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NONEXISTENCE OF ENTIRE POSITIVE SOLUTION FOR A CONFORMAL k-HESSIAN INEQUALITY

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Abstract. In this paper, we study the nonexistence of entire positive solution for a conformal k-Hessian inequality in $\mathbb{R}^n$ via the method of proof by contradiction.

Keywords: conformal Hessian inequality; entire positive solution

MSC 2010: 35J60, 35B08, 35B09

1. Introduction

In this paper, we study a conformal k-Hessian inequality

$$\sigma_k(A^g) \geqslant u^\alpha, \tag{1.1}$$

where $\sigma_k(A^g) = \sigma_k(\lambda(A^g))$, u is the unknown function, α is a constant, and σ_k denotes the k-Hessian operator or the kth order elementary symmetric polynomial given by

$$\sigma_k(\lambda) = \sum_{i_1<\ldots<i_s} \prod_{s=1}^{k} \lambda_{i_s}, \tag{1.2}$$

where $k = 1, \ldots, n$, $i_1, \ldots, i_s \in \{1, \ldots, n\}$, and $\lambda = (\lambda_1, \ldots, \lambda_n)$ denotes eigenvalues of the matrix A^g. In (1.1), A^g is the conformal Schouten tensor of $(\mathcal{M}, g)$ given by

$$A^g := \frac{1}{n-2}\Big(\mathrm{Ric}_g - \frac{\mathrm{R}_g}{2(n-1)} g\Big) = \nabla^2 u - \tfrac{1}{2}|\nabla u|^2 g_0 + \nabla u \otimes \nabla u + A^{g_0} \tag{1.3}$$

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and $\mathcal{M}, g$, respectively, denote the manifold and a conformal metric $g = g_u = \mathrm{e}^{-2u} g_0$, where Ric_g and R_g, respectively, denote the Ricci tensor and scalar curvature, ∇ is the Levi-Civita connection of $\mathcal{M}$, ∇u and $\nabla^2 u$ denote the covariant gradient and covariant Hessian of the function of u, g_0 is a given metric on $\mathcal{M}$, and A^{g_0} is the Schouten tensor of $(\mathcal{M}, g_0)$ given by

$$A^{g_0} = \frac{1}{n-2}\Big(\mathrm{Ric}_{g_0} - \frac{\mathrm{R}_{g_0}}{2(n-1)} g_0\Big).$$

The leading term $\sigma_k(A^g)$ in the inequality (1.1) is related to the k-Yamabe problem, see [21], [24]. When $k = 1$, it is related to the well-known Yamabe problem, see [1], [2], [20], [22], [25]. Note that for Euclidean space $\mathbb{R}^n$, the Schouten tensor A^g has the form

$$A^g = D^2 u - \big(\tfrac{1}{2}|Du|^2 I - Du \otimes Du\big), \tag{1.4}$$

where Du and $D^2 u$ denote the gradient vector and Hessian matrix of u, respectively, I is the $n \times n$ identity matrix.

In this paper, we consider the admissible solution of the inequality (1.1) in $\mathbb{R}^n$ with A^g given by (1.4). According to Caffarelli-Nirenberg-Spruck (see [6]), we say that u is an admissible function of (1.1) if $\lambda(A^g) \in \Gamma^k := \{\lambda \in \mathbb{R}^n \colon \sigma_s(\lambda) > 0, s = 1, \ldots, k\}$. We give the following nonexistence result of the positive admissible solution of the inequality (1.1) in $\mathbb{R}^n$.

Theorem 1.1. *The inequality* (1.1) *with A^g in the form* (1.4) *has no entire positive admissible solution in $\mathbb{R}^n$ for $n \geqslant 3$, $\alpha \geqslant 0$ and $n = 2$, $\alpha > k$.*

The result in Theorem 1.1 is sharp when $n = 2$. In the case when $n = 2$ and $k = 1$, some obvious examples will be given in Section 2 to show that there are entire positive admissible solutions of (1.1) if α does not satisfy $\alpha > 1$.

Taking $k = 1$ in Theorem 1.1, we have the following nonexistence result for an inequality involving the Laplace operator.

Corollary 1.1. *The inequality*

$$\Delta u - \Big(\frac{n}{2} - 1\Big)|Du|^2 \geqslant u^\alpha \tag{1.5}$$

has no entire positive solution in $\mathbb{R}^n$ for $n \geqslant 3$, $\alpha \geqslant 0$ and $n = 2$, $\alpha > 1$.

Remark 1.1. When $n = 2$, the inequality (1.5) reduces to $\Delta u \geqslant u^\alpha$. Therefore, when $n = 2$, the result in Corollary 1.1 for $\alpha > 1$ corresponds to the well-known results in [4], [12], [15], [16]. If u^α in the inequalities (1.1) and (1.5) is replaced by Cu^α with any positive constant C, the conclusions of Theorem 1.1 and Corollary 1.1 still hold.

We recall some related studies on the entire solutions of Hessian equations and Hessian inequalities. The classification of the entire nonnegative solutions of the equation $-\Delta u = u^\alpha$ in $\mathbb{R}^n$ is established for $1 \leqslant \alpha < (n+2)/(n-2)$ in [7] and for $\alpha = (n+2)/(n-2)$ in [5], respectively. The similar classification results are extended in [13] to admissible solutions of the conformal k-Hessian equations $\sigma_k(A^g) = u^\alpha$ in $\mathbb{R}^n$ for $\alpha \in [0,\infty)$, where $k \in N^+$ and $g = u^{-2}\,\mathrm{d}x^2$ with $u > 0$ is a locally conformally flat metric in $\mathbb{R}^n$, where A^g is given by $A^g = g^{-1}A^{g_0}$. In [18], the same classification result for the special case of $n = 2k+1$ is also obtained by suitable choices of the test functions and the arguments of integration. Using the method as in [7], [18], a nonexistence result for the Hessian inequality $\sigma_k(-D^2u) \geqslant u^\alpha$ is proved in [17]. The conformal k-Hessian inequality (1.1) with $A^g = u(D^2u) - \frac{1}{2}|Du|^2 I$ is considered in [19], where the nonexistence result for $2k < n$ and $\alpha \in [k,\infty)$ is formulated. Note that A^g in (1.4) has a different structure from that in [19], which leads to a different nonexistence result in Theorem 1.1 here from that of [19].

The nonexistence of the inequality (1.1) in Theorem 1.1 implies that the equation

$$\sigma_k(D^2u - A(x,u,Du)) = B(x,u,Du) \tag{1.6}$$

with $A = \frac{1}{2}|Du|^2 I - Du \otimes Du$ and $B = u^\alpha$ has no entire positive admissible subsolution in $\mathbb{R}^n$. Note that the equation in the general form (1.6) has been studied in [9], [10], [23], where the existence results of classical solutions for Dirichlet boundary value problem and the oblique boundary value problem on bounded domains are established. One can refer to [3], [8], [11] for various nonexistence results of entire positive subsolutions for equation (1.6) arising from the conformal geometry with different special forms of A and B. In a sequel, we will study the corresponding nonexistence for the equation (1.6) with $A = \frac{1}{2}|Du|^2 I - Du \otimes Du$ and $B = \mathrm{e}^{\alpha u}$, where α is a constant.

The organization of this paper is as follows. In Section 2, we first recall Maclaurin's inequality for k-Hessian operators. Then we give the proof of Theorem 1.1 by appropriate choice of test functions and the method of integration by parts. The proof is divided into two cases, where Schwarz's inequality and Young's inequality are properly used. At the end, we give examples of entire positive admissible solutions of (1.1) for some $\alpha \leqslant k = 1$ in the two dimensional case.

2. Proof of Theorem 1.1

In this section, we give the proof of the nonexistence result, Theorem 1.1. We first establish relations between the k-Hessian operator and the Laplace operator by using Maclaurin's inequality. In this way, we reduce the conformal k-Hessian inequality (1.1) to an inequality involving the Laplace operator. Then by multiplying a cut-off function and integrating the inequality over $\mathbb{R}^n$, we divide the proof into two cases, $\alpha > k$, $n \geqslant 2$ and $0 < \alpha \leqslant k$, $n \geqslant 3$. By a detailed analysis, we get contradictions.

First, we recall Maclaurin's inequality in [14].

Lemma 2.1. *For $\lambda \in \Gamma^k$, $1 \leqslant l \leqslant k \leqslant n$, we have*

$$\left[\frac{\sigma_k(\lambda)}{C_n^k}\right]^{1/k} \leqslant \left[\frac{\sigma_l(\lambda)}{C_n^l}\right]^{1/l}. \tag{2.1}$$

The inequality (2.1) is a direct consequence of Newton's inequality, see the proof of Lemma 15.12 in [14]. Note that in [14], the term in the brackets on the left hand side of (2.1) is defined to be $\sigma_k(\lambda)$ (which is different from our definition of the k-Hessian operator).

Taking $l = 1$ in Lemma 2.1, we can obtain a relationship between the k-Hessian operator and the Laplace operator, which is crucial in studying the nonexistence for the inequality (1.1). We now give the proof of the main result, Theorem 1.1.

Proof of Theorem 1.1. Suppose that $u > 0$ is an admissible solution of (1.1). We will deduce the contradiction.

Since

$$\frac{C_n^1}{(C_n^k)^{1/k}} = \left(k!\,\frac{n}{n}\,\frac{n}{n-1}\cdots\frac{n}{n-k+1}\right)^{1/k} \geqslant 1$$

when $\lambda(A^g) \in \Gamma^k$, from Lemma 2.1 we have

$$\sigma_k^{1/k}(A^g) \leqslant \sigma_1(A^g). \tag{2.2}$$

Since A^g has the form (1.4), it can be readily calculated that

$$\sigma_1(A^g) = \Delta u - \left(\frac{n}{2} - 1\right)|Du|^2. \tag{2.3}$$

Using (1.1), (2.2) and (2.3), we have

$$u^{\alpha/k} \leqslant \Delta u - \left(\frac{n}{2} - 1\right)|Du|^2. \tag{2.4}$$

Multiplying both sides of (2.4) by $u^\delta\eta^\theta$ and integrating over $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} u^{\alpha/k+\delta}\eta^\theta \,\mathrm{d}x \leqslant -\left(\frac{n}{2} - 1\right)\int_{\mathbb{R}^n} u^\delta\eta^\theta|Du|^2 \,\mathrm{d}x + \int_{\mathbb{R}^n} u^\delta\eta^\theta\Delta u \,\mathrm{d}x, \tag{2.5}$$

where δ, θ are constants to be determined, $\eta \in C^2$ is a cut-off function satisfying

$$\eta \equiv 1, \qquad \text{in } B_R, \tag{2.6}$$

$$0 \leqslant \eta \leqslant 1, \quad \text{in } B_{2R}, \tag{2.7}$$

$$\eta \equiv 0, \qquad \text{in } \mathbb{R}^n \setminus B_{2R}, \tag{2.8}$$

$$|D\eta| \leqslant \frac{C}{R}, \quad \text{in } \mathbb{R}^n, \tag{2.9}$$

where B_R denotes a ball in $\mathbb{R}^n$ centered at 0 with radius R, C is a positive constant. In order to deal with the last term of (2.5), we use the integration by parts to get

$$\begin{aligned} \int_{\mathbb{R}^n} u^\delta \eta^\theta \Delta u \,\mathrm{d}x = & -\int_{B_{2R}} (D_i u) D_i(u^\delta \eta^\theta) \,\mathrm{d}x \\ = & -\delta \int_{\mathbb{R}^n} |Du|^2 u^{\delta-1}\eta^\theta \,\mathrm{d}x - \theta \int_U u^\delta \eta^{\theta-1}(D_i\eta)(D_i u) \,\mathrm{d}x, \end{aligned} \tag{2.10}$$

where

$$U := \mathrm{supp}|D\eta| = \{x \in \mathbb{R}^n\colon\ R < |x| < 2R\}.$$

Note that we use the standard summation convention meaning that the repeated indices indicate summation from 1 to n unless otherwise specified.

We next split the proof into the following two cases of α and n:

(i) $\alpha > k$ and $n \geqslant 2$;

(ii) $0 \leqslant \alpha \leqslant k$ and $n \geqslant 3$.

We shall make different choices of the constants δ and θ in cases (i) and (ii), respectively.

In case (i), we fix the constants δ and θ such that

$$\delta > \max\Big\{\frac{n-2}{k}\alpha - n + 1, 0\Big\}, \quad \theta > \frac{2(\alpha + k\delta)}{\alpha - k}.$$

Note that we always have $\delta > 0$ and $\theta > 2$. For the last term in (2.10), using Schwarz's inequality, we get

$$\begin{aligned} -\theta \int_U u^\delta \eta^{\theta-1}(D_i\eta)(D_i u) \,\mathrm{d}x &\leqslant \theta \int_U u^\delta \eta^{\theta-1}|D\eta||Du| \,\mathrm{d}x \\ &\leqslant \varepsilon_0\theta \int_{\mathbb{R}^n} |Du|^2 u^{\delta-1}\eta^\theta \,\mathrm{d}x + \theta C_{\varepsilon_0} \int_U |D\eta|^2 u^{\delta+1}\eta^{\theta-2} \,\mathrm{d}x, \end{aligned} \tag{2.11}$$

where ε_0 is any positive constant and C_{ε_0} denotes some positive constant depending on ε_0. Inserting (2.11) and (2.10) into (2.5), we get

$$\begin{aligned} \int_{\mathbb{R}^n} u^{\alpha/k+\delta}\eta^\theta \,\mathrm{d}x \leqslant & -\Big(\frac{n}{2} - 1\Big) \int_{\mathbb{R}^n} u^\delta \eta^\theta |Du|^2 \,\mathrm{d}x \\ & + (\varepsilon_0\theta - \delta) \int_{\mathbb{R}^n} |Du|^2 u^{\delta-1}\eta^\theta \,\mathrm{d}x + \theta C_{\varepsilon_0} \int_U |D\eta|^2 u^{\delta+1}\eta^{\theta-2} \,\mathrm{d}x. \end{aligned} \tag{2.12}$$

Hence, by taking

$$\varepsilon_0 \leqslant \delta/\theta$$

in (2.12), we get

$$\int_{\mathbb{R}^n} u^{\alpha/k+\delta}\eta^\theta \,\mathrm{d}x \leqslant \theta C_{\varepsilon_0}\int_U |D\eta|^2 u^{\delta+1}\eta^{\theta-2}\,\mathrm{d}x. \tag{2.13}$$

Then ε_0 is now fixed. Applying Young's inequality to the last term in (2.13), we have

$$\int_{\mathbb{R}^n} u^{\alpha/k+\delta}\eta^\theta \,\mathrm{d}x \leqslant \theta\varepsilon_1 C_{\varepsilon_0}\int_U \frac{(u^{\delta+1}\eta^p)^s}{s}\,\mathrm{d}x + \theta C_{\varepsilon_1}C_{\varepsilon_0}\int_U \frac{(\eta^q|D\eta|^2)^t}{t}\,\mathrm{d}x, \tag{2.14}$$

where p, q are positive constants satisfying

$$p+q=\theta-2,\quad s>1,\ t>1,\ \frac{1}{s}+\frac{1}{t}=1,$$

where ε_1 is any positive constant, C_{ε_1} denotes some positive constant depending on ε_1. Setting

$$s=\frac{\alpha/k+\delta}{\delta+1}>1,$$

we get

$$t=\frac{\alpha+k\delta}{\alpha-k}>1,\quad p=\frac{\theta(\delta+1)}{\alpha/k+\delta}>0,\quad q=\theta-2-p>0,\quad qt=\theta-\frac{2(\alpha+k\delta)}{\alpha-k}>0.$$

Inserting p, q, s and t into (2.14), we have

$$\begin{aligned}\int_{\mathbb{R}^n} u^{\alpha/k+\delta}\eta^\theta \,\mathrm{d}x \leqslant{}& \frac{\theta\varepsilon_1 C_{\varepsilon_0}k(\delta+1)}{\alpha+k\delta}\int_U u^{\alpha/k+\delta}\eta^\theta\,\mathrm{d}x\\ &+\frac{\theta C_{\varepsilon_1}C_{\varepsilon_0}(\alpha-k)}{\alpha+k\delta}\int_U \eta^{\theta-2(\alpha+k\delta)/(\alpha-k)}|D\eta|^{2(\alpha+k\delta)/(\alpha-k)}\,\mathrm{d}x.\end{aligned} \tag{2.15}$$

Taking

$$\varepsilon_1 < \frac{\alpha+k\delta}{\theta C_{\varepsilon_0}k(\delta+1)}$$

and using (2.9), we get from (2.15) that

$$\int_{\mathbb{R}^n} u^{\alpha/k+\delta}\eta^\theta \,\mathrm{d}x \leqslant \frac{\theta C_{\varepsilon_1}C_{\varepsilon_0}(\alpha-k)\omega_n C}{\alpha+k\delta-\theta\varepsilon_1 C\varepsilon_0 k(\delta+1)}R^{n-2(\alpha+k\delta)/(\alpha-k)}, \tag{2.16}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. Now the constant ε_1 is fixed. Recall that $\delta > \max\{\alpha(n-2)/k - n + 1, 0\}$. If $\alpha(n-2)/k - n + 1 \geqslant 0$, we have

$$(2.17)\quad \begin{aligned} n - \frac{2(\alpha + k\delta)}{\alpha - k} &= n - 2 - \frac{2k(\delta+2)}{\alpha - k} \\ &< n - 2 - \frac{2k(\alpha(n-2)/k - n + 3)}{\alpha - k} = -(n-2) - \frac{2k}{\alpha - k} < 0. \end{aligned}$$

If $\alpha(n-2)/k - n + 1 < 0$, we have $k/\alpha > (n-2)/(n-1)$ and

$$(2.18)\quad \begin{aligned} n - \frac{2(\alpha + k\delta)}{\alpha - k} &< n - \frac{2\alpha}{\alpha - k} \\ &= n - \frac{2}{1 - k/\alpha} < n - \frac{2}{1 - (n-2)/(n-1)} = -n + 2 \leqslant 0. \end{aligned}$$

Then $n - 2(\alpha + k\delta)/(\alpha - k)$ is negative in both the above cases. Letting $R \to \infty$ in (2.16), we get

$$(2.19)\qquad \int_{\mathbb{R}^n} u^{\alpha/k+\delta}\eta^{\theta}\,\mathrm{d}x \leqslant 0.$$

On the other hand, from the positivity of u and the property of the cut-off function η, the integration in (2.19) should be positive. We then get a contradiction in case (i).

In case (ii), we further consider the two subcases:

(a) $0 < \alpha \leqslant k$ and $n \geqslant 3$;

(b) $\alpha = 0$ and $n \geqslant 3$.

In case (ii) (a), we fix the constants δ and θ such that

$$\delta > \max\Big\{\frac{\alpha}{k}\Big(\frac{n}{4} - 1\Big), 1\Big\}, \quad \theta > 4\Big(1 + \frac{k\delta}{\alpha}\Big).$$

For the last term in (2.10), we have

$$(2.20)\quad \begin{aligned} -\theta\int_U u^{\delta}\eta^{\theta-1}(D_i\eta)(D_iu)\,\mathrm{d}x &\leqslant \theta\int_U u^{\delta}\eta^{\theta-1}|D\eta||Du|\,\mathrm{d}x \\ &\leqslant \theta\varepsilon_2\int_U |Du|^2u^{\delta-\alpha/2k}\eta^{\theta}\,\mathrm{d}x + \theta C_{\varepsilon_2}\int_U |D\eta|^2u^{\delta+\alpha/2k}\eta^{\theta-2}\,\mathrm{d}x \\ &\leqslant \theta\varepsilon_2\int_{\mathbb{R}^n} |Du|^2u^{(1-\alpha/2k)\delta}u^{\alpha/2k(\delta-1)}\eta^{\theta}\,\mathrm{d}x + \theta C_{\varepsilon_2}\int_U |D\eta|^2u^{\delta+\alpha/2k}\eta^{\theta-2}\,\mathrm{d}x \\ &\leqslant \frac{\theta\varepsilon_2\varepsilon_3(2k-\alpha)}{2k}\int_{\mathbb{R}^n} |Du|^2u^{\delta}\eta^{\theta}\,\mathrm{d}x + \frac{\theta\varepsilon_2C_{\varepsilon_3}\alpha}{2k}\int_{\mathbb{R}^n} |Du|^2u^{\delta-1}\eta^{\theta}\,\mathrm{d}x \\ &\quad + \theta C_{\varepsilon_2}\int_U |D\eta|^2u^{\delta+\alpha/2k}\eta^{\theta-2}\,\mathrm{d}x, \end{aligned}$$

where Schwarz's inequality is used to obtain the second inequality, Young's inequality $ab \leqslant \varepsilon_3 a^p/p + C_{\varepsilon_3} b^q/q$ with exponent pair $(p,q) = (2k/(2k-\alpha), 2k/\alpha)$ is used to obtain the last inequality, ε_2, ε_3 are positive constants, C_{ε_2}, C_{ε_3} denote the positive constants depending on ε_2, ε_3 respectively. Inserting (2.20) and (2.10) into (2.5), we get

$$\tag{2.21} \begin{aligned} \int_{\mathbb{R}^n} u^{\alpha/k+\delta}\eta^\theta \,\mathrm{d}x &\leqslant \Big(\frac{\theta\varepsilon_2\varepsilon_3(2k-\alpha)}{2k} - \frac{n-2}{2}\Big)\int_{\mathbb{R}^n} |Du|^2 u^\delta \eta^\theta \,\mathrm{d}x \\ &\quad + \Big(\frac{\theta\varepsilon_2 C_{\varepsilon_3}\alpha}{2k} - \delta\Big)\int_{\mathbb{R}^n} |Du|^2 u^{\delta-1}\eta^\theta \,\mathrm{d}x + \theta C_{\varepsilon_2}\int_U |D\eta|^2 u^{\delta+\alpha/2k}\eta^{\theta-2}\,\mathrm{d}x. \end{aligned}$$

By successively choosing ε_3 and ε_2 such that

$$\varepsilon_3 \leqslant \frac{k(n-2)}{\theta(2k-\alpha)}, \quad \varepsilon_2 \leqslant \min\Big\{\frac{2k\delta}{\theta C_{\varepsilon_3}\alpha}, 1\Big\},$$

we can discard the first two terms on the right hand side of (2.21). Now (2.21) becomes

$$\tag{2.22} \begin{aligned} \int_{\mathbb{R}^n} u^{\alpha/k+\delta}\eta^\theta \,\mathrm{d}x &\leqslant \theta C_{\varepsilon_2}\int_U |D\eta|^2 u^{\delta+\alpha/2k}\eta^{\theta-2}\,\mathrm{d}x \\ &= \theta C_{\varepsilon_2}\int_U (u^{\delta+\alpha/2k}\eta^{\theta(\alpha+2k\delta)/(2\alpha+2k\delta)})(\eta^{\theta\alpha/(2\alpha+2k\delta)-2}|D\eta|^2)\,\mathrm{d}x \\ &\leqslant \frac{\theta\varepsilon_4 C_{\varepsilon_2}(\alpha+2k\delta)}{2\alpha+2k\delta}\int_U u^{\alpha/k+\delta}\eta^\theta\,\mathrm{d}x \\ &\quad + \frac{\theta C_{\varepsilon_2}C_{\varepsilon_4}\alpha}{2\alpha+2k\delta}\int_U \eta^{\theta-4(1+k\delta/\alpha)}|D\eta|^{4(1+k\delta/\alpha)}\,\mathrm{d}x, \end{aligned}$$

where Young's inequality $ab \leqslant \varepsilon_4 a^p/p + C_{\varepsilon_4} b^q/q$ with the exponent pair $(p,q) = ((\alpha/k+\delta)/(\alpha/2k+\delta), (2\alpha+2k\delta)/\alpha)$ is used to obtain the last inequality, ε_4 is any positive constant, C_{ε_4} denotes some positive constant depending on ε_4. By choosing

$$\varepsilon_4 < \frac{2\alpha+2k\delta}{\theta C_{\varepsilon_2}(\alpha+2k\delta)}$$

and using (2.9), we get from (2.22) that

$$\tag{2.23} \int_{\mathbb{R}^n} u^{\alpha/k+\delta}\eta^\theta\,\mathrm{d}x \leqslant \frac{\theta C_{\varepsilon_2}C_{\varepsilon_4}\alpha\omega_n C}{2\alpha+2k\delta-\theta\varepsilon_4 C_{\varepsilon_2}(\alpha+2k\delta)} R^{n-4(1+k\delta/\alpha)},$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. By the choice of δ, we have $n-4(1+k\delta/\alpha) < 0$. Letting $R \to \infty$ in (2.23), we can obtain

$$\tag{2.24} \int_{\mathbb{R}^n} u^{\alpha/k+\delta}\eta^\theta\,\mathrm{d}x \leqslant 0.$$

We then get a contradiction in case (ii) (a).

In case (ii) (b), we fix the constants δ and θ such that

$$\delta = 0, \quad \theta > n.$$

The last term in (2.10) becomes

$$\begin{aligned}(2.25)\qquad -\theta \int_U \eta^{\theta-1}(D_i\eta)(D_iu)\,\mathrm{d}x &\leqslant \theta\varepsilon_5 \int_U |Du|^2\eta^\theta\,\mathrm{d}x + \theta C_{\varepsilon_5}\int_U |D\eta|^2\eta^{\theta-2}\,\mathrm{d}x \\ &\leqslant \theta\varepsilon_5 \int_{\mathbb{R}^n} |Du|^2\eta^\theta\,\mathrm{d}x + \varepsilon_6 C_{\varepsilon_5}(\theta-2)\int_U \eta^\theta\,\mathrm{d}x + 2C_{\varepsilon_5}C_{\varepsilon_6}\int_U |D\eta|^\theta\,\mathrm{d}x,\end{aligned}$$

where Schwarz's inequality is used to obtain the first inequality, Young's inequality $ab \leqslant \varepsilon_6 a^p/p + C_{\varepsilon_6} b^q/q$ with the exponent pair $(p,q) = (\theta/(\theta-2), \theta/2)$ is used to obtain the second inequality, ε_5, ε_6 are positive constants, C_{ε_5}, C_{ε_6} denote the positive constants depending on ε_5, ε_6, respectively. Combining (2.5), (2.10) and (2.25), we have

$$\begin{aligned}(2.26)\qquad \int_{\mathbb{R}^n} \eta^\theta\,\mathrm{d}x \leqslant \Big(\theta\varepsilon_5 - \frac{n-2}{2}\Big)\int_{\mathbb{R}^n} |Du|^2\eta^\theta\,\mathrm{d}x \\ + \varepsilon_6 C_{\varepsilon_5}(\theta-2)\int_U \eta^\theta\,\mathrm{d}x + 2C_{\varepsilon_5}C_{\varepsilon_6}\int_U |D\eta|^\theta\,\mathrm{d}x.\end{aligned}$$

By successively choosing ε_5 and ε_6 such that

$$\varepsilon_5 \leqslant \frac{n-2}{2\theta}, \quad \varepsilon_6 < \frac{1}{C_{\varepsilon_5}(\theta-2)},$$

we get from (2.26) that

$$(2.27)\qquad \int_{\mathbb{R}^n} \eta^\theta\,\mathrm{d}x \leqslant \frac{2C_{\varepsilon_5}C_{\varepsilon_6}}{1-\varepsilon_6 C_{\varepsilon_5}(\theta-2)}\int_U |D\eta|^\theta\,\mathrm{d}x \leqslant \frac{2C_{\varepsilon_5}C_{\varepsilon_6}\omega_n C}{1-\varepsilon_6 C_{\varepsilon_5}(\theta-2)}R^{n-\theta},$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. Letting $R \to 0$ in (2.27), since $\theta > n$, we can get

$$(2.28)\qquad \int_{\mathbb{R}^n} \eta^\theta\,\mathrm{d}x \leqslant 0.$$

We then get a contradiction in case (ii)(b). We have completed the proof of Theorem 1.1. □

Note that Corollary 1.1 is a direct consequence of Theorem 1.1 when $k = 1$. We omit its proof.

We have discussed the nonexistence of positive admissible solutions for (1.1) when $n \geqslant 2$, $\alpha \in (k, \infty)$ and $n \geqslant 3$, $\alpha \in [0, k]$. To close this section, we show three simple examples of positive entire admissible solutions of (1.1) for some $\alpha \leqslant k = 1$ in the two dimensional case.

Example 2.1. When $n = 2$, $k = 1$, $\alpha = 1$, then the function

$$u(x) = A\mathrm{e}^{Bx^2+C} \tag{2.29}$$

with constants $A > 0$, $B \geqslant 1/4$ and any constant $C \geqslant 0$ satisfies the inequality (1.1) in $\mathbb{R}^2$ with A^g in (1.4), namely

$$\begin{aligned} \sigma_1[u_{ij} - (\tfrac{1}{2}|Du|^2\delta_{ij} - u_iu_j)] &= 4AB\mathrm{e}^{Bx^2+C} + 4AB^2x^2\mathrm{e}^{Bx^2+C} \\ &\geqslant A\mathrm{e}^{Bx^2+C} = u. \end{aligned} \tag{2.30}$$

Moreover, if $A \geqslant 1$, $B \geqslant 1/4$ and $C \geqslant 1$ in (2.29), since $A\mathrm{e}^{Bx^2+C} \geqslant 1$, we have, in place of (2.30),

$$\sigma_1[u_{ij} - (\tfrac{1}{2}|Du|^2\delta_{ij} - u_iu_j)] \geqslant A\mathrm{e}^{Bx^2+C} \geqslant u^\alpha \tag{2.31}$$

for any $\alpha \in (-\infty, 1]$. In this case, the function in (2.29) satisfies $\sigma_1(A^g) \geqslant u^\alpha$ in $\mathbb{R}^2$, where A^g is defined in (1.4) and α is any constant in $(-\infty, 1]$.

Example 2.2. When $n = 2$, $k = 1$, $\alpha = 0$, then the function

$$u(x) = Ax^2 + B \tag{2.32}$$

with the constant $A \geqslant 1/4$ and any constant B satisfies the inequality (1.1) in $\mathbb{R}^2$ with A^g in (1.4), namely

$$\sigma_1\big[u_{ij} - \big(\tfrac{1}{2}|Du|^2\delta_{ij} - u_iu_j\big)\big] = 4A \geqslant 1. \tag{2.33}$$

Moreover, if $A \geqslant 1/4$ and $B \geqslant 1$ in (2.28), since $Ax^2 + B \geqslant 1$, we have, in place of (2.33),

$$\sigma_1\big[u_{ij} - \big(\tfrac{1}{2}|Du|^2\delta_{ij} - u_iu_j\big)\big] \geqslant u^\alpha \tag{2.34}$$

for any $\alpha \in (-\infty, 0]$. In this case, the function in (2.32) satisfies $\sigma_1(A^g) \geqslant u^\alpha$ in $\mathbb{R}^2$, where A^g is defined in (1.4) and α is any constant in $(-\infty, 0]$.

Example 2.3. When $n = 2$, $k = 1$, $\alpha = -1$, then the function

$$u(x) = \sqrt{Ax^2 + B} \tag{2.35}$$

with constants $A \geqslant 1$ and $B > 0$ satisfies the inequality (1.1) in $\mathbb{R}^2$ with A^g in (1.4), namely

$$\begin{aligned} \sigma_1[u_{ij} - (\tfrac{1}{2}|Du|^2\delta_{ij} - u_i u_j)] &= \frac{AB}{(Ax^2+B)^{3/2}} + \frac{A}{\sqrt{Ax^2+B}} \\ &\geqslant \frac{1}{\sqrt{Ax^2+B}} = u^{-1}. \end{aligned} \tag{2.36}$$

Moreover, if $A \geqslant 1$ and $B \geqslant 1$ in (2.35), since $\sqrt{Ax^2+B} \geqslant 1$, we can replace (2.36) by

$$\sigma_1[u_{ij} - (\tfrac{1}{2}|Du|^2\delta_{ij} - u_i u_j)] \geqslant \frac{1}{\sqrt{Ax^2+B}} \geqslant u^{\alpha} \tag{2.37}$$

for any $\alpha \in (-\infty, -1]$. In this case, the function in (2.35) satisfies $\sigma_1(A^g) \geqslant u^{\alpha}$ in $\mathbb{R}^2$, where A^g is defined in (1.4) and α is any constant in $(-\infty, -1]$.

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