega_n(x)| \leqslant M$ for all $n \in \mathbb{N}$ and $x \in [a,b]$, and*

$$\sum_{k=1}^{\infty}\frac{1}{|\beta_k|} < \infty,$$

then for every $f \in \mathrm{HK}([a,b])$,

$$\Gamma(f) = \sum_{k=1}^{\infty}\langle f,\omega_k\rangle\frac{1}{\beta_k}\omega_k$$

uniformly on $[a,b]$.

P r o o f. Due to Theorem 5.5, we only need to prove that the inequallity (5.3) holds. In a way similar to the proof of Theorem 4.6 (1), it follows that $\overline{L([a,b])}^{\|\cdot\|_{[a,b]}} = \mathrm{HK}([a,b])$. Let $u \in \mathrm{HK}([a,b])$, then there exists (s_n) in $L([a,b])$ such that $\|s_n - u\|_{[a,b]} \to 0$. This implies that

$$\big||\langle s_n,\omega_k\rangle| - |\langle u,\omega_k\rangle|\big| \leqslant |\langle s_n - u,\omega_k\rangle| \leqslant \|s_n - u\|_{[a,b]}\|\omega_k\|_{\mathrm{BV}} \underset{n\to\infty}{\longrightarrow} 0.$$

Therefore $|\langle u,\omega_k\rangle| = \lim\limits_{n\to\infty}|\langle s_n,\omega_k\rangle|$. Suppose that $\|u\|_{[a,b]} > 0$, then there exists $N \in \mathbb{N}$ such that $\|s_n\|_1 = \|s_n\|_{[a,b]} < 2\|u\|_{[a,b]}$ for every $n \geqslant N$. Thus

$$|\langle s_n,\omega_k\rangle| \leqslant \int_a^b |s_n(t)|\,|\omega_k(t)|\,\mathrm{d}t \leqslant M\|s_n\|_1 < 2M\|u\|_{[a,b]}$$

for all $k \in \mathbb{N}$ and $n \geqslant N$. Consequently, $|\langle u,\omega_k\rangle| = \lim\limits_{N\leqslant n\to\infty}|\langle s_n,\omega_k\rangle| \leqslant 2M\|u\|_{[a,b]}$ for all $k \in \mathbb{N}$, and so

$$\left|\sum_{k=1}^{n}\langle u,\omega_k\rangle\frac{1}{\beta_k}\omega_k(x)\right| \leqslant 2M^2\|u\|_{[a,b]}\sum_{k=1}^{\infty}\frac{1}{|\beta_k|}.$$

□

Example 5.7. Separated and periodic boundary conditions.

The separated boundary conditions are those that in (3.2) correspond to the matrix

$$\begin{pmatrix} m_1 & n_1 & 0 & 0 \\ 0 & 0 & p_2 & q_2 \end{pmatrix} \tag{5.4}$$

and the periodic conditions correspond to the matrix

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}. \tag{5.5}$$

If U is defined by the matrix given in (5.4) or in (5.5) then L is a symmetric operator. Indeed, by Proposition 4.4, $\langle Lu, \overline{v}\rangle = W_b(u,v) - W_a(u,v) + \langle u, L\overline{v}\rangle$ for all $u, v \in \mathcal{D}_*$. Suppose that separated conditions hold. Thus

$$\begin{aligned} m_1 u(a) + n_1 u'(a) &= 0, \qquad & m_1 v(a) + n_1 v'(a) &= 0, \\ p_2 u(b) + q_2 u'(b) &= 0, \qquad & p_2 v(b) + q_2 v'(b) &= 0. \end{aligned}$$

If $W_a(u,v) \neq 0$ then $m_1 = n_1 = 0$, which contradicts our assumption that the problem (B) has only a trivial solution. Therefore $W_a(u,v) = 0$. The equality $W_b(u,v) = 0$ is proved in a similar way. Now, if we consider periodic conditions then we have that $u(a) = u(b)$, $u'(a) = u'(b)$, $v(a) = v(b)$ and $v'(a) = v'(b)$. Therefore, $W_b(u,v) - W_a(u,v) = 0$. Thus, in any case, we have $\langle Lu, v\rangle = \langle u, Lv\rangle$ for all $u, v \in \mathcal{D}_*$.

Example 5.8. Let f be a function defined on $[0,1]$ as

$$f(x) = \begin{cases} \dfrac{2\pi}{x}\sin\Big(\dfrac{\pi}{x^2}\Big), & \text{if } x \in (0,1]; \\ 0, & \text{if } x = 0. \end{cases}$$

This is an unbounded HK-integrable function on $[0,1]$. Consider the boundary value problem

$$\begin{cases} -y'' + y = f & \text{a.e.}, \\ y(0) = 0, \\ y(1) = 1. \end{cases} \tag{5.6}$$

By Theorem 3.2, this problem has a unique solution, moreover from Remark 3.3 and Theorem 3.5, the solution of the problem (5.6) is given by

$$y(t) = \Gamma(f)(t) - (\mathrm{e} - \mathrm{e}^{-1})\mathrm{e}^{-t} + (\mathrm{e} - \mathrm{e}^{-1})\mathrm{e}^{t}.$$

The boundary conditions of this problem are separated, thus by Example 5.7, L is symmetric, and for every $\mu \in \sigma_p(L)$, $\dim\ker(\lambda - L) = 1$. For each $k \in \mathbb{N}$, let $\mu_k = 1 + k^2\pi^2$ and $\varphi_k(x) = \sqrt{2}\sin(k\pi x)$. Then $\sigma_p(L) \setminus \{0\} = \{\mu_k \colon k \in \mathbb{N}\}$, $\sum\limits_{k=1}^{\infty} 1/\mu_k < \infty$ and (φ_k) is a sequence in $\mathcal{D}_*$ such that for each $k \in \mathbb{N}$, $\|\varphi_k\|_2 = 1$, $L\varphi_k = \mu_k\varphi_k$ and $|\varphi_k(x)| \leqslant 1$ for all $x \in [a,b]$. Consequently, by Theorem 5.6,

$$y(t) = \sum_{k=1}^{\infty}\langle f, \varphi_k\rangle \frac{1}{\mu_k}\varphi_k(t) - (\mathrm{e} - \mathrm{e}^{-1})\mathrm{e}^{-t} + (\mathrm{e} - \mathrm{e}^{-1})\mathrm{e}^{t}.$$

The function f is not Lebesgue integrable on $[0,1]$. Hence, this example is not covered by any result using the Lebesgue integral. Thus, the results presented in this document are more extensive.

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