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PSEUDOMETRICS ON EXT-SEMIGROUPS

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Abstract. This paper considers certain pseudometric structures on Ext-semigroups and gives a unified characterization of several topologies on Ext-semigroups. It is demonstrated that these Ext-semigroups are complete topological semigroups. To this end, it is proved that a metric induces a pseudometric on a quotient space with respect to an equivalence relation if it has certain invariance. We give some properties of this pseudometric space and prove that the topology induced by the pseudometric coincides with the one induced by the quotient map.

Keywords: pseudometric; topological group; extension; Ext-group

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1. Introduction

In 1973, in order to study quasidiagonality for operators and for C^*-algebras, Brown, Douglas and Fillmore recognized that it may be related to the topological structure on Ext-groups, see [4]. They topologized the group $\mathrm{Ext}(X)$ by point-wise convergence in the norm topology and announced that the closure of zero equaled the quasidiagonal extensions, where X is a compact metric space. This result was proved by Brown in 1980, see [3].

In [1] Arveson considered the point-norm topologies on completely positive linear maps and extensions of a separable C^*-algebra and proved that the set of completely positive linear maps which is liftable is point-norm closed. He hence obtained the fact that invertible extensions are point-norm closed for extensions of a separable C^*-algebra by the compact operators. In [10] Salinas generalized Brown's investiga-

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tions to the case of extensions of separable nuclear C^*-algebras by the compact operators, and subsequently to the case of relative quasidiagonal extensions in [11], [12]. In [6] Dadarlat defined a topology on the Kasparov group $KK(A,B)$ in terms of Cuntz pairs and approximate unitary equivalence for separable C^*-algebras A and B. Then he proved that this topology, the Pimsner topology, and the Brown-Salinas topology coincide for separable C^*-algebras A and B.

In [13], [14], [15] Schochet systematically studied topologies on the Kasparov groups $KK^{\mathrm{i}}(A,B)$. He particularly proved that the Brown-Salinas topology, the Zekri topology, and the two Cuntz topologies are identical. He also verified that $KK^{\mathrm{i}}(A,B)$ have a natural structure of pseudopolonais topological groups. One remarkable result of his work is that the well-known UCT (see [9]) is an exact sequence in the category of topological groups:

$$0 \to \mathrm{Ext}(K_*(A), K_*(B)) \to \mathrm{KK}^*(A,B) \to \mathrm{Hom}(K_*(A), K_*(B)) \to 0.$$

Notice that Schochet's work mainly focused on stable Ext-groups for nonunital extensions. It is known that the original BDF-theory is engaged in classifying unital extensions. However, classifications of unital extensions are essentially different to these of nonunital extensions, not only due to invariants but also due to methods, since the classic Ext-groups and UCT only hold for the nonunital case. Motivated by the references mentioned above, we aim to topologize Ext-semigroups and UCTs for unital extensions, and give a unified characterization of several topologies on Ext-semigroups.

For our purpose, we first study a pseudometric on a quotient space induced by an equivalence relation in Section 2. It is natural that the pseudometric should be compatible with the equivalence relation. It is investigated that this requirement is related to a property analogous to certain invariance. We call this property minimal invariance (see Definition 2.3). Subsequently, we consider the topological property of this pseudometric quotient space. One of the results is that we prove that the topology induced by the pseudometric coincides with the one induced by the quotient map. In Section 3, by the results of Section 2, it is proved that these Ext-semigroups are complete topological semigroups. In the subsequent paper we will apply these topologies to topologizing UCTs for unital extensions of C^*-algebras.

2. Pseudometrics on quotient spaces

In order to describe the topologies on Ext-groups uniformly, we consider pseudometrics on quotient spaces in this section. We are concerned with the question when a (pseudo)metric induces a pseudometric on a quotient space.

Recall that a pseudometric on a space X is a function on $X \times X$ with values in nonnegative real numbers and satisfies the following conditions for any $x, y, z \in X$:

(i) $d(x, y) = d(y, x)$;

(ii) $d(x, z) \leqslant d(x, y) + d(y, z)$;

(iii) $d(x, y) = 0$ if $x = y$.

Suppose that (X, d) is a (pseudo)metric space with an equivalence relation $\sim$. Let $X/{\sim}$ be the quotient space consisting of the equivalence classes. For $x \in X$, we denote by $\widetilde{x}$ the equivalence class of x. Let $\pi\colon X \to X/{\sim}$ be the quotient map defined by $\pi(x) = \widetilde{x}$ for any $x \in X$. Then d induces a function $\widetilde{d}$ on the space $(X/{\sim}) \times (X/{\sim})$ by

$$\tilde{d}(\tilde{x}, \tilde{y}) = \inf\{d(x', y')\colon\ x' \sim x, y' \sim y\}.$$

Obviously, for any $\widetilde{x}, \widetilde{y} \in X/{\sim}$ we have $\widetilde{d}(\widetilde{x}, \widetilde{y}) = \widetilde{d}(\widetilde{y}, \widetilde{x})$ and $\widetilde{d}(\widetilde{x}, \widetilde{y}) = 0$ if $\widetilde{x} = \widetilde{y}$.

Next, we consider the following questions:

(1) When is $\widetilde{d}$ a pseudometric on $X/{\sim}$?

(2) When does the topology induced by $\widetilde{d}$ coincide with the quotient topology?

(3) When is π open?

In fact, $\widetilde{d}$ is not a pseudometric in general. We need some restriction to make $\widetilde{d}$ a pseudometric. Before answering the above questions, we need to give some properties of $\widetilde{d}$. Firstly, we notice the following fundamental fact of the infimum. Though it is known, we list it here for the sake of convenience.

Lemma 2.1. *Suppose that A, B are two nonempty sets and f is a real function on $A \times B$. Then the following equalities hold:*

$$\inf_{y\in B}\inf_{x\in A} f(x, y) = \inf_{x\in A, y\in B} f(x, y) = \inf_{x\in A}\inf_{y\in B} f(x, y).$$

Proposition 2.2. *For (X, d) given above, the following statements are equivalent:*

(1) $\inf\limits_{x'\sim x} d(x', y) = \inf\limits_{y'\sim y} d(x, y')$ *for any* $x, y \in X$;

(2) $\inf\limits_{x'\sim x} d(x', y) = \inf\limits_{x'\sim x} d(x', y_0)$ *for any* $x, y \in X$ *and any* $y_0 \sim y$;

(3) $\inf\limits_{x'\sim x, y'\sim y} d(x', y') = \inf\limits_{x'\sim x} d(x', y)$ *for any* $x, y \in X$;

(4) $\inf\limits_{x'\sim x, y'\sim y} d(x', y') = \inf\limits_{x'\sim x} d(x', y_0)$ *for any* $x, y \in X$ *and any* $y_0 \sim y$.

Proof. (1)⇒(2) Suppose $y_0 \in X$ such that $y_0 \sim y$. By (1), we have

$$\inf_{x'\sim x} d(x', y_0) = \inf_{y''\sim y_0} d(x, y''),\ \ \inf_{x'\sim x} d(x', y) = \inf_{y'\sim y} d(x, y').$$

Since $y_0 \sim y$, it follows that $y'' \sim y_0$ if and only if $y'' \sim y$. Hence

$$\inf_{y'' \sim y_0} d(x, y'') = \inf_{y' \sim y} d(x, y').$$

Therefore $\inf\limits_{x' \sim x} d(x', y) = \inf\limits_{x' \sim x} d(x', y_0)$.

(2)⇒(3) By (2) and Lemma 2.1, we have

$$\inf_{x' \sim x, y' \sim y} d(x', y') = \inf_{y' \sim y} \inf_{x' \sim x} d(x', y') = \inf_{y' \sim y} \inf_{x' \sim x} d(x', y) = \inf_{x' \sim x} d(x', y).$$

(3)⇒(1) By (3), it follows that

$$\inf_{x' \sim x} d(x', y) = \inf_{x' \sim x, y' \sim y} d(x', y') \leqslant \inf_{y' \sim y} d(x, y').$$

This implies that $\inf\limits_{x' \sim x} d(x', y) \leqslant \inf\limits_{y' \sim y} d(x, y')$ for any x, y. Hence,

$$\inf_{y' \sim y} d(x, y') = \inf_{y' \sim y} d(y', x) \leqslant \inf_{x' \sim x} d(y, x') = \inf_{x' \sim x} d(x', y).$$

Therefore $\inf\limits_{x' \sim x} d(x', y) = \inf\limits_{y' \sim y} d(x, y')$.

(2)⇒(4) Similarly to the proof of (2)⇒(3), for any $x, y \in X$ and any $y_0 \sim y$ we have

$$\inf_{x' \sim x, y' \sim y} d(x', y') = \inf_{y' \sim y} \inf_{x' \sim x} d(x', y') = \inf_{y' \sim y} \inf_{x' \sim x} d(x', y_0) = \inf_{x' \sim x} d(x', y_0).$$

(4)⇒(2) Since $y' \sim y$, by (4) it follows that

$$\inf_{x' \sim x} d(x', y) \leqslant \inf_{x' \sim x, y' \sim y} d(x', y') = \inf_{x' \sim x} d(x', y_0).$$

Hence,

$$\inf_{x' \sim x} d(x', y) \leqslant \inf_{x' \sim x} d(x', y_0).$$

Exchanging the positions of y and y_0, we obtain

$$\inf_{x' \sim x} d(x', y_0) \leqslant \inf_{x' \sim x} d(x', y).$$

Therefore (2) holds. □

Definition 2.3. We say that the metric d has the minimal invariance with respect to the equivalence relation $\sim$ if for any $x, y \in X$,

$$\inf\{d(x', y) \colon\ x' \sim x\} = \inf\{d(x, y') \colon\ y' \sim y\}.$$

Next, we prove that $\widetilde{d}$ becomes a pseudometric if d has the minimal invariance.

Theorem 2.4. *If d has the minimal invariance, then $(X/{\sim}, \widetilde{d})$ constitutes a pseudometric space.*

P r o o f. We only need to prove the triangle inequality:

$$\widetilde{d}(\widetilde{x},\widetilde{y}) \leqslant \widetilde{d}(\widetilde{x},\widetilde{z}) + \widetilde{d}(\widetilde{z},\widetilde{y})$$

for any $x, y, z \in X$.

Suppose that $x' \sim x$, $y' \sim y$ and $z' \sim z$. By the triangle inequality of d, we have

$$d(x',y') \leqslant d(x',z') + d(z',y').$$

It follows from Lemma 2.1 and Proposition 2.2 that

$$\begin{aligned}\inf_{x'\sim x, y'\sim y} d(x',y') &\leqslant \inf_{y'\sim y}\inf_{x'\sim x}(d(x',z')+d(z',y')) = \inf_{y'\sim y}(\inf_{x'\sim x} d(x',z')+d(z',y'))\\ &= \inf_{x'\sim x} d(x',z') + \inf_{y'\sim y} d(z',y')\\ &= \inf_{x'\sim x, z''\sim z'} d(x',z'') + \inf_{y'\sim y, z''\sim z'} d(z'',y')\\ &= \inf_{x'\sim x, z''\sim z} d(x',z'') + \inf_{y'\sim y, z''\sim z} d(z'',y').\end{aligned}$$

Therefore $\widetilde{d}(\widetilde{x},\widetilde{y}) \leqslant \widetilde{d}(\widetilde{x},\widetilde{z}) + \widetilde{d}(\widetilde{z},\widetilde{y})$. □

Corollary 2.5. *Suppose that $\{\widetilde{x}_n\}$ is a sequence in $X/\sim$ and $\widetilde{x}_0 \in X/\sim$. If d has the minimal invariance, then $\{\widetilde{x}_n\}$ converges to $\widetilde{x}_0$ under $\widetilde{d}$ if and only if there is a sequence $\{y_n\}$ in X such that $y_n \sim x_n$ for all n and $\{y_n\}$ converges to x_0 under d.*

P r o o f. ($\Leftarrow$) Suppose that there exists $\{y_n\}$ in X such that $y_n \sim x_n$ for all n and $\{y_n\}$ converges to x_0 under d. Since $\widetilde{d}(\widetilde{x}_n,\widetilde{x}_0) \leqslant d(y_n,x_0)$, then $\widetilde{x}_n \xrightarrow{\widetilde{d}} \widetilde{x}_0$.

($\Rightarrow$) Suppose that $\widetilde{x}_n \xrightarrow{\widetilde{d}} \widetilde{x}_0$. By Proposition 2.2, we have

$$\widetilde{d}(\widetilde{x}_n,\widetilde{x}_0) = \inf_{y\sim x_n} d(y,x_0).$$

Hence, for every n there is $y_n \in X$ such that $y_n \sim x_n$ and

$$d(y_n,x_0) < \widetilde{d}(\widetilde{x}_n,\widetilde{x}_0) + 1/n.$$

Therefore $\{y_n\}$ converges to x_0 as n tends to infinity. □

There are many examples that the quotient spaces are pseudometric spaces, especially when the metrics have certain translation invariance. The following example is a typical case.

Example 2.6. Let X be a normed linear space and let M be a linear subspace of X. Define an equivalence relation on X by $x \sim y \Leftrightarrow x - y \in M$. Set $d(x,y) = \|x - y\|$. Then d induces the function

$$\widetilde{d}(\widetilde{x}, \widetilde{y}) = \inf_{x' \sim x, y' \sim y} \|x' - y'\|.$$

One can easily check that $\inf_{x' \sim x} d(x', y) = \inf_{y' \sim y} d(x, y')$ for any $x, y \in X$. That is to say that d has the minimal invariance with respect to the above equivalence relation. Hence, $\widetilde{d}$ is a pseudometric on $X/{\sim}$.

Define $\|\widetilde{x}\| = \widetilde{d}(\widetilde{x}, \widetilde{0})$. Then by Proposition 2.2,

$$\|\widetilde{x}\| = \inf_{x' \sim x} d(x', 0) = \inf_{x' \sim x} \|x'\|.$$

This is the usual quotient seminorm on quotient spaces of normed linear spaces. Moreover, if M is a closed subspace, then $\|\widetilde{x}\|$ is a norm on $X/{\sim}$.

Example 2.7. Let B be a C^*-algebra with an ideal I. Suppose that A is a separable C^*-algebra. Denote by $CP(A,B)$ and $CP(A,B/I)$ completely positive linear maps from A into B and B/I, respectively. For any φ and ψ in $CP(A,B)$, we say they are equivalent if they induce the same map from A into the quotient algebra B/I. Define a metric on $CP(A,B)$ by $d(\varphi,\psi) = \sum\limits_{n=1}^{\infty} 2^{-n} \|\varphi(a_n) - \psi(a_n)\|$, where $\{a_n\}$ is a dense sequence in the unit ball of A. Then by [1], Lemma 3.1, the metric d has the minimal invariance with respect to the equivalence relation. It follows that the induced map $\widetilde{d}$ is a pseudometric on $CP(A,B)/{\sim}$, which is a complete metric subspace of $CP(A,B/I)$ by [1], Theorem 6.

In the following, we assume that (X,d) is a (pseudo)metric space and d has the minimal invariance with respect to an equivalence relation $\sim$ on X. Let A be a subset of X and $x \in X$. Set $R(A) = \{x \in X \colon x \sim a \text{ for some } a \in A\}$ and $O_\varepsilon(x) = \{y \in X \colon d(y,x) < \varepsilon\}$ for $\varepsilon > 0$.

Lemma 2.8. *Suppose that $x, x' \in X$ such that $x \sim x'$. Then $R(O_\varepsilon(x)) = R(O_\varepsilon(x'))$ for any $\varepsilon > 0$.*

P r o o f. Let $z \in R(O_\varepsilon(x))$. Then there exists $y \in O_\varepsilon(x)$ such that $z \sim y$. Since $d(x,y) < \varepsilon$, $\inf_{x'' \sim x} d(x'', y) < \varepsilon$. By the minimal invariance, we have $\inf_{y' \sim y} d(x, y') < \varepsilon$. Since $x' \sim x$, $\inf_{y' \sim y} d(x', y') < \varepsilon$ by Proposition 2.2. Hence, there is $y_1 \in X$ such that $y_1 \sim y$ and $d(x', y_1) < \varepsilon$. Since $z \sim y \sim y_1$ and $y_1 \in O_\varepsilon(x')$, it follows that $z \in R(O_\varepsilon(x'))$. Therefore $R(O_\varepsilon(x)) \subset R(O_\varepsilon(x'))$.

By the symmetry of x and x', we obtain $R(O_\varepsilon(x')) \subset R(O_\varepsilon(x))$. So the conclusion holds. $\square$

Lemma 2.9. *Let G be an open subset of X. Then $R(G)$ is open.*

P r o o f. For any $y_0 \in R(G)$, there is $x_0 \in G$ such that $y_0 \sim x_0$. Since G is open, there is $\varepsilon > 0$ such that $O_\varepsilon(x_0) \subset G$. Then $y_0 \in R(O_\varepsilon(x_0))$ and $R(O_\varepsilon(x_0)) \subset R(G)$.

In the following, we prove that $R(O_\varepsilon(x_0))$ is open and hence $R(G)$ is open.

Let $z \in R(O_\varepsilon(x_0))$. Then there is $x \in X$ such that $x \sim z$ and $d(x, x_0) < \varepsilon$. Hence, we have

$$\inf_{x' \sim x} d(x', x_0) = \inf_{x'' \sim x_0} d(x, x'') < \varepsilon.$$

Since $x \sim z$, by Proposition 2.2 we have

$$\inf_{x'' \sim x_0} d(z, x'') = \inf_{x'' \sim x_0} d(x, x'') < \varepsilon.$$

Hence, there is $x_1 \in X$ such that $x_1 \sim x_0$ and $d(z, x_1) < \varepsilon$. So $z \in O_\varepsilon(x_1)$ for some $x_1 \sim x_0$.

On the other hand, by Lemma 2.8, $R(O_\varepsilon(x_0)) = R(O_\varepsilon(x_1))$. Note that $O_\varepsilon(x_1) \subset R(O_\varepsilon(x_1))$. This implies that

$$z \in O_\varepsilon(x_1) \subset R(O_\varepsilon(x_0)).$$

Therefore $R(O_\varepsilon(x_0))$ is open. □

Recall that the quotient topology on $X/{\sim}$ induced by the map π is the family

$$\{U \subset X/{\sim}\colon\ \pi^{-1}(U) \text{ is open in } (X, d)\}$$

of subsets of $X/{\sim}$, where $\pi\colon\ X \to X/{\sim}$ is the quotient map.

It should be noted that when $X/{\sim}$ is equipped with the quotient topology, π is open if and only if $R(G)$ is open for any open subset G of X.

Theorem 2.10. *The topology on $X/{\sim}$ induced by $\widetilde{d}$ coincides with the quotient topology on $X/{\sim}$.*

P r o o f. Let τ_d be the topology induced by d on X. Suppose that σ is the quotient topology on $X/{\sim}$ and τ is the topology induced by $\widetilde{d}$ on $X/{\sim}$.

Since $\widetilde{d}(\widetilde{x}, \widetilde{y}) \leqslant d(x, y)$, the quotient map $\pi\colon\ X \to X/{\sim}$ is continuous. Hence τ is weaker than σ.

For the inverse direction, we first prove that

$$O_\varepsilon(\widetilde{x}) = \pi(R(O_\varepsilon(x))$$

for any $x \in X$ and any $\varepsilon > 0$.

Suppose that $y \in X$ such that $\widetilde{d}(\widetilde{x}, \widetilde{y}) < \varepsilon$. Then $\inf_{x' \sim x} d(x', y) < \varepsilon$ and hence there is $x'' \in X$ such that $x'' \sim x'$ and $d(x'', y) < \varepsilon$. Since $x'' \sim x$, it follows that $y \in R(O_\varepsilon(x)$. Furthermore, $\widetilde{y} \in \pi(R(O_\varepsilon(x))$. So $O_\varepsilon(\widetilde{x}) \subset \pi(R(O_\varepsilon(x))$.

Conversely, let $z \in R(O_\varepsilon(x))$. Then there is $z_0 \in O_\varepsilon(x)$ such that $z \sim z_0$. The fact that $d(z_0, x) < \varepsilon$ implies that $\widetilde{d}(\widetilde{z}_0, \widetilde{x}) < \varepsilon$. Since $z \sim z_0$, we have $\widetilde{d}(\widetilde{z}, \widetilde{x}) < \varepsilon$. Therefore $\widetilde{z} \in O_\varepsilon(\widetilde{x})$.

Now we prove that σ is weaker than τ. For any $F \in \sigma$, then $\pi^{-1}(F) \in \tau_d$. Assume that $\widetilde{x}_0 \in F$ and then $x_0 \in \pi^{-1}(F)$. Hence there is $\varepsilon > 0$ such that $O_\varepsilon(x_0) \subset \pi^{-1}(F)$. Moreover, $R(O_\varepsilon(x_0)) \subset R(\pi^{-1}(F))$. Note that $R(\pi^{-1}(F)) = \pi^{-1}(F)$. Therefore $\pi(R(O_\varepsilon(x_0)) \subset F$. By the above proof, we have

$$\pi(R(O_\varepsilon(x_0)) = O_\varepsilon(\widetilde{x}_0) \in \tau.$$

Hence, $F \in \tau$. This implies that $\sigma \subset \tau$. $\square$

Theorem 2.11. *The quotient map* $\pi\colon (X, d) \to (X/\sim, \widetilde{d})$ *is open.*

Proof. Note that for any subset G of X, $\pi^{-1}(\pi(G)) = R(G)$. Hence, by Lemma 2.9, $\pi\colon (X, d) \to (X/\sim, \sigma)$ is open, where σ is the quotient topology on $X/\sim$ induced by the quotient map π. By Theorem 2.10, $(X/\sim, \widetilde{d})$ is homeomorphic to $(X/\sim, \sigma)$. Therefore $\pi\colon (X, d) \to (X/\sim, \widetilde{d})$ is open. $\square$

Suppose that X has a binary operation, denoted by $\diamond$, which preserves the equivalence relation, i.e. $a \diamond c \sim b \diamond d$ if $a \sim b$ and $c \sim d$. Then there is an operation (still denoted by $\diamond$) defined by: $\widetilde{x} \diamond \widetilde{y} = \widetilde{x \diamond y}$ in the quotient space. On the relation of continuity of the two operations we have the proposition below.

Proposition 2.12. *If* (X, d) *has a continuous binary operation preserving the equivalence relation, then the induced binary operation on the quotient space* $(X/\sim, \widetilde{d})$ *is also continuous.*

Proof. Suppose that $\{\widetilde{x}_n\}$ and $\{\widetilde{y}_n\}$ converge to $\{\widetilde{x}_0\}$ and $\{\widetilde{y}_0\}$ in $(X/\sim, \widetilde{d})$, respectively. By Corollary 2.5, there are x'_n and y'_n in X such that

$$x'_n \sim x_n, \quad y'_n \sim y_n, \quad x'_n \xrightarrow{d} x_0, \quad y'_n \xrightarrow{d} y_0.$$

Hence $x'_n \diamond y'_n \sim x_n \diamond y_n$ and $x'_n \diamond y'_n \xrightarrow{d} x_0 \diamond y_0$.

Since $\widetilde{x}_n \diamond \widetilde{y}_n = \widetilde{x_n \diamond y_n} = \widetilde{x'_n \diamond y'_n}$, by Corollary 2.5 again, it follows that $\widetilde{x}_n \diamond \widetilde{y}_n \xrightarrow{\widetilde{d}} \widetilde{x}_0 \diamond \widetilde{y}_0$. Therefore, the induced binary operation is continuous. $\square$

For the completeness of the quotient space, one can check the following proposition.

Proposition 2.13. *If (X, d) is a complete metric space, then $(X/\sim, \widetilde{d})$ is also complete.*

3. Topologies on Ext-semigroups

In this section, we try to topologize several Ext-semigroups uniformly, especially the Ext-semigroups of unital extensions. From this, these Ext-semigroups are equipped with complete topological structures.

Firstly, we need to recall some definitions and notations of C^*-algebra extension. One can see [2], [16], [17], [18], [19], [20], [21], [22] for more details.

Let A and B be C^*-algebras. An extension of A by B is a short exact sequence

$$e\colon\ 0 \longrightarrow B \xrightarrow{\alpha} E \xrightarrow{\beta} A \longrightarrow 0.$$

Denote this extension by e or (E, α, β) and the set of all such extensions by $\mathcal{E}xt(A, B)$.

The extension (E, α, β) is called *trivial* if the above sequence splits, i.e., if there is a homomorphism $\gamma\colon A \to E$ such that $\beta \circ \gamma = \mathrm{id}_A$. We call (E, α, β) *essential* if $\alpha(A)$ is an essential ideal in E.

Let $0 \longrightarrow B \xrightarrow{\alpha} E \xrightarrow{\beta} A \longrightarrow 0$ be an extension of A by B. Then there is a unique homomorphism $\sigma\colon E \to M(B)$ such that $\sigma \circ \alpha = \iota$, where $M(B)$ is the multiplier algebra of B, and ι is the inclusion map from B into $M(B)$.

The Busby invariant of (E, α, β) is a homomorphism τ from A into the corona algebra $\mathcal{Q}(B) = M(B)/B$ defined by $\tau(a) = \pi(\sigma(b))$ for $a \in A$, where $\pi\colon M(B) \to \mathcal{Q}(B)$ is the quotient map, and $b \in E$ such that $\beta(b) = a$. Note that an extension is essential if and only if its Busby invariant τ is an injective homomorphism.

If A is unital and the Busby invariant is unital, then (E, α, β) is called unital.

Suppose that A and B are C^*-algebras. There are several equivalence relations of extensions of A by B. Let $e_i\colon\ 0 \to B \to E_i \to A \to 0$ be an extension with the Busby invariant τ_i for $i = 1, 2$.

Two extensions e_1 and e_2 are called (strongly) unitarily equivalent, denoted by $e_1 \overset{s}{\sim} e_2$, if there exists a unitary $u \in M(B)$ such that $\tau_2(a) = \pi(u)\tau_1(a)\pi(u)^*$ for all $a \in A$. Denote by $\mathbf{Ext}(A, B)$ or $\mathbf{Ext}_s(A, B)$ the set of (strong) unitary equivalence classes of extensions of A by B. If A is unital, we denote by $\mathbf{Ext}_s^{\mathrm{u}}(A, B)$ the set of unitary equivalence classes of unital essential extensions of A by B.

Two extensions e_1 and e_2 are called *weakly unitarily equivalent*, denoted by $e_1 \overset{w}{\sim} e_2$, if there exists a unitary $v \in \mathcal{Q}(B)$ such that $\tau_2(a) = v\tau_1(a)v^*$ for all $a \in A$. Denote by

$\mathbf{Ext}_w(A,B)$ [$\mathbf{Ext}_w^{\mathrm{u}}(A,B)$ when A is unital] the set of equivalence classes of extensions [unital extensions] of A by B under weak unitary equivalence.

Similarly, we denote essential extensions by adding a superscript "e" on these sets. Then there are several analogs, e.g. $\mathbf{Ext}^e(A,B)$, $\mathbf{Ext}_s^{eu}(A,B)$, $\mathbf{Ext}_w^e(A,B)$ etc.

Let H be a separable infinite-dimensional Hilbert space and $\mathcal{K}$ the ideal of compact operators in $B(H)$. If B is a stable C^*-algebra (i.e. $B\otimes\mathcal{K}\cong B$), then the sum of two extensions τ_1 and τ_2 is defined to be the homomorphism $\tau_1\oplus\tau_2$, where

$$\tau_1\oplus\tau_2\colon\ A\to\mathcal{Q}(B)\oplus\mathcal{Q}(B)\subseteq M_2(\mathcal{Q}(B))\cong\mathcal{Q}(B)$$

and the isomorphism $M_2(\mathcal{Q}(B))\cong\mathcal{Q}(B)$ is induced by an inner isomorphism from $M_2(M(B))$ onto $M(B)$.

The above sets of equivalence classes of extensions are commutative semigroups with respect to this addition when B is stable. One can similarly define these semigroups replacing B by $B\otimes\mathcal{K}$ if B is not stable.

A trivial extension τ is called strongly unital if there exists a unital homomorphism from A to $M(B)$ lifting τ.

Denote by $E\mathrm{xt}(A,B)$ the quotient of $\mathbf{Ext}_s(A,B)$ by the subsemigroup of trivial extensions. If A is unital, $E\mathrm{xt}_s^{\mathrm{u}}(A,B)$ [or $E\mathrm{xt}_w^{\mathrm{u}}(A,B)$] is the quotient of $\mathbf{Ext}^{\mathrm{u}}(A,B)$ [or $\mathbf{Ext}_w^{\mathrm{u}}(A,B)$] by the subsemigroup of strong unital trivial extensions. Denote by $[\tau]$ [or $[\tau]_s$, $[\tau]_w$] the equivalence class of τ in $E\mathrm{xt}(A,B)$ [or $E\mathrm{xt}_s^{\mathrm{u}}(A,B)$, $E\mathrm{xt}_w^{\mathrm{u}}(A,B)$].

Let $e_1,e_2\in\mathcal{E}xt(A,B)$. If e_1 and e_2 are equal in $E\mathrm{xt}(A,B)$ [or $E\mathrm{xt}_s^{\mathrm{u}}(A,B)$, $E\mathrm{xt}_w^{\mathrm{u}}(A,B)$], then e_1 and e_2 are called stably unitarily equivalent, denoted by $e_1\overset{ss}{\sim}e_2$.

Let e be an extension of A by B with the Busby invariant τ. Then e is called absorbing [unital-absorbing when A is unital] if τ is unitarily equivalent to $\tau\oplus\sigma$ for any trivial [strong unital trivial] extension σ.

Suppose that A and D are C^*-algebras and let $\mathrm{Hom}(A,D)$ be the set of homomorphisms form A to D. There are three topologies on $\mathrm{Hom}(A,D)$ as follows:

(1) The topology of pointwise convergence;
(2) The compact-open topology;
(3) The topology of uniform convergence on compact sets.

By [13], Proposition 2.1, the above topologies on $\mathrm{Hom}(A,D)$ coincide and are induced by the metric if A is separable:

$$d(f,g)=\sum_{i=1}^{\infty}\frac{\|(f-g)(a_i)\|}{2^i\|a_i\|},$$

where $\{a_i\}$ is a dense sequence consisting of nonzero elements in A. These topologies are the same as the one given by [1], also see Example 2.7. For these topologies, one can check the following properties.

Proposition 3.1. *Suppose that A and D are two C^*-algebras with A separable. Then* $\mathrm{Hom}(A, D)$ *is a complete metric space under the metric defined above. Furthermore,*

(1) $\mathrm{Hom}^{\mathrm{i}}(A, D) = \{f \in \mathrm{Hom}(A, D)\colon f$ *is injective*$\}$ *is closed;*

(2) $\mathrm{Hom}^{\mathrm{u}}(A, D) = \{f \in \mathrm{Hom}(A, D)\colon f$ *is unital*$\}$ *is closed when A and D are unital;*

(3) $\mathrm{Hom}^{\mathrm{iu}}(A, D) = \{f \in \mathrm{Hom}(A, D)\colon f$ *is injective and unital*$\}$ *is closed when A and D are unital.*

In the following, we assume that A and B are C^*-algebras with A separable and B stable. Similarly to the addition of extensions, one can define a binary operation on $\mathrm{Hom}(A, \mathcal{Q}(B))$ via an inner isomorphism. To be specific, take two isometries s_1, s_2 in $M(B \otimes \mathcal{K})$ with $s_1 s_1^* + s_2 s_2^* = 1$ and then the binary operation is defined by

$$(f \oplus g)(a) = \pi(s_1) f(a) \pi(s_1^*) \oplus \pi(s_2) g(a) \pi(s_2^*)$$

for any f, g in $\mathrm{Hom}(A, \mathcal{Q}(B))$ and a in A, where π is the quotient map from $M(B \otimes \mathcal{K})$ into $\mathcal{Q}(B)$. Notice that $\mathrm{Hom}(A, \mathcal{Q}(B))$ is not an abelian semigroup under this operation in general.

Proposition 3.2. *Suppose that $\oplus$ is the above binary operation defined on the metric space* $\mathrm{Hom}(A, \mathcal{Q}(B))$ *equipped with the preceding metric d. Then*

(1) *the operation $\oplus$ is continuous in the metric;*

(2) *the metric d has the properties*

$$d(f \oplus \varrho, g \oplus \sigma) \leqslant d(f, g) + d(\varrho, \sigma), \quad d(f \oplus h, g \oplus h) = d(f, g)$$

for any f, g, h, ϱ, σ in $\mathrm{Hom}(A, \mathcal{Q}(B))$.

P r o o f. It suffices to show that (2) holds since continuity of the operation follows from the inequality in (2).

Note that $(f \oplus g)(a) = V(f(a) \oplus g(a))V^*$ for every a in A, where $V = (\pi(s_1), \pi(s_2))$ implements that inner isomorphism. Then it follows that:

$$\begin{aligned} \|(f \oplus \varrho)(a) - (g \oplus \sigma)(a)\| &= \|V[(f(a) - g(a)) \oplus (\varrho(a) - \sigma(a))]V^*\| \\ &= \max\{\|f(a) - g(a)\|,\ \|\varrho(a) - \sigma(a)\|\} \\ &\leqslant \|f(a) - g(a)\| + \|\varrho(a) - \sigma(a)\|. \end{aligned}$$

The above discussion also implies that

$$\|(f \oplus h)(a) - (g \oplus h)(a)\| = \|f(a) - g(a)\|.$$

As a result, the metric d has the properties mentioned above. □

In fact, $\mathrm{Hom}(A, \mathcal{Q}(B))$ is the set of extensions of A by B under strong isomorphism (see [2], Section 15.4). In the same way, $\mathrm{Hom}^{\mathrm{i}}(A, \mathcal{Q}(B))$ [or $\mathrm{Hom}^{\mathrm{iu}}(A, \mathcal{Q}(B))$] is the set of essential [or unital essential] extensions of A by B. They are complete metric spaces under the above metric d.

Since the metric d has minimal invariance on each of the preceding metric spaces with respect to the unitary equivalence and the weak unitary equivalence, respectively, d induces pseudometric structures on these spaces. By Propositions 2.12, 2.13, 3.1, and 3.2, we have the following conclusion.

Theorem 3.3. *Equipped with the induced pseudometrics (still denoted by d), these semigroups $\mathbf{Ext}_*(A, B)$, $\mathbf{Ext}_*^e(A, B)$, $\mathbf{Ext}_*^{eu}(A, B)$, where $* = s$, or w, constitute complete topological semigroups.*

Next, we will consider how to topologize the Ext-groups, that is $E\mathrm{xt}(A, B)$, $E\mathrm{xt}_s^{\mathrm{u}}(A, B)$, and $E\mathrm{xt}_w^{\mathrm{u}}(A, B)$. But we do not intend to show that the metrics have minimal invariance with respect to the stable unitary equivalence of extensions. Here we will achieve the goal with the aid of absorbing extensions.

For the sake of convenience, we let

$$E^a(A, B) = \{\tau \in \mathrm{Hom}(A, \mathcal{Q}(B))\colon\ \tau \text{ is absorbing}\},$$
$$E^{\mathrm{ua}}(A, B) = \{\tau \in \mathrm{Hom}^{\mathrm{u}}(A, \mathcal{Q}(B))\colon\ \tau \text{ is unital-absorbing}\}.$$

Recall [7] that an extension $e\colon\ 0 \to B \to E \to A \to 0$ is called *purely large* if for every $x \in E \setminus B$, the C^*-algebra $\overline{xBx^*}$ contains a subalgebra which is stable and is full in B.

By [7], [8], there is a characterization of absorbing extensions and purely large extensions in the following.

Lemma 3.4. *Let A and B be separable C^*-algebras with A nuclear. Suppose that*

$$e\colon\ 0 \longrightarrow B \longrightarrow E \xrightarrow{\psi} A \longrightarrow 0$$

is a nonunital essential extension. Then the following statements are equivalent:

(1) *The extension e is absorbing.*
(2) *The extension e is purely large.*
(3) *For every $\varepsilon > 0$, $x \in E^+ \setminus B$ and $b \in B^+$ with $\|\psi(x)\| = 1$ and $\|b\| = 1$, there exists $r \in B$ such that $\|r\| = 1$ and $\|rxr^* - b\| < \varepsilon$.*

Theorem 3.5. *Assume that A and B satisfy the conditions in Lemma 3.4. Equipped with the above metric d, $E^a(A, B)$ and $E^{\mathrm{ua}}(A, B)$ are complete metric spaces.*

P r o o f. Suppose that $\{\tau_n\}$ is a sequence of absorbing extensions with limit τ in $E(A,B)$. Let π be the quotient homomorphism form $M(B)$ into $\mathcal{Q}(B)$ and

$$e_n\colon\ 0 \to B \to E_n \to A \to 0$$

the standard extension of τ_n, where $E_n = \pi^{-1}(\tau_n(A))$ and $\psi_n = \tau_n^{-1} \circ \pi$. Then there is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E_n & \xrightarrow{\psi_n} & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \tau_n} & & \\ 0 & \longrightarrow & B & \longrightarrow & M(B) & \xrightarrow{\pi} & \mathcal{Q}(B) & \longrightarrow & 0. \end{array}$$

Similarly, for the extension τ there is an exact short sequence $e\colon\ 0 \to B \to E \to A \to 0$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E & \xrightarrow{\psi} & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \tau} & & \\ 0 & \longrightarrow & B & \longrightarrow & M(B) & \xrightarrow{\pi} & \mathcal{Q}(B) & \longrightarrow & 0, \end{array}$$

where $E = \pi^{-1}(\tau(A))$ and $\psi = \tau^{-1} \circ \pi$.

For every $\varepsilon > 0$, $x \in E^+ \setminus B$ and $b \in B^+$ with $\|\psi(x)\| = 1$ and $\|b\| = 1$, we set $a = \psi(x)$. Then $a \in A^+$ and $\tau(a) = \pi(x)$. Since $\{\tau_n(a)\}$ converges to $\tau(a)$, there is n such that $\|\tau_n(a) - \tau(a)\| < \varepsilon/2$. Choose $y \in M(B)^+$ such that $\tau_n(a) = \pi(y)$. Then $y \in E_n^+$ and

$$\|\pi(y)\| = \|\tau_n(a)\| = \|a\| = 1.$$

Hence,

$$\|\pi(x-y)\| = \inf_{z\in B} \|x - (y+z)\| < \varepsilon/2.$$

It follows that there is $z_0 \in B^+$ such that $\|x - (y+z_0)\| < \varepsilon/2$ and $\|\pi(y+z_0)\| = 1$.

Since e_n is purely large, for $y + z_0 \in E_n^+$, $b \in B^+$ and $\varepsilon > 0$, by Lemma 3.4 there exists $r \in B$ such that $\|r\| = 1$ and $\|r(y+z_0)r^* - b\| < \varepsilon/2$. Hence,

$$\begin{aligned} \|rxr^* - b\| &\leqslant \|rxr^* - r(y+z_0)r^*\| + \|r(y+z_0)r^* - b\| \\ &\leqslant \|x - (y+z_0)\| + \|r(y+z_0)r^* - b\| \leqslant \varepsilon/2 + \varepsilon/2 < \varepsilon. \end{aligned}$$

By Lemma 3.4 again, e is purely large. Therefore $E^a(A,B)$ is complete.

For unital-absorbing extensions, there is an analogue of Lemma 3.4 and hence by a similar argument of the above we conclude that $E^{\mathrm{ua}}(A,B)$ is complete. □

If there is a trivial absorbing [unital-absorbing] extension, by [2] we have the following isomorphisms:

$$E\mathrm{xt}(A,B) = E^{\mathrm{a}}(A,B)/\overset{s}{\sim}, \quad E\mathrm{xt}_s^{\mathrm{u}}(A,B) = E^{\mathrm{ua}}(A,B)/\overset{s}{\sim},$$
$$E\mathrm{xt}_w^{\mathrm{u}}(A,B) = E^{\mathrm{ua}}(A,B)/\overset{w}{\sim}.$$

Let π, π_s and π_w denote the tree quotient maps, respectively.

Equipped with the metric d, $E^{\mathrm{a}}(A,B)$ and $E^{\mathrm{ua}}(A,B)$ are metric spaces and d has minimal invariance with respect to the unitary equivalence and the weak unitary equivalence, respectively. Hence, it induces three pseudometrics $\widetilde{d}$, $\widetilde{d}_s$, $\widetilde{d}_w$ on $E\mathrm{xt}(A,B)$, $E\mathrm{xt}_s^{\mathrm{u}}(A,B)$ and $E\mathrm{xt}_w^{\mathrm{u}}(A,B)$, respectively.

With the help of Theorems 2.10, 2.11, Propositions 2.12, 2.13, 3.1, 3.2 and Theorem 3.5, we summarize the topological properties of these quotient semigroups in the following.

Theorem 3.6. *Suppose that A and B are C^*-algebras with A separable. Assume that there is a trivial absorbing [or unital-absorbing] extension of A by B. Then:*

(1) *$E\mathrm{xt}(A,B)$, $E\mathrm{xt}_s^{\mathrm{u}}(A,B)$ and $E\mathrm{xt}_w^{\mathrm{u}}(A,B)$ are topological semigroups.*
(2) *The topologies induced by $\widetilde{d}$, $\widetilde{d}_s$, $\widetilde{d}_w$ coincide with the quotient topologies induced by π, π_s and π_w on $E\mathrm{xt}(A,B)$, $E\mathrm{xt}_s^{\mathrm{u}}(A,B)$ and $E\mathrm{xt}_w^{\mathrm{u}}(A,B)$, respectively. In addition, these topologies satisfy the first axiom of countability.*
(3) *The quotient maps π, π_s and π_w are open in the three situations.*
(4) *If A is nuclear and B is also separable, then $E\mathrm{xt}(A,B)$, $E\mathrm{xt}_s^{\mathrm{u}}(A,B)$ and $E\mathrm{xt}_w^{\mathrm{u}}(A,B)$ are complete.*

Remark 3.7. Special cases of the above semigroups also appeared in [3], [4], [5], [6], [10], [11], [12], [13] and they are topologized in several ways. Here they are equipped with topologies uniformly.

It is obvious that the completeness of the above topological semigroups comes from Theorem 3.5. One may notice that this theorem requires that B is separable. Since whether an absorbing extension can be lifted to an absorbing completely positive map into the multiplier algebra is unknown, one can not use the completeness of completely positive maps to show that the set of absorbing extensions is complete. But without the help of the separability of B, we can also directly show that the three topological groups are complete.

Theorem 3.8. *Let A be a separable nuclear C^*-algebra and B a σ-unital C^*-algebra. Suppose that $E\mathrm{xt}(A,B)$, $E\mathrm{xt}_s^{\mathrm{u}}(A,B)$ and $E\mathrm{xt}_w^{\mathrm{u}}(A,B)$ are endowed with the pseudometrics $\widetilde{d}$, $\widetilde{d}_s$, $\widetilde{d}_w$, respectively. Then they are complete topological groups.*

P r o o f. By [2], Corollary 15.8.4, the assumption implies that these semigroups become abelian groups. It suffices to show that the conclusion holds for the case of $(E\mathrm{xt}_s^{\mathrm{u}}(A,B),\widetilde{d}_s)$ because the proofs of the others are similar.

Suppose that $\{[\tau_n]_s\}$ is a Cauchy sequence in $E\mathrm{xt}_s^{\mathrm{u}}(A,B)$. By passing to a subsequence, if necessary, one can assume that $\widetilde{d}_s([\tau_n]_s,[\tau_{n+1}]_s) < 1/2^{n+1}$. By the definition of $\widetilde{d}_s$ and the mathematical induction, we can choose $\sigma_n \in E^{\mathrm{au}}(A,B)$ one by one such that $[\tau_n]_s = [\sigma_n]_s$ and

$$\widetilde{d}_s([\sigma_n]_s,[\sigma_{n+1}]_s) < 1/2^n.$$

Then $\{\sigma_n\}$ is a Cauchy sequence in $E^{\mathrm{au}}(A,B)$.

By the completeness of $E^{\mathrm{u}}(A,B)$, there is an essential unital extension σ such that $\sigma_n \xrightarrow{d} \sigma$ in $E^{\mathrm{u}}(A,B)$. By the Kasparov absorbing theorem, there is an absorbing unital trivial extension σ_0. Then

$$\sigma_n \oplus \sigma_0 \xrightarrow{d} \sigma \oplus \sigma_0.$$

Hence it follows that

$$\widetilde{d}_s([\tau_n]_s,[\sigma\oplus\sigma_0]_s) = \widetilde{d}_s([\sigma_n\oplus\sigma_0]_s,[\sigma\oplus\sigma_0]_s) \leqslant d(\sigma_n\oplus\sigma_0,\sigma\oplus\sigma_0) \to 0$$

in $E\mathrm{xt}_s^{\mathrm{u}}(A,B)$ as $n\to\infty$. Therefore $E\mathrm{xt}_s^{\mathrm{u}}(A,B)$ is complete.

As regards the invariance under translation, by Proposition 3.2 and similar argument in [11], Lemma 5.1, one have

$$\widetilde{d}_s([\tau_1]_s\oplus[\sigma_1]_s,[\tau_2]_s\oplus[\sigma_2]_s) \leqslant \widetilde{d}_s([\tau_1]_s,[\tau_2]_s) + \widetilde{d}_s([\sigma_1]_s,[\sigma_2]_s),$$
$$\widetilde{d}_s([\tau_1]_s\oplus[\sigma]_s,[\tau_2]_s\oplus[\sigma]_s) = \widetilde{d}_s([\tau_1]_s,[\tau_2]_s)$$

for any τ_i, σ_i, σ in $\mathrm{Hom}(A,\mathcal{Q}(B))$. This implies that the pseudometric $\widetilde{d}_s$ is translation invariant for the additive operation in $E\mathrm{xt}_s^{\mathrm{u}}(A,B)$.

Finally, from the translation invariance, it follows that the inverse operation is also continuous. Therefore $E\mathrm{xt}_s^{\mathrm{u}}(A,B)$ is a topological group. $\square$

Remark 3.9. As for the separability of these topological Ext-groups, it should be pointed out that the above Ext-groups may not be separable in general even if A is separable and B is σ-unital. The following is a counterexample. But when B is separable, these topological Ext-groups are separable (see [13]).

Example 3.10. Let A be the C^*-algebra $C_0(0,1)$ consisting of all continuous functions on the open interval $(0,1)$ which vanish on the end points. Let $B = \bigoplus\limits_{\alpha\in\Lambda}\mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space and Λ is an index set with cardinality uncountable. Then A is separable and B is σ-unital. By the Kasparov theorem and [13], there are isomorphisms of topological groups:

$$\mathrm{Ext}(A, B) = KK^1(A, B) = KK(\mathbb{C}, B).$$

Using the UCT, one can compute that

$$KK(\mathbb{C}, B) = \mathrm{Hom}(K_0(\mathbb{C}), K_0(B)) = \mathrm{Hom}\Big(\mathbb{Z}, \bigoplus_{\alpha\in\Lambda} \mathbb{Z}\Big).$$

Since the topological group $\mathrm{Hom}\Big(\mathbb{Z}, \bigoplus_{\alpha\in\Lambda} \mathbb{Z}\Big)$ is discrete and uncountable, $KK(\mathbb{C}, B)$ is not separable.

References

[1] *W. Arveson*: Notes on extensions of C^*-algebras. Duke Math. J. *44* (1977), 329–355. zbl MR doi
[2] *B. Blackadar*: K-Theory for Operator Algebras. Mathematical Sciences Research Institute Publications 5, Cambridge University Press, Cambridge, 1998. zbl MR
[3] *L. G. Brown*: The universal coefficient theorem for Ext and quasidiagonality. Operator Algebras and Group Representations, Vol. I. Monographs and Studies in Mathematics 17, Pitman, Boston, 1984, pp. 60–64. zbl MR
[4] *L. G. Brown, R. G. Douglas, P. A. Fillmore*: Extensions of C^*-algebras, operators with compact self-commutators, and K-homology. Bull. Am. Math. Soc. *79* (1973), 973–978. zbl MR doi
[5] *L. G. Brown, R. G. Douglas, P. A. Fillmore*: Extensions of C^*-algebras and K-homology. Ann. Math. (2) *105* (1977), 265–324. zbl MR doi
[6] *M. Dadarlat*: On the topology of the Kasparov groups and its applications. J. Func. Anal. *228* (2005), 394–418. zbl MR doi
[7] *G. A. Elliott, D. Kucerovsky*: An abstract Voiculescu-Brown-Douglas-Fillmore absorption theorem. Pac. J. Math. *198* (2001), 385–409. zbl MR doi
[8] *D. Kucerovsky, P. W. Ng*: The corona factorization property and approximate unitary equivalence. Houston J. Math. *32* (2006), 531–550. zbl MR
[9] *J. Rosenberg, C. Schochet*: The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor. Duke Math. J. *55* (1987), 431–474. zbl MR doi
[10] *N. Salinas*: Homotopy invariance of $\mathrm{Ext}(\mathcal{A})$. Duke Math. J. *44* (1977), 777–794. zbl MR doi
[11] *N. Salinas*: Quasitriangular extensions of C^*-algebras and problems on joint quasitriangularity of operators. J. Oper. Theory *10* (1983), 167–205. zbl MR
[12] *N. Salinas*: Relative quasidiagonality and KK-theory. Houston J. Math. *18* (1992), 97–116. zbl MR
[13] *C. L. Schochet*: The fine structure of the Kasparov groups I: Continuity of the KK-pairing. J. Func. Anal. *186* (2001), 25–61. zbl MR doi
[14] *C. L. Schochet*: The fine structure of the Kasparov groups II: Topologizing the UCT. J. Func. Anal. *194* (2002), 263–287. zbl MR doi
[15] *C. L. Schochet*: The fine structure of the Kasparov groups III: Relative quasidiagonality. J. Oper. Theory *53* (2005), 91–117. zbl MR

[16] *C. Wei*: Universal coefficient theorems for the stable Ext-groups. J. Funct. Anal. *258* (2010), 650–664. zbl MR doi
[17] *C. Wei*: Classification of extensions of A$\mathbb{T}$-algebras. Int. J. Math. *22* (2011), 1187–1208. zbl MR doi
[18] *C. Wei*: On the classification of certain unital extensions of C^*-algebras. Houston J. Math. *41* (2015), 965–991. zbl MR
[19] *C. Wei, S. Liu*: On the structure of multiplier algebras. Rocky Mt. J. Math. *47* (2017), 997–1012. zbl MR doi
[20] *C. Wei, L. Wang*: Hereditary C*-subalgebras and comparison of positive elements. Sci. China, Math. *53* (2010), 1565–1570. zbl MR doi
[21] *C. Wei, L. Wang*: Isomorphism of extensions of $C(\mathbb{T}^2)$. Sci. China, Math. *54* (2011), 281–286. zbl MR doi
[22] *R. Xing, C. Wei, S. Liu*: Quotient semigroups and extension semigroups. Proc. Indian Acad. Sci., Math. Sci. *122* (2012), 339–350. zbl MR doi

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