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BOUNDARY VALUE PROBLEM FOR AN INFINITE SYSTEM OF SECOND ORDER DIFFERENTIAL EQUATIONS IN ℓ_p SPACES

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Abstract. The concept of measures of noncompactness is applied to prove the existence of a solution for a boundary value problem for an infinite system of second order differential equations in ℓ_p space. We change the boundary value problem into an equivalent system of infinite integral equations and result is obtained by using Darbo's type fixed point theorem. The result is illustrated with help of an example.

Keywords: Darbo's fixed point theorem; equicontinuous sets; infinite system of second order differential equations; infinite system of integral equations; measures of noncompactness

MSC 2010: 34A34, 34G20, 47H08

1. Introduction and preliminaries

In 1930 Kuratowski (see [12]) introduced the concept of measure of noncompactness which was further extended to general Banach spaces by Banaś and Goebel (see [3]). In 1955 Darbo (see [7]) proved a fixed point theorem for condensing operators using the concept of measures of noncompactness, which generalized the classical Schauder fixed point theorem and Banach contraction principle. The method of fixed point arguments has been widely used to study the existence of solutions of functional equations, like Banach contraction principle in [1] and Schauder's fixed point theorem in [11], [13]. But if compactness and Lipschitz condition are not satisfied these results cannot be used. Measure of noncompactness comes handy in such situations.

The Hausdorff measure of noncompactness is used frequently in finding the existence of solutions for various functional equations and is defined as follows:

Definition 1.1 ([3]). Let (Ω, d) be a metric space and A a bounded subset of Ω. Then the *Hausdorff measure of noncompactness* (the ball-measure of noncompact-

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ness) of the set A, denoted by $\chi(A)$, is defined to be the infimum of the set of all real $\varepsilon > 0$ such that A can be covered by a finite number of balls of radii $< \varepsilon$, that is

$$\chi(A) = \inf\Big\{\varepsilon > 0\colon\ A \subset \bigcup_{i=1}^{n} B(x_i, r_i),\ x_i \in \Omega,\ r_i < \varepsilon,\ i = 1, \ldots, n,\ n \in \mathbb{N}\Big\}$$

where $B(x_i, r_i)$ denotes the ball of radius r_i centered at x_i.

Let $(X, \|\cdot\|)$ be a Banach space; for any $E \subset X$, $\overline{E}$ denotes closure of E and $\operatorname{conv}(E)$ denotes the closed convex hull of E. We denote the family of nonempty bounded subsets of X by $\mathfrak{M}_X$ and the family of nonempty and relatively compact subsets of X by $\mathfrak{N}_X$. Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{R}$ the set of real numbers; for $\mathbb{R}_+ = [0, \infty)$, the axiomatic definition of measure of noncompactness is given below

Definition 1.2 ([5])**.** A mapping $\mu\colon \mathfrak{M}_X \to \mathbb{R}_+$ is said to be the measure of noncompactness in X, if the following conditions hold:

(i) The family $\operatorname{Ker}\mu = \{E \in \mathfrak{M}_X\colon\ \mu(E) = 0\}$ is nonempty and $\operatorname{Ker}\mu \subset \mathfrak{N}_X$;
(ii) $E_1 \subset E_2 \Rightarrow \mu(E_1) \leqslant \mu(E_2)$;
(iii) $\mu(\overline{E}) = \mu(E)$;
(iv) $\mu(\operatorname{conv} E) = \mu(E)$;
(v) $\mu(\lambda E_1 + (1-\lambda)E_2) \leqslant \lambda\mu(E_1) + (1-\lambda)\mu(E_2)$ for $0 \leqslant \lambda \leqslant 1$;
(vi) if (E_n) is a sequence of closed sets from $\mathfrak{M}_X$ such that $E_{n+1} \subset E_n$ and $\lim\limits_{n\to\infty} \mu(E_n) = 0$ then the intersection set $E_\infty = \bigcap\limits_{n=1}^{\infty} E_n$ is nonempty.

Further properties of measures of noncompactness can be found in [3], [5].

The fixed point theorem of Darbo's (see [7]) is stated below:

Lemma 1.3 ([7])**.** *Let E be a nonempty, bounded, closed, and convex subset of a Banach space X and let $T\colon\ E \to E$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that $\mu(T(E)) \leqslant k\mu(E)$ for any nonempty subset E of X. Then T has a fixed point in the set E.*

The idea of equicontinuous sets is defined as follows:

Definition 1.4. Let (Ω_1, d) and (Ω_2, d) be two metric spaces, and $\mathcal{T}$ the family of functions from Ω_1 to Ω_2. The family $\mathcal{T}$ is equicontinuous at a point $m_0 \in \Omega_1$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(f(m), f(m_0)) < \varepsilon$ for all $f \in \mathcal{T}$ and all $m \in \Omega_1$ such that $d(m, m_0) < \delta$. The family is pointwise equicontinuous if it is equicontinuous at each point of Ω_1.

For fixed $p \geqslant 1$, we denote by ℓ_p the Banach sequence space with a norm defined as:

$$\|x\|_p = \|(x_n)\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$$

for $x = (x_n) \in \ell_p$. In order to apply Lemma 1.3 in a given Banach space X, we need a formula expressing the measures of noncompactness in a simple manner. Such formulas are known only for few sequence spaces (see [3], [5]).

For the Banach sequence space $(\ell_p, \|\cdot\|_p)$, Hausdorff measure of noncompactness is given by

$$\chi(E) = \lim_{n\to\infty} \left\{ \sup_{(e_k)\in E} \left(\sum_{k\geqslant n} |e_k|^p\right)^{1/p} \right\} \tag{1.1}$$

where $E \in \mathfrak{M}_{\ell_p}$. The above formula will be used in the sequel of the paper.

In recent years many researchers have worked on the infinite system of second order differential equations of the form

$$\frac{\mathrm{d}^2 u_i}{\mathrm{d}t^2} = -f_i(t, u_1, u_2, \ldots), \quad u_i(0) = u_i(T) = 0, \quad i \in \mathbb{N},\ t \in [0, T] \tag{1.2}$$

and obtained conditions for the existence of solutions of (1.2) in different Banach spaces (see [2], [6], [18], [19]).

Measures of noncompactness has been used to obtain conditions under which an infinite system of differential equations has a solution in the given Banach space (see [2], [4], [5], [6], [15], [16], [17], [19], [20]).

We consider the infinite system of second order differential equations of the form

$$\frac{\mathrm{d}^2 v_j}{\mathrm{d}t^2} - v_j = f_j(t, v(t)) \tag{1.3}$$

where $t \in [0, T]$, $v(t) = (v_j(t))_{j=1}^{\infty}$ and $j = 1, 2, \ldots$

The above system will be studied together with the boundary conditions

$$v_j(0) = 0, \quad v_j(T) = 0. \tag{1.4}$$

The solution is investigated using the infinite system of integral equations and Green's function (see [10]). Such systems appear in the study of the theory of neural sets, theory of branching processes and theory of dissociation of polymers (see [8], [9]).

In this paper, we find conditions under which the system given in (1.3) under the boundary conditions (1.4) has a solution in the Banach sequence space ℓ_p, for that we define an equivalent infinite system of integral equations. The result is supported by an example.

2. Main results

Let $I=[0,T]$, by $C(I,\mathbb{R})$ we denote the space of continuously differentiable functions on I and by $C^2(I,\mathbb{R})$ we denote the space of twice continuously differentiable functions on I. A function $v\in C^2(I,\mathbb{R})$, is a solution of (1.3) if and only if v is a solution of the infinite system of integral equations

$$v_j(t)=\int_0^T G(t,s)f_j(s,v(s))\,\mathrm{d}s,\quad t\in I \tag{2.1}$$

where $f_j(t,v)\in C(I,\mathbb{R})$, $j=1,2,3,\dots$ and Green's function $G(s,t)$ is defined on I^2 as:

$$G(t,s)=\begin{cases}\dfrac{\sinh(t)\sinh(T-s)}{\sinh(T)}; & 0\leqslant s<t\leqslant T,\\[2mm] \dfrac{\sinh(s)\sinh(T-t)}{\sinh(T)}; & 0\leqslant t<s\leqslant T.\end{cases} \tag{2.2}$$

Using standard methods, it can be easily shown that

$$G(t,s)\leqslant\frac{1}{2}\tanh\left(\tfrac{1}{2}T\right) \tag{2.3}$$

for all $(t,s)\in I^2$.

From equations (2.1) and (2.2) we have

$$v_j(t)=\int_0^t\frac{\sinh(t)\sinh(T-s)}{\sinh(T)}f_j(s,v(s))\,\mathrm{d}s+\int_t^T\frac{\sinh(s)\sinh(T-t)}{\sinh(T)}f_j(s,v(s))\,\mathrm{d}s.$$

Differentiating the above equation, we get

$$\frac{\mathrm{d}v_j}{\mathrm{d}t}=\int_0^t\frac{\cosh(t)\sinh(T-s)}{\sinh(T)}f_j(s,v(s))\,\mathrm{d}s+\int_t^T\frac{-\sinh(s)\cosh(T-t)}{\sinh(T)}f_j(s,v(s))\,\mathrm{d}s.$$

Further differentiation gives

$$\begin{aligned}\frac{\mathrm{d}^2v_j}{\mathrm{d}t^2}&=\int_0^t\frac{\sinh(t)\sinh(T-s)}{\sinh(T)}f_j(s,v(s))\,\mathrm{d}s+\frac{\cosh(t)\sinh(T-t)}{\sinh(T)}f_j(t,v(t))\\&\quad+\int_t^T\frac{\sinh(s)\sinh(T-t)}{\sinh(T)}f_j(s,v(s))\,\mathrm{d}s+\frac{\sinh(t)\cosh(T-t)}{\sinh(T)}f_j(t,v(t))\\&=\int_0^T G(t,s)f_j(s,v(s))\,\mathrm{d}s\\&\quad+\frac{1}{\sinh(T)}(\sinh(t)\cosh(T-t)+\cosh(t)\sinh(T-t))f_j(t,v(t))\\&=v_j(t)+f_j(t,v(t)).\end{aligned}$$

Thus $v_j(t)$ given in (2.1) satisfies (1.3).

Hence, finding existence of a solution for the system (1.3) with boundary conditions (1.4) is equivalent to finding the existence of a solution for the infinite system of integral equations (2.1).

The functions $v = v(t)$ act continuously from the interval I into the space ℓ_p, the class of such functions $C(I, \ell_p)$ is a Banach space endowed by the classical supremium norm

$$\|v\|_C = \sup\{\|v(t)\|_p \colon t \in I\}.$$

R e m a r k 2.1. If X is a Banach space and χ_X denotes its Hausdorff measure of noncompactness, then the Hausdorff measure of noncompactness of a subset E of $C(I, X)$ in the Banach space of continuous functions is given by (see [3], [14])

$$\chi(E) = \sup\{\chi_X(X(t)) \colon t \in I\},$$

where E is equicontinuous on the interval $I = [0, T]$.

In order to find conditions under which the system (2.1) has a solution the following assumptions are made:

(A_1) The functions f_j are real valued, defined on the set $I \times \mathbb{R}^\infty$, $j = 1, 2, 3, \ldots$

(A_2) The operator $\mathcal{F}$ defined on the space $I \times \ell_p$ as

$$(t, v) \mapsto (\mathcal{F}v)(t) = (f_1(t, v), f_2(t, v), f_3(t, v), \ldots)$$

transforms the space $I \times \ell_p$ into ℓ_p.

The class of all functions $\{(\mathcal{F}v)(t)\}_{t \in I}$ is equicontinuous at each point of the space ℓ_p, that is for each $v \in \ell_p$ fixed arbitrarily and for a given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\|_p < \varepsilon \tag{2.4}$$

for each $t \in I$ and for any $u \in \ell_p$ such that $\|u - v\|_p < \delta$.

(A_3) For each $t \in I$, $v(t) = (v_j(t)) \in \ell_p$, the following inequality holds:

$$|f_j(t, v(t))|^p \leqslant g_j(t) + h_j(t)|v_j|^p, \quad n \in \mathbb{N} \tag{2.5}$$

where $h_j(t)$ and $g_j(t)$ are real valued continuous functions on I. Moreover, we assume that the function g_j, $j = 1, 2, \ldots$ is continuous on I and the function series $\sum\limits_{k \geqslant 1} g_k(t)$ is uniformly convergent, while the function sequence $(h_j(t))_{j \in \mathbb{N}}$ is equibounded on I.

The function $g = g(t)$ defined on the interval I as $g(t) = \sum\limits_{j=1}^{\infty} g_j(t)$ is continuous under the assumption (A_3), and the constants defined as

$$G = \max\{g(t)\colon\ t \in I\}, \quad H = \sup\{h_j(t)\colon\ t \in I,\ j \in \mathbb{N}\}$$

are finite.

Theorem 2.2. *Under the assumptions* (A_1)–(A_3), *with* $H^{1/p} T \tanh(\frac{1}{2}T) < 2$, *the infinite system of integral equations* (2.1) *has at least one solution* $v(t) = (v_j(t))$ *in the space* ℓ_p, *i.e.* $(v_j(t)) \in \ell_p$, *for each* $t \in I$.

Proof. We consider the space $C(I, \ell_p)$ of all continuous functions on $I = [0, T]$ with supremum norm given as

$$\|v\| = \sup_{t \in I}\{\|v(t)\|_p\}.$$

Define the operator $\mathcal{F}$ on the space $C(I, \ell_p)$ by

$$\begin{aligned} (\mathcal{F}v)(t) = ((\mathcal{F}v)_j(t)) &= \Bigl(\int_0^T G(t,s) f_j(s, v(s))\,\mathrm{d}s\Bigr) \\ &= \Bigl(\int_0^T G(t,s) f_1(s, v(s))\,\mathrm{d}s, \int_0^T G(t,s) f_2(s, v(s))\,\mathrm{d}s, \dots\Bigr). \end{aligned} \tag{2.6}$$

The operator $\mathcal{F}$ as defined in (2.6) transforms the space $C(I, \ell_p)$ into itself, which we will show. Fix $v = v(t) = (v_j(t))$ in $C(I, \ell_p)$, then for arbitrary $t \in I$ using assumption (A_3), inequality (2.3) and Hölder's inequality we have

$$\begin{aligned} \|(\mathcal{F}v)(t)\|_p^p &= \sum_{j=1}^{\infty} \Bigl|\int_0^T G(t,s) f_j(s, v(s))\,\mathrm{d}s\Bigr|^p \\ &\leqslant \sum_{j=1}^{\infty} \Bigl(\int_0^T |G(t,s)|^p |f_j(s, v(s))|^p\,\mathrm{d}s\Bigr)\Bigl(\int_0^T \mathrm{d}s\Bigr)^{p/q} \\ &\leqslant T^{p/q} \sum_{j=1}^{\infty} \Bigl(\int_0^T |G(t,s)|^p (g_j(s) + h_j(s)|v_j(s)|^p)\,\mathrm{d}s\Bigr) \\ &\leqslant \Bigl(\frac{1}{2}\tanh(\tfrac{1}{2}T)\Bigr)^p T^{p/q} \sum_{j=1}^{\infty} \Bigl(\int_0^T g_j(s)\,\mathrm{d}s + \int_0^T h_j(s)|v_j(s)|^p\,\mathrm{d}s\Bigr) \\ &\leqslant \Bigl(\frac{T^{1/q}}{2}\tanh(\tfrac{1}{2}T)\Bigr)^p \Bigl(\int_0^T \sum_{j=1}^{\infty}(g_j(s))\,\mathrm{d}s + \int_0^T \sum_{j=1}^{\infty}(h_j(s)|v_j(s)|^p)\,\mathrm{d}s\Bigr). \end{aligned}$$

Now, using the Lebesgue dominated convergence theorem we get

$$\begin{aligned}\|(\mathcal{F}(v))(t)\|_p^p &\leqslant \Big(\frac{T^{1/q}}{2}\tanh(\tfrac{1}{2}T)\Big)^p\Big(\int_0^T g(s)\,\mathrm{d}s + H\int_0^T\sum_{j=1}^{\infty}|v_j(s)|^p\,\mathrm{d}s\Big)\\ &\leqslant \Big(\frac{T^{1/q}}{2}\tanh(\tfrac{1}{2}T)\Big)^p(GT + HT(\|v\|_p)^p)\\ &= \Big(\frac{T}{2}\tanh(\tfrac{1}{2}T)\Big)^p(G + H(\|v\|_p)^p).\end{aligned}$$

Therefore,

$$\|\mathcal{F}(v)(t)\|_p^p \leqslant \Big(\frac{T}{2}\tanh(\tfrac{1}{2}T)\Big)^p(G + H(\|v\|_p)^p). \tag{2.7}$$

Hence, $\mathcal{F}v$ is bounded on the interval I. Thus $\mathcal{F}$ transforms the space $C(I,\ell_p)$ into itself. From (2.7) we get

$$\|(\mathcal{F}(v))(t)\|_p \leqslant \frac{T}{2}\tanh(\tfrac{1}{2}T)(G + H(\|v\|_p)^p)^{1/p}. \tag{2.8}$$

Now, using (2.1) and following the procedure as above we get

$$\begin{aligned}\|v\|_p^p \leqslant \Big(\frac{T}{2}\tanh\,(\tfrac{1}{2}T)\Big)^p&(G + H(\|v\|_p)^p)\\ &\Rightarrow (2^p - H(T\tanh(\tfrac{1}{2}T))^p)(\|v\|_p)^p \leqslant G(T\tanh(\tfrac{1}{2}T))^p\\ &\Rightarrow \|v\|_p^p \leqslant \frac{G(T\tanh(\frac{1}{2}T))^p}{2^p - H(T\tanh(\frac{1}{2}T))^p}\\ &\Rightarrow \|v\|_p \leqslant \frac{G^{1/p}(T\tanh(\frac{1}{2}T))}{(2^p - H(T\tanh(\frac{1}{2}T))^p)^{1/p}}.\end{aligned}$$

Thus, the positive number

$$r = \frac{G^{1/p}(T\tanh(\frac{1}{2}T))}{(2^p - H(T\tanh(\frac{1}{2}T))^p)^{1/p}}$$

is the optimal solution of the inequality

$$\frac{T}{2}\tanh(\tfrac{1}{2}T)(G + HR^p)^{1/p} \leqslant R.$$

Hence, by (2.8) the operator $\mathcal{F}$ transforms the ball $B_r \subset C(I,\ell_p)$ into itself.

We now show that $\mathcal{F}$ is continuous on B_r. Let $\varepsilon > 0$ be arbitrarily fixed and let $v = (v(t)) \in B_r$ be any arbitrarily fixed function, then if $u = (u(t)) \in B_r$ is any function such that $\|u - v\|_p < \varepsilon$, then for any $t \in I$ we have

$$\begin{aligned}\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\|_p^p &= \sum_{j=1}^{\infty}\biggl|\int_0^T G(t,s)(f_j(s,u(s)) - f_j(s,v(s)))\,\mathrm{d}s\biggr|^p\\ &\leqslant \sum_{j=1}^{\infty}\int_0^T |G(t,s)|^p|f_j(s,u(s)) - f_j(s,v(s))|^p\,\mathrm{d}s\biggl(\int_0^T \mathrm{d}s\biggr)^{p/q}\\ &\leqslant T^{p/q}\sum_{j=1}^{\infty}\int_0^T |G(t,s)|^p|f_j(s,u(s)) - f_j(s,v(s))|^p\,\mathrm{d}s.\end{aligned}$$

Now, by using (2.3) and the assumption (A_2) of equicontinuity, we get

$$\begin{aligned}(2.9)\qquad &\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\|_p^p\\ &\leqslant T^{p/q}\Bigl(\frac{1}{2}\tanh(\tfrac{1}{2}T)\Bigr)^p\sum_{j=1}^{\infty}\int_0^T |f_j(s,u(s)) - f_j(s,v(s))|^p\,\mathrm{d}s\\ &= \Bigl(\frac{T^{1/q}}{2}\tanh(\tfrac{1}{2}T)\Bigr)^p\lim_{m\to\infty}\sum_{j=1}^{m}\int_0^T |f_j(s,u(s)) - f_j(s,v(s))|^p\,\mathrm{d}s\\ &= \Bigl(\frac{T^{1/q}}{2}\tanh(\tfrac{1}{2}T)\Bigr)^p\lim_{m\to\infty}\int_0^T\sum_{j=1}^{m} |f_j(s,u(s)) - f_j(s,v(s))|^p\,\mathrm{d}s.\end{aligned}$$

Define the function $\delta(\varepsilon)$ as

$$\delta(\varepsilon) = \sup\{|f_j(s,u(s)) - f_j(s,v(s))|\colon\ u,v \in \ell_p,\ \|u - v\|_p \leqslant \varepsilon,\ t \in I,\ j \in \mathbb{N}\}.$$

Then clearly $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, since the family $\{(fv)(t)\colon\ t \in I\}$ is equicontinuous at every point $v \in \ell_p$.

Therefore, by (2.9) and using the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\|_p^p &\leqslant \Bigl(\frac{T^{1/q}}{2}\tanh(\tfrac{1}{2}T)\Bigr)^p\int_0^T (\delta(\varepsilon))^p\,\mathrm{d}s\\ &= \Bigl(\frac{T}{2}\tanh(\tfrac{1}{2}T)\Bigr)^p(\delta(\varepsilon))^p.\end{aligned}$$

This implies that the operator $\mathcal{F}$ is continuous on the ball B_r.

Since $G(t,s)$ as defined in (2.2) is uniformly continuous on I^2, so by the definition of the operator $\mathcal{F}$ it is easy to show that $\{\mathcal{F}u\colon\ u \in B_r\}$ is equicontinuous on I. Let $B_{r_1} = \operatorname{conv}(\mathcal{F}B_r)$, then $B_{r_1} \subset B_r$ and the functions from the set B_{r_1} are equicontinuous on I.

Let $E \subset B_{r_1}$, then E is equicontinuous on I. If $v \in E$ is a function then for arbitrarily fixed $t \in I$ we have by assumption (A_3)

$$\sum_{j=k}^{\infty} |(\mathcal{F}v)_j(t)|^p = \sum_{j=k}^{\infty} \left| \int_0^T G(t,s) f_j(s, v(s))\, \mathrm{d}s \right|^p \leqslant \sum_{j=k}^{\infty} \Big(\int_0^T |G(t,s)||f_j(s,v(s))|\, \mathrm{d}s \Big)^p.$$

Using Hölder's inequality and (2.3), we get

$$\begin{aligned} \sum_{j=k}^{\infty} |(\mathcal{F}v)_j(t)|^p &\leqslant \sum_{j=k}^{\infty} \Big(\int_0^T |G(t,s)|^p |f_j(s,v(s))|^p\, \mathrm{d}s \Big) \Big(\int_0^T \mathrm{d}s \Big)^{p/q} \\ &\leqslant T^{p/q} \Big(\frac{1}{2} \tanh(\tfrac{1}{2}T) \Big)^p \sum_{j=k}^{\infty} \Big(\int_0^T |f_j(s,v(s))|^p\, \mathrm{d}s \Big). \end{aligned}$$

Again, using the Lebesgue dominated convergence theorem and the assumption (A_3), we get

$$\begin{aligned} \sum_{j=k}^{\infty} |(\mathcal{F}v)_j(t)|^p &\leqslant \Big(\frac{T^{1/q}}{2} \tanh(\tfrac{1}{2}T) \Big)^p \int_0^T \sum_{j=k}^{\infty} (g_j(s) + h_j(s)|v_j(s)|^p)\, \mathrm{d}s \\ &= \Big(\frac{T^{1/q}}{2} \tanh(\tfrac{1}{2}T) \Big)^p \Big(\int_0^T \sum_{j=k}^{\infty} g_j(s)\, \mathrm{d}s + \int_0^T \sum_{j=k}^{\infty} h_j(s)|v_j(s)|^p\, \mathrm{d}s \Big) \\ &\leqslant \Big(\frac{T^{1/q}}{2} \tanh(\tfrac{1}{2}T) \Big)^p \Big(\int_0^T \sum_{j=k}^{\infty} g_j(s)\, \mathrm{d}s + H \int_0^T \sum_{j=k}^{\infty} |v_j(s)|^p\, \mathrm{d}s \Big). \end{aligned}$$

Taking supremum over all $v \in E$, we obtain

$$\begin{aligned} \sup_{v \in E} \sum_{j=k}^{\infty} |(\mathcal{F}v)_j(t)|^p &\leqslant \Big(\frac{T^{1/q}}{2} \tanh(\tfrac{1}{2}T) \Big)^p \\ &\quad \times \Big(\int_0^T \sum_{j=k}^{\infty} g_j(s)\, \mathrm{d}s + H \sup_{v \in E} \int_0^T \sum_{j=k}^{\infty} |v_j(s)|^p\, \mathrm{d}s \Big). \end{aligned}$$

Using the definition of the Hausdorff measure of noncompactness in the ℓ_p space and noting that E is the set of equicontinuous functions on I, by applying Remark 2.1, we get

$$(\chi(\mathcal{F}E))^p \leqslant H \Big(\frac{T}{2} \tanh(\tfrac{1}{2}T) \Big)^p (\chi(E))^p \Rightarrow \chi(\mathcal{F}E) \leqslant H^{1/p} \Big(\frac{T}{2} \tanh(\tfrac{1}{2}T) \Big) \chi(E).$$

Hence, if

$$H^{1/p} \frac{T}{2} \tanh(\tfrac{1}{2}T) < 1 \Rightarrow H^{1/p} T \tanh(\tfrac{1}{2}T) < 2$$

then, by Lemma 1.3, the operator $\mathcal{F}$ on the set B_{r_1} has a fixed point, which completes the proof of the theorem. $\square$

N o t e. The value of T is chosen in such a way that the condition

$$H^{1/p}T\tanh(\tfrac{1}{2}T) < 2$$

is satisfied.

The above result is illustrated by the following example:

E x a m p l e 2.3. Consider the infinite system of second order differential equations in ℓ_2

$$\frac{\mathrm{d}^2 v_n}{\mathrm{d}t^2} - v_n = \frac{t3^{-nt}}{n} + \sum_{k=n}^{\infty} \frac{\cos t}{(1+2n)\sqrt{(k-1)!}} \cdot \frac{v_k(t)(1-\sqrt{k-n}\,v_k(t))}{\sqrt{k-n+1}} \tag{2.10}$$

for $n = 1, 2, \ldots$

S o l u t i o n: Comparing the infinite system of differential equations (2.10) with (1.3) we have

$$f_n(t,v) = \frac{t3^{-nt}}{n} + \sum_{k=n}^{\infty} \frac{\cos t}{(1+2n)\sqrt{(k-1)!}} \cdot \frac{v_k(t)(1-\sqrt{k-n}\,v_k(t))}{\sqrt{k-n+1}}. \tag{2.11}$$

Clearly f_j, $j = 1, 2, \ldots$ is a real valued function, so assumption (A_1) of Theorem 2.2 is satisfied. We now show that assumption (A_2) of Theorem 2.2 is also satisfied, that is

$$|f_n(t,v)|^2 \leqslant g_n(t) + h_n(t)|v_n|^2. \tag{2.12}$$

Using the Cauchy-Schwarz inequality and equation (2.10) we have

$$\begin{aligned}
|f_n(t,v)|^2 &= \Big|\frac{t3^{-nt}}{n} + \sum_{k=n}^{\infty} \frac{\cos t}{(1+2n)\sqrt{(k-1)!}} \cdot \frac{v_k(t)(1-\sqrt{k-n}\,v_k(t))}{\sqrt{k-n+1}}\Big|^2 \\
&\leqslant 2\Big(\frac{t^2 3^{-2nt}}{n^2} + \Big(\sum_{k=n}^{\infty} \frac{\cos t}{(1+2n)\sqrt{(k-1)!}} \cdot \frac{v_k(t)(1-\sqrt{k-n}\,v_k(t))}{\sqrt{k-n+1}}\Big)^2\Big) \\
&\leqslant 2\frac{t^2 3^{-2nt}}{n^2} + 2\sum_{k=n}^{\infty} \frac{\cos^2 t}{(1+2n)^2(k-1)!} \cdot \sum_{k=n}^{\infty}\Big(\frac{v_k(t)(1-\sqrt{k-n}\,v_k(t))}{\sqrt{k-n+1}}\Big)^2.
\end{aligned}$$

Again, using the fact that

$$\frac{\alpha(1-\alpha\beta)}{\beta} \leqslant \frac{1}{(2\beta)^2}, \quad \beta \neq 0 \tag{2.13}$$

for any real α, β we have

$$\begin{aligned}
|f_n(t,v)|^2 &\leqslant 2\frac{t^2 3^{-2nt}}{n^2} + 2\frac{\cos^2 t}{(1+2n)^2}\,\mathrm{e}\Big(v_n^2 + \sum_{k=n+1}^{\infty}\Big(\frac{v_k(t)(1-\sqrt{k-n}\,v_k(t))}{\sqrt{k-n+1}}\Big)^2\Big) \\
&\leqslant 2\frac{t^2 3^{-2nt}}{n^2} + 2\frac{\mathrm{e}\cos^2 t}{(1+2n)^2}\Big(v_n^2 + \sum_{k=n+1}^{\infty}\Big(\frac{v_k(t)(1-\sqrt{k-n}\,v_k(t))}{\sqrt{k-n}}\Big)^2\Big) \\
&\leqslant 2\frac{t^2 3^{-2nt}}{n^2} + 2\frac{\mathrm{e}\cos^2 t}{(1+2n)^2}v_n^2 + 2\frac{\mathrm{e}\cos^2 t}{(1+2n)^2}\sum_{k=n+1}^{\infty}\Big(\frac{1}{(2\sqrt{k-n})^2}\Big)^2 \\
&\leqslant 2\frac{t^2 3^{-2nt}}{n^2} + \frac{1}{8}\frac{\mathrm{e}\cos^2 t}{(1+2n)^2}\frac{\pi^2}{6} + 2\frac{\mathrm{e}\cos^2 t}{(1+2n)^2}v_n^2.
\end{aligned}$$

Hence, by taking

$$g_n(t) = 2\frac{t^2 3^{-2nt}}{n^2} + \frac{\pi^2}{48}\frac{\mathrm{e}\cos^2 t}{(1+2n)^2}, \quad h_n(t) = 2\frac{\mathrm{e}\cos^2 t}{(1+2n)^2}$$

it is clear that $g_n(t)$ and $h_n(t)$ are real valued continuous functions on I.

Also, for each $t \in I$

$$|g_n(t)| \leqslant 2\frac{T^2}{n^2} + \frac{\pi^2}{48}\frac{\mathrm{e}}{(1+2n)^2} \leqslant \Big(2T^2 + \frac{\pi^2\mathrm{e}}{48}\Big)\frac{1}{n^2}.$$

Thus, by Weierstrass test for uniform convergence of function series we see that $\sum\limits_{k\geqslant 1} g_k(t)$ is uniformly convergent on I.

Further, we have

$$|h_j(t)| \leqslant \frac{2\mathrm{e}}{(1+2n)^2} \quad \forall\, t \in I.$$

Hence, the function sequence $(h_j(t))$ is equibounded on I.

Thus (2.11) is satisfied and hence the assumption (A_3) is satisfied.

Also

$$G = \sup\Big\{\sum_{k\geqslant 1} g_k(t)\colon\ t \in I\Big\} = \Big(2T^2 + \frac{\pi^2\mathrm{e}}{12}\Big)\frac{\pi^2}{6}$$

and

$$H = \sup\{h_j(t)\colon\ t \in I\} = \frac{2\mathrm{e}}{9}.$$

The assumption (A_2) is also satisfied as for fixed $t \in T$ and $(v_j(t)) = (v_1(t), v_2(t), \ldots) \in \ell_2$ we have

$$\sum_{j=1}^{\infty}|f_j(t,v)|^2 = \sum_{j=1}^{\infty} g_j(t) + \sum_{j=1}^{\infty} h_j(t)|v_j(t)|^2 \leqslant G + H\sum_{j=1}^{\infty}|v_j(t)|^2.$$

Hence, the operator $f = (f_j)$ transforms the space (I, ℓ_2) into ℓ_2.

Also, for $\varepsilon > 0$ and $u = (u_j), v = (v_j) \in \ell_2$ with $\|u - v\|_2 < \varepsilon$, we have

$$\begin{aligned}
&\|(fu)(t) - (fv)(t)\|_2^2 \\
&\quad = \sum_{n=1}^{\infty} |f_n(t, u(t)) - f_n(t, v(t))|^2 \\
&\quad = \sum_{n=1}^{\infty} \biggl| \sum_{k=n}^{\infty} \frac{(\cos t) u_k(t)(1 - \sqrt{k-n}\, u_k(t))}{(1+2n)\sqrt{k-n+1}\sqrt{(k-1)!}} - \frac{(\cos t) v_k(t)(1 - \sqrt{k-n}\, v_k(t))}{(1+2n)\sqrt{k-n+1}\sqrt{(k-1)!}} \biggr|^2 \\
&\quad \leqslant \sum_{n=1}^{\infty} \frac{1}{(1+2n)^2} \biggl| \sum_{k=n}^{\infty} \frac{u_k(t)(1 - \sqrt{k-n}\, u_k(t)) - v_k(t)(1 - (k-n) v_k(t))}{\sqrt{k-n+1}\sqrt{(k-1)!}} \biggr|^2 \\
&\quad \leqslant \sum_{n=1}^{\infty} \frac{1}{(1+2n)^2} \biggl(\sum_{k=n}^{\infty} \biggl| \frac{(u_k(t) - v_k(t))(1 - (k-n)(u_k(t) + v_k(t)))}{\sqrt{(k-1)!}\sqrt{k-n+1}} \biggr| \biggr)^2.
\end{aligned}$$

Using Hölder's inequality we get

$$\begin{aligned}
\|(fu)(t) - (fv)(t)\|_2^2 &\leqslant \sum_{n=1}^{\infty} \frac{1}{(1+2n)^2} \sum_{k=n}^{\infty} \frac{1}{(k-1)!} \\
&\quad \times \sum_{k=n}^{\infty} \biggl| \frac{(u_k(t) - v_k(t))(1 - \sqrt{k-n}(u_k(t) + v_k(t)))}{\sqrt{k-n+1}} \biggr|^2 \\
&\leqslant \mathrm{e} \sum_{n=1}^{\infty} \frac{1}{(1+2n)^2} \biggl(|u_n(t) - v_n(t)|^2 \\
&\quad + \sum_{k=n+1}^{\infty} |u_k(t) - v_k(t)|^2 \biggl| \frac{1 - \sqrt{k-n}(u_k(t) + v_k(t))}{\sqrt{k-n}} \biggr|^2 \biggr) \\
&\leqslant \mathrm{e} \sum_{n=1}^{\infty} \frac{1}{(1+2n)^2} \biggl(|u_n(t) - v_n(t)|^2 + \frac{\pi^2}{48} \biggr) \quad \text{using (2.13)} \\
&\leqslant \mathrm{e} \biggl(\sum_{n=1}^{\infty} |u_n(t) - v_n(t)|^2 + \frac{\pi^2}{48} \sum_{n=1}^{\infty} \frac{1}{(1+2n)^2} \biggr).
\end{aligned}$$

Thus, for any $t \in I$, we have

$$\|(fu)(t) - (fv)(t)\|_2 \leqslant \sqrt{\mathrm{e}\Bigl(\varepsilon^2 + \frac{\pi^4}{384}\Bigr)}.$$

Therefore, the family $\{(fv)(t)\colon\ t \in I\}$ is equicontinuous.

Finally, we see that the condition $H^{1/2} T \tanh(\frac{1}{2}T) < 2$ is satisfied for all T. So, by Theorem 2.2 there exists at least one solution to the given infinite system of differential equations (2.10) in $C(I, \ell_2)$.

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References

[1] *R. P. Agarwal, M. Benchohra, S. Hamani*: A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. Acta Appl. Math. *109* (2010), 973–1033. zbl MR doi

[2] *A. Aghajani, E. Pourhadi*: Application of measure of noncompactness to ℓ_1-solvability of infinite systems of second order differential equations. Bull. Belg. Math. Soc.-Simon Stevin *22* (2015), 105–118. zbl MR

[3] *J. Banaś, K. Goebel*: Measures of Noncompactness in Banach Spaces. Lecture Notes in Pure and Applied Mathematics 60. Marcel Dekker, New York, 1980. zbl MR

[4] *J. Banaś, M. Lecko*: Solvability of infinite systems of differential equations in Banach sequence spaces. J. Comput. Appl. Math. *137* (2001), 363–375. zbl MR doi

[5] *J. Banaś, M. Mursaleen*: Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations. Springer, New Delhi, 2014. zbl MR doi

[6] *J. Banaś, M. Mursaleen, S. M. H. Rizvi*: Existence of solutions to a boundary-value problem for an infinite system of differential equations. Electron. J. Differ. Equ. *2017* (2017), Paper No. 262, 12 pages. zbl MR

[7] *G. Darbo*: Punti uniti in trasformazioni a codominio non compatto. Rend. Sem. Mat. Univ. Padova *24* (1955), 84–92. (In Italian.) zbl MR

[8] *K. Deimling*: Ordinary Differential Equations in Banach Spaces. Lecture Notes in Mathematics 596. Springer, Berlin, 1977. zbl MR doi

[9] *K. Deimling*: Nonlinear Functional Analysis. Dover Books on Mathematics. Dover Publications, Mineola, 2010. zbl MR doi

[10] *D. G. Duffy*: Green's Functions with Applications. Studies in Advanced Mathematics, CRC Press, Boca Raton, 2015. zbl MR doi

[11] *J. Klamka*: Schauder's fixed-point theorem in nonlinear controllability problems. Control Cybern. *29* (2000), 153–165. zbl MR

[12] *C. Kuratowski*: Sur les espaces complets. Fundamenta *15* (1930), 301–309. (In French.) zbl doi

[13] *Z. Liu, S. M. Kang*: Applications of Schauder's fixed-point theorem with respect to iterated functional equations. Appl. Math. Lett. *14* (2001), 955–962. zbl MR doi

[14] *E. Malkowsky, V. Rakočević*: An introduction into the theory of sequence spaces and measures of noncompactness. Four Topics in Mathematics (B. Stanković, ed.). Zbornik Radova *9(17)*. Matematički Institut SANU, Beograd, 2000, pp. 143–234. zbl MR

[15] *S. A. Mohiuddine, H. M. Srivastava, A. Alotaibi*: Application of measures of noncompactness to the infinite system of second-order differential equations in ℓ_p spaces. Adv. Difference Equ. *2016* (2016), Paper No. 317, 13 pages. zbl MR doi

[16] *M. Mursaleen*: Application of measure of noncompactness to infinite systems of differential equations. Can. Math. Bull. *56* (2013), 388–394. zbl MR doi

[17] *M. Mursaleen, S. A. Mohiuddine*: Applications of measures of noncompactness to the infinite system of differential equations in ℓ_p spaces. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *75* (2012), 2111–2115. zbl MR doi

[18] *M. Mursaleen, S. M. H. Rizvi*: Solvability of infinite systems of second order differential equations in c_0 and ℓ_1 by Meir-Keeler condensing operators. Proc. Am. Math. Soc. *144* (2016), 4279–4289. zbl MR doi

[19] *M. Mursaleen, S. M. H. Rizvi, B. Samet*: Solvability of a class of boundary value problems in the space of convergent sequences. Appl. Anal. *97* (2018), 1829–1845. zbl MR doi
[20] *H. M. Srivastava, A. Das, B. Hazarika, S. A. Mohiuddine*: Existence of solutions of infinite systems of differential equations of general order with boundary conditions in the spaces c_0 and ℓ_1 via the measure of noncompactness. Math. Methods Appl. Sci. *41* (2018), 3558–3569. zbl MR doi

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