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Applications of Mathematics, Vol. 65 (2020), No. 2, 153–172

Persistent URL: `http://dml.cz/dmlcz/148107`

65 (2020) APPLICATIONS OF MATHEMATICS No. 2, 153–172

ON POLYNOMIAL ROBUSTNESS OF FLUX RECONSTRUCTIONS

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Received June 15, 2019. Published online February 26, 2020.

Abstract. We deal with the numerical solution of elliptic not necessarily self-adjoint problems. We derive a posteriori upper bound based on the flux reconstruction that can be directly and cheaply evaluated from the original fluxes and we show for one-dimensional problems that local efficiency of the resulting a posteriori error estimators depends on $p^{1/2}$ only, where p is the discretization polynomial degree. The theoretical results are verified by numerical experiments.

Keywords: a posteriori error estimate; p-robustness; elliptic problem

MSC 2020: 65N15, 65N30

1. Introduction

A posteriori error estimates are important and practical tools in numerical mathematics. They serve two main purposes in numerical discretization of PDEs: to provide information about the discretization error for the current choice of discretization parameters and to provide the localization of the sources of high errors for upcoming possible adaptive procedures. For the survey of main a posteriori techniques for PDE discretizations see e.g. [2], [4], [9], [17], [21] and references cited therein. The applications and comparisons of a posteriori error estimates can be found in e.g. [13].

Since higher order methods and hp-adaptive techniques start to be more and more popular, the question of robustness with respect to the discretization polynomial degree becomes very important. On the other hand and in contrast to the number of existing results devoted to the robustness with respect to the mesh-size, there are not many theoretical results devoted to the robustness with respect to the polynomial degree. A posteriori error techniques based on the local Neumann problem for

The work was supported from European Regional Development Fund-Project "Center for Advanced Applied Science" (No. CZ.02.1.01/0.0/0.0/16_019/0000778).

DOI: 10.21136/AM.2020.0152-19

hp-adaptive discretizations are discussed e.g. in [1] and [3]. For the analysis of the polynomial dependence of the technique based on the local residual estimators see e.g. [14]. It shall be pointed out that the efficiency of individual estimators proved in [14] behaves as p^1, where p is the underlying polynomial degree used in the finite element method (FEM) discretization.

Important class of approaches for deriving guaranteed a posteriori upper bounds is based on the hypercircle theorem, see [15], where the reconstruction of fluxes should be fully equilibrated, i.e. they should satisfy exactly certain differential equation. By the residual splitting using the dual variable, the restrictive condition of exact solution of full equilibration of the fluxes can be replaced by a milder assumption that the fluxes should be in $H(\mathrm{div})$ only, see e.g. [16]. The extension of these ideas to nonconforming discretizations can be found in e.g. [8], [20]. The quality of the resulting error estimate depends heavily on the choice of the flux reconstruction. Among many approaches for flux reconstructions, the local mixed finite element technique is very popular, since it enables to reconstruct the fluxes based on local relatively cheap problems and since the resulting reconstruction is completely polynomially robust, i.e. the resulting estimators are efficient independently of the polynomial degree. The core of the proof of the polynomial robustness can be found in [7]. The extension of these ideas to wide class of discretization methods can be found in [11].

We assume in this paper even more simple and cheaper reconstruction following the ideas from [10] that can be easily evaluated directly, i.e. without the necessity to solve any local problems. The main aim of this paper is to show its practical usefulness by proving that the resulting local estimators for one-dimensional problems are efficient up to extremely mild polynomial dependence $p^{1/2}$.

This paper is organized as follows: Section 2 contains the continuous problem setting and the corresponding FEM discretization. Auxiliary results are presented in Section 3. A posteriori error upper bound is derived in Section 4 and corresponding efficiency results are proved in Section 5. Finally, Section 6 contains the numerical experiments illustrating the results derived in Section 5.

2. Continuous problem and its discretization

2.1. Continuous problem. Let $\Omega \subset \mathbb{R}^d$ be a bounded polyhedral domain with Lipschitz continuous boundary $\partial\Omega$. We use standard notation for Lebesgue and Sobolev spaces. Let us consider the following boundary value problem: find $u\colon \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta u + b\cdot\nabla u + cu &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \partial\Omega, \end{aligned} \tag{2.1}$$

where $f \in L^2(\Omega)$ and $b \in \mathbb{R}^d$, $c \in \mathbb{R}$ are constants such that $c \geqslant 0$. Moreover, we assume that the convective constant b is of mediocre size at most, i.e. at most $|b| \sim 1$, to prevent the problem becoming convection dominated. Convection dominated problems represent a very challenging task, see e.g. [18] and the references cited therein, and they are beyond the scope of this paper. Let us denote weak space derivative of u by u' for $d = 1$.

Let $(\cdot, \cdot)$ and $\|\cdot\|$ be the $L^2(\Omega)$ scalar product and norm, respectively. Let us denote the function space $V = H_0^1(\Omega)$.

Definition 2.1. We say that the function $u \in V$ is a weak solution of (2.1) if

$$(\nabla u, \nabla v) + (b \cdot \nabla u + cu, v) = (f, v) \quad \forall\, v \in V. \tag{2.2}$$

According to the Lax-Milgram lemma, there exists a unique solution of problem (2.2).

2.2. Discrete problem. We consider a space partition $\mathcal{T}_h$ consisting of a finite number of closed, d-dimensional simplices K with mutually disjoint interiors and covering $\overline{\Omega}$, i.e. $\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$. We denote the vertices of the mesh by a and edges (or faces) by e. In the rest of the paper we talk about boundary objects of co-dimension 1 as about edges, but we mean vertices, edges or faces depending on the dimension d. For each edge e, let $n = n_e$ denote a unit normal vector to e with arbitrary but fixed direction for the inner edges and with outer direction on $\partial\Omega$. We assume conforming properties of the mesh, i.e. neighbouring elements share an entire edge. We set $h_K = \operatorname{diam}(K)$ and $h = \max_K h_K$. We assume shape regularity of elements, i.e. $h_K/\varrho_K \leqslant C$ for all $K \in \mathcal{T}_h$, where ϱ_K is the radius of the largest d-dimensional ball inscribed into K and constant the C does not depend on $\mathcal{T}_h$ for $h \in (0, h_0)$. Moreover, we assume the local quasi-uniformity of the mesh, i.e. we assume $h_K \leqslant C h_{K'}$ for neighbouring elements K and K' and constant the C does not depend on $\mathcal{T}_h$ for $h \in (0, h_0)$ again.

In order to simplify the notation, we set $(\cdot, \cdot)_M$ and $\|\cdot\|_M$ the local $L^2(M)$-scalar products and norms, respectively, where $M \subset \overline{\Omega}$ is a union of elements $K \in \mathcal{T}_h$.

We define classical finite element space

$$V_h = \{v \in H_0^1(\Omega)\colon\ v|_K \in P_p(K)\}, \tag{2.3}$$

where the space $P_p(K)$ denotes the space of polynomials up to the degree $p \geqslant 1$.

Now we are able to define finite element solution of problem (2.2).

Definition 2.2. We say that the function $u_h \in V_h$ is a discrete solution of (2.2) if

$$(\nabla u_h, \nabla v_h) + (b \cdot \nabla u_h + c u_h, v_h) = (f, v_h) \quad \forall\, v_h \in V_h. \tag{2.4}$$

The existence and uniqueness of the discrete solution follows again from the Lax-Milgram lemma.

Although the functions from V_h are globally continuous, we will need to work with piece-wise continuous functions as well. We define one-sided values, jumps and mean values on the inner edges respectively as

$$\begin{aligned} v(x-) &= \lim_{s\to 0+} v(x - ns), \quad v(x+) = \lim_{s\to 0+} v(x+ns), \\ [v](x) &= v(x-) - v(x+), \quad \langle v\rangle(x) = \frac{1}{2}(v(x-) + v(x+)). \end{aligned} \tag{2.5}$$

For the boundary edges we define

$$v(x-) = \langle v\rangle(x) = \lim_{s\to 0+} v(x-ns), \quad [v](x) = 0. \tag{2.6}$$

3. Auxiliary results

Let $\{\widehat{\phi}_s \in P_s(-1,1)\}_{s=0}^{\infty}$ be Legendre orthogonal polynomials, i.e. $\widehat{\phi}_s \perp P_{s-1}(-1,1)$ with respect to $L^2(-1,1)$-scalar product, normalized by $\widehat{\phi}_s(1) = 1$. The lowest degree examples are $\widehat{\phi}_0(x) = 1$ and $\widehat{\phi}_1(x) = x$. Let $\{\widehat{\chi}_s \in P_s(-1,1)\}_{s=1}^{\infty}$ be Radau polynomials defined by

$$\widehat{\chi}_s = \frac{\widehat{\phi}_s + \widehat{\phi}_{s-1}}{2} \tag{3.1}$$

and $\{\widehat{\psi}_s \in P_s(-1,1)\}_{s=2}^{\infty}$ be Lobatto polynomials defined by

$$\widehat{\psi}_s = \widehat{\phi}_s - \widehat{\phi}_{s-2}. \tag{3.2}$$

Lemma 3.1. *The Legendre polynomials satisfy*

$$\|\widehat{\phi}_s\|^2_{L^2(-1,1)} = \frac{2}{2s+1}, \quad \widehat{\phi}_s'(1) = \frac{s(s+1)}{2}. \tag{3.3}$$

The Radau polynomials defined by (3.1) *satisfy*

$$\widehat{\chi}_s(-1) = 0, \quad \widehat{\chi}_s(1) = 1, \quad \widehat{\chi}_s \perp P_{s-2}(-1,1), \quad \|\widehat{\chi}_s\|^2_{L^2(-1,1)} = \frac{2s}{4s^2-1}. \tag{3.4}$$

The Lobatto polynomials defined by (3.2) *satisfy*

$$\widehat{\psi}_s(1) = 0,\ \widehat{\psi}_s(-1) = 0,\ \widehat{\psi}_s \perp P_{s-3}(-1,1),\ \|\widehat{\psi}_s\|^2_{L^2(-1,1)} = \frac{8s-4}{(2s+1)(2s-3)}, \tag{3.5}$$

and

$$\widehat{\psi}'_s = (2s-1)\widehat{\phi}_{s-1}, \quad \|\widehat{\psi}'_s\|^2_{L^2(-1,1)} = 4s-2. \tag{3.6}$$

P r o o f. The relation for the norm of Legendre polynomials can be found in e.g. [19]. Moreover, the Legendre polynomials satisfy the three-term recurrence

$$(s+1)\widehat{\phi}_{s+1}(x) = (2s+1)x\widehat{\phi}_s(x) - s\widehat{\phi}_{s-1}(x), \tag{3.7}$$

see e.g. [19]. Differentiating the three-term recurrence, inserting $x = 1$ and using $\widehat{\phi}_s(1) = 1$, we obtain

$$(s+1)\widehat{\phi}'_{s+1}(1) = (2s+1) + (2s+1)\widehat{\phi}'_s(1) - s\widehat{\phi}'_{s-1}(1). \tag{3.8}$$

Then the relation for $\widehat{\phi}'_s(1)$ follows by induction. Relations (3.4) and (3.5) can be directly verified from (3.1) and (3.2), respectively, and from the properties of Legendre polynomials. Now, let us show that $\widehat{\psi}'_s = C\widehat{\phi}_{s-1}$, where $C = C(s)$ is a constant. Since $\widehat{\psi}'_s \in P_{s-1}(-1,1)$, it is sufficient to show that $\widehat{\psi}'_s \perp P_{s-2}(-1,1)$. Using (3.5), we get

$$\int_{-1}^{1} \widehat{\psi}'_s w \,\mathrm{d}x = -\int_{-1}^{1} \widehat{\psi}_s w' \,\mathrm{d}x - \widehat{\psi}_s(-1)w(-1) + \widehat{\psi}_s(1)w(1) = 0 \quad \forall\, w \in P_{s-2}(-1,1). \tag{3.9}$$

From this it follows that

$$C\widehat{\phi}_{s-1}(1) = \widehat{\psi}'_s(1) = \widehat{\phi}'_s(1) - \widehat{\phi}'_{s-2}(1). \tag{3.10}$$

Applying (3.3), we arrive at $C = 2s-1$. The relation for the norm of $\widehat{\psi}'_s$ then follows from the relation for the norm of Legendre polynomials. □

The Lobatto polynomials ψ_s on $K = [a_L, a_R]$ are defined by transformation of $\widehat{\psi}_s$ from the reference interval $[-1,1]$,

$$\psi_s(x) = \widehat{\psi}_s\Big(\frac{2(x-a_L)}{h_K} - 1\Big), \quad x \in K. \tag{3.11}$$

The Legendre polynomials ϕ_s and the Radau polynomials χ_s are defined on $K \in \mathcal{T}_h$ analogously.

Lemma 3.2. *Let $v \in V$. Then there exists $v_h \in V_h$ and constant $C_{\rm Fl} > 0$ independent of local mesh-size h_K and polynomial degree $p \geqslant 1$ such that*

$$\|v - v_h\|_K \leqslant C_{\rm Fl} \frac{h_K}{p} \|\nabla v\|_K. \tag{3.12}$$

P r o o f. The result can be found in [5]. □

For some cases, the value of the constant $C_{\rm Fl}$ from Lemma 3.2 can be determined exactly. We will show the value of $C_{\rm Fl}$ for $d = 1$.

Lemma 3.3. *Let $d = 1$ and $v \in V$. Then there exists $v_h \in V_h$ such that estimate* (3.12) *holds with*

$$C_{\rm Fl} = \frac{p}{\sqrt{(2p+3)(2p-1)}}. \tag{3.13}$$

P r o o f. Let us decompose $v|_K \in H^1(K)$ as

$$v|_K = \varphi + \sum_{s=2}^{\infty} \alpha_s \psi_s, \tag{3.14}$$

where $\{\alpha_s\}_{s=2}^{\infty} \subset \mathbb{R}$, $\varphi \in P_1(K)$ is the linear interpolation at the end points of K and $\psi_s \in P_s(K)$ are Lobatto basis (bubble) function defined on $K \in \mathcal{T}_h$ by (3.11). Let us construct suitable v_h element-wise as

$$v_h|_K = \varphi + \sum_{s=2}^{p} \alpha_s \psi_s. \tag{3.15}$$

Applying (3.2) and the orthogonality of Legendre polynomials ϕ_s, we get

$$\begin{aligned} \|v - v_h\|_K^2 &= \Big\| \sum_{s=p+1}^{\infty} \alpha_s \psi_s \Big\|_K^2 = \Big\| \sum_{s=p+1}^{\infty} \alpha_s (\phi_s - \phi_{s-2}) \Big\|_K^2 \\ &= \sum_{s=p+1}^{\infty} \alpha_s^2 (\|\phi_s\|_K^2 + \|\phi_{s-2}\|_K^2) - 2 \sum_{s=p+1}^{\infty} \alpha_s \alpha_{s+2} \|\phi_s\|_K^2 \end{aligned} \tag{3.16}$$

$$\leqslant \sum_{s=p+1}^{\infty} \alpha_s^2(\|\phi_s\|_K^2 + \|\phi_{s-2}\|_K^2) + \sum_{s=p+1}^{\infty} \alpha_s^2\|\phi_s\|_K^2$$
$$+ \sum_{s=p+1}^{\infty} \alpha_{s+2}^2\|\phi_s\|_K^2 \leqslant 2 \sum_{s=p+1}^{\infty} \alpha_s^2(\|\phi_s\|_K^2 + \|\phi_{s-2}\|_K^2)$$
$$= 2 \sum_{s=p+1}^{\infty} \alpha_s^2\|\psi_s\|_K^2.$$

From Lemma 3.1, it follows for Lobatto polynomials scaled to $[-1,1]$ that

$$\|\widehat{\psi}_s\|_{(-1,1)}^2 = \frac{2}{(2s+1)(2s-3)}\|\widehat{\psi}_s'\|_{(-1,1)}^2. \tag{3.17}$$

Since the ratio between the original element K and the reference domain $[-1,1]$ is $h_K/2$, we get after transformation from $[-1,1]$ to K that

$$\|\psi_s\|_K^2 = \frac{h_K^2}{2(2s+1)(2s-3)}\|\psi_s'\|_K^2. \tag{3.18}$$

Inserting this relation into (3.16), we obtain

$$\begin{aligned}\|v - v_h\|_K^2 &\leqslant 2 \sum_{s=p+1}^{\infty} \alpha_s^2\|\psi_s\|_K^2 = 2 \sum_{s=p+1}^{\infty} \alpha_s^2 \frac{h_K^2}{2(2s+1)(2s-3)}\|\psi_s'\|_K^2 \\ &\leqslant \frac{h_K^2}{(2p+3)(2p-1)} \sum_{s=p+1}^{\infty} \alpha_s^2\|\psi_s'\|_K^2.\end{aligned} \tag{3.19}$$

Since $\widehat{\psi}_s' = (2s-1)\widehat{\phi}_{s-1}$, $s \geqslant 2$, the derivatives of Lobatto basis and constants are mutually orthogonal. Then we get

$$\sum_{s=p+1}^{\infty} \alpha_s^2\|\psi_s'\|_K^2 \leqslant \|\varphi'\|_K^2 + \sum_{s=2}^{\infty} \alpha_s^2\|\psi_s'\|_K^2 = \left\|\varphi' + \sum_{s=2}^{\infty} \alpha_s\psi_s'\right\|_K^2 = \|v'\|_K^2. \tag{3.20}$$

□

4. Flux reconstruction, error measure and its upper bound

4.1. Flux reconstruction. Since the discretization by FEM is conforming, the exact solution u as well as the discrete solution u_h belong to common space $V = H_0^1(\Omega)$. This quality, i.e. the exact and the discrete solutions belong to common space, does not hold for the gradient of the solution, since $\nabla u \in H(\mathrm{div}, \Omega)$ and $\nabla u_h \notin H(\mathrm{div}, \Omega)$ in general. Our aim is to find suitable reconstruction $\sigma_h = \sigma_h(\nabla u_h) \in H(\mathrm{div}, \Omega)$ such that $\sigma_h \approx \nabla u_h$.

Let $RT_p(K)$ be the local Raviart-Thomas space of order p for element $K \in \mathcal{T}_h$, i.e. $RT_p(K) = P_p(K)^d + x\overline{P}_p(K)$, where $\overline{P}_p(K)$ is a subspace of $P_p(K)$ containing only the polynomial terms of degree p. For $d = 1$, $RT_p(K)$ space is simplified to $P_{p+1}(K)$. The details about Raviart-Thomas spaces and about FEM-like spaces for approximation $H(\mathrm{div}, \Omega)$ in general can be found in [6]. We define the reconstruction σ_h element-wise. We seek $\sigma_h|_K \in RT_p(K)$ such that

$$\begin{aligned} \sigma_h|_e \cdot n &= \langle \nabla u_h \rangle|_e \cdot n \quad \forall\, e \subset K, \\ (\sigma_h, z_h)_K &= (\nabla u_h, z_h)_K \quad \forall\, z_h \in P_{p-1}(K)^d. \end{aligned} \tag{4.1}$$

The conditions in (4.1) represent the natural degrees of freedom for $RT_p(K)$, see [6], Proposition 2.3.4. Applying basis corresponding to these degrees of freedom enables to assemble σ_h directly without the necessity to solve any local linear problems, which results in extremely cheap evaluation of the reconstruction σ_h. This property will be demonstrated later in Lemma 5.1 for $d = 1$.

We should point out that the resulting function σ_h has continuous normal components on inter-element edges and therefore the composition of local contributions of σ_h is in $H(\mathrm{div}, \Omega)$, see e.g. [6].

Important property of σ_h is the orthogonality of $f + \mathrm{div}\, \sigma_h - b \cdot \nabla u_h - cu_h$ on V_h that follows from the discrete problem formulation (2.4) and from (4.1)

$$\begin{aligned} (f + \mathrm{div}\, \sigma_h - b \cdot \nabla u_h - cu_h, v_h) &= (f, v_h) - (b \cdot \nabla u_h + cu_h, v_h) - (\sigma_h, \nabla v_h) \\ &= (f, v_h) - (b \cdot \nabla u_h + cu_h, v_h) - (\nabla u_h, \nabla v_h) = 0 \quad \forall\, v_h \in V_h. \end{aligned} \tag{4.2}$$

R e m a r k 4.1. Relation (4.2) represents a weaker version of the equilibrated flux property

$$(f + \mathrm{div}\, \sigma_h - b \cdot \nabla u_h - cu_h, v_h)_K = 0 \quad \forall\, v_h \in P_p(K), \tag{4.3}$$

used in e.g. [11].

R e m a r k 4.2. The important ingredient for relation (4.2) is that u_h is the exact solution of the discrete problem (2.4). Such a solution is not available for the reconstruction in practical computations, since many other sources of errors come into play (algebraic errors, quadrature errors, rounding errors, etc.). Including these sources of errors will result in the necessity to enhance relation (4.2) by corresponding remainders, e.g. the algebraic error could be represented by the additional term corresponding to the algebraic residuum. A posteriori error estimate including algebraic error can be found in e.g. [12]. For simplicity, we assume in this paper that the exact solution u_h of problem (2.4) is available.

4.2. Upper bound. We define the error measure for $w \in V$ as the dual norm of residual

$$\operatorname{Err}(w) = \sup_{0\neq v\in V} \frac{(f,v) - (\nabla w, \nabla v) - (b\cdot\nabla w + cw, v)}{\|\nabla v\|}. \tag{4.4}$$

R e m a r k 4.3. For the most simple case $b = 0$, $c = 0$, the error measure is equivalent to H^1-seminorm, i.e. $\operatorname{Err}(w) = \|\nabla u - \nabla w\|$.

The aim of this section is to bound the error measure $\operatorname{Err}(u_h)$ from above. Let $v \in V$ be arbitrary, let $u_h \in V_h$ be the discrete solution given by (2.4) and let σ_h be the reconstruction obtained from u_h by (4.1). Then

$$\begin{aligned}(f,v) &- (\nabla u_h, \nabla v) - (b\cdot\nabla u_h + cu_h, v)\\ &= (f + \operatorname{div}\sigma_h - b\cdot\nabla u_h - cu_h, v) + (\sigma_h - \nabla u_h, \nabla v).\end{aligned} \tag{4.5}$$

We estimate the terms on the right-hand side individually. We apply (4.2) and Lemma 3.2 on the first term and we get

$$\begin{aligned}(f + \operatorname{div}\sigma_h - b\cdot\nabla u_h - cu_h, v) &= \inf_{v_h\in V_h} (f + \operatorname{div}\sigma_h - b\cdot\nabla u_h - cu_h, v - v_h)\\ &\leqslant \sum_K C_{\mathrm{Fl}}\frac{h_K}{p}\|f + \operatorname{div}\sigma_h - b\cdot\nabla u_h - cu_h\|_K\|\nabla v\|_K.\end{aligned} \tag{4.6}$$

The second term can be estimated by the Cauchy inequality

$$(\sigma_h - \nabla u_h, \nabla v) \leqslant \sum_K \|\sigma_h - \nabla u_h\|_K\|\nabla v\|_K. \tag{4.7}$$

Applying these individual estimates together, we get

$$\begin{aligned}&((f - b\cdot\nabla u_h - cu_h, v) - (\nabla u_h, \nabla v))^2\\ &\leqslant \sum_K \Big(C_{\mathrm{Fl}}\frac{h_K}{p}\|f + \operatorname{div}\sigma_h - b\cdot\nabla u_h - cu_h\|_K + \|\sigma_h - \nabla u_h\|_K\Big)^2\|\nabla v\|^2.\end{aligned} \tag{4.8}$$

Let us denote partial estimators

$$\eta_{R,K} = C_{\mathrm{Fl}} \frac{h_K}{p} \|f + \operatorname{div} \sigma_h - b \cdot \nabla u_h - c u_h\|_K, \tag{4.9}$$
$$\eta_{F,K} = \|\sigma_h - \nabla u_h\|_K.$$

From these considerations follows the upper a posteriori error estimate.

Theorem 4.1. *Let $u_h \in V_h$ be the discrete solution obtained by* (2.4) *and σ_h be the reconstruction obtained from u_h by* (4.1). *Then*

$$\operatorname{Err}(u_h)^2 \leqslant \eta^2 = \sum_K (\eta_{R,K} + \eta_{F,K})^2. \tag{4.10}$$

Remark 4.4. The constant C_{Fl} contained in $\eta_{R,K}$ is unknown in general. This constant can be determined in some special cases, e.g. the application of Lemma 3.3 instead of Lemma 3.2 gives the modification of the estimator $\eta_{R,K}$ for $d = 1$

$$\eta_{R,K} = \frac{h_K}{\sqrt{(2p+3)(2p-1)}} \|f + \sigma_h' - b u_h' - c u_h\|_K, \tag{4.11}$$

where all the terms in (4.11) are known. Then both the estimators $\eta_{R,K}$ and $\eta_{F,K}$ are fully computable.

5. Local error measures and its lower bound in one dimension

In this section we assume $d = 1$. The aim of this section is to show that the local individual estimators $\eta_{R,K}$ and $\eta_{F,K}$ from a posteriori estimate (4.10) are locally efficient and how this efficiency depends on the polynomial degree p. It means that these local estimators provide local lower bounds to the local error measure up to some powers of p and some generic constant $C > 0$ that may depend on constants coming from the original continuous problem (the size of the domain Ω, etc.) or on the constants coming from the discretization (mesh shape regularity constant, etc.). However, this constant should be independent of the exact solution u, discrete solution u_h, local mesh sizes h_K, and polynomial degree p. Dependence of the estimate up to this generic constant will be denoted by $\lesssim$.

For the purpose of the efficiency analysis we suppose a traditional assumption that $f \in V_h$. Otherwise, classical oscillation term

$$\sup_{0 \neq v \in V} \frac{(f - f_h, v)}{\|v'\|} \tag{5.1}$$

appears additionally in the efficiency results, where f_h is L^2-orthogonal projection of f on V_h.

To be able to apply the result in a local way, we need the following notation. Let ω_a be a patch consisting of elements sharing common vertex a and ω_K be a patch consisting of elements sharing at least a vertex with K. Let $M \subset \overline{\Omega}$, e.g. $M = K$ or $M = \omega_K$. We define a local version of the space V by

$$V_M = \{v \in V\colon \operatorname{supp}(v) \subset M\} \tag{5.2}$$

and a corresponding local version of Err

$$\mathrm{Err}_M(w) = \sup_{0 \neq v \in V_M} \frac{(f, v) - (w', v') - (bw' + cw, v)}{\|v'\|}. \tag{5.3}$$

Typically, we use $\mathrm{Err}_K(u_h)$, $\mathrm{Err}_{\omega_a}(u_h)$ or $\mathrm{Err}_{\omega_K}(u_h)$. Since the patch ω_K is composed from three elements at most, it is possible to see that

$$\sum_K \mathrm{Err}_K(u_h)^2 \leqslant \sum_K \mathrm{Err}_{\omega_K}(u_h)^2 \lesssim \mathrm{Err}(u_h)^2. \tag{5.4}$$

We divide the proof of the local efficiency of the individual partial estimators $\eta_{R,K}$ and $\eta_{F,K}$ into next auxiliary lemmas.

Lemma 5.1. *Let $d = 1$. Let us denote a polynomial $r_L \in P_{p+1}(K)$ such that $r_L(a_L) = 1$, $r_L(a_R) = 0$ and $r_L \perp P_{p-1}(K)$ for the element $K = [a_L, a_R]$. The polynomial $r_R \in P_{p+1}(K)$ associated with a_R instead of a_L is defined analogically, i.e. $r_R(a_R) = 1$, $r_R(a_L) = 0$ and $r_R \perp P_{p-1}(K)$. Then the reconstruction σ_h defined by* (4.1) *can be expressed by*

$$\sigma_h|_K = u_h'|_K + \frac{1}{2} n[u_h'](a_L) r_L - \frac{1}{2} n[u_h'](a_R) r_R. \tag{5.5}$$

Proof. Inserting a_L and a_R into (5.5), we obtain $\sigma_h(a_L) = \langle u_h' \rangle(a_L)$ and $\sigma_h(a_R) = \langle u_h' \rangle(a_R)$, respectively. That corresponds to the first condition in (4.1). Using the orthogonality of polynomials r_L and r_R on $P_{p-1}(K)$, we gain the second condition in (4.1). □

Remark 5.1. The polynomials r_L and r_R are known as Radau polynomials, e.g. $r_R = \chi_{p+1}$, where χ_{p+1} is transformation of the reference Radau polynomial $\widehat{\chi}_{p+1}$ defined in Section 3. They can be alternatively defined as polynomials with zeros in the Radau quadrature nodes. They represent natural basis functions associated with *edge* degrees of freedom in (4.1) for $d = 1$.

Lemma 5.2. *Let $d = 1$, $f \in V_h$, $u_h \in V_h$ and let σ_h be the reconstruction obtained from u_h' by* (4.1). *Then*

$$\eta_{F,K} = \|\sigma_h - u_h'\|_K \lesssim p^{1/2}\,\mathrm{Err}_{\omega_K}(u_h). \tag{5.6}$$

P r o o f. Let us denote the end points of K as a_L and a_R, i.e. $K = [a_L, a_R]$. Then applying Lemma 5.1 and Lemma 3.1 and scaling between reference interval $[-1, 1]$ and K, we get

$$\begin{aligned}\|\sigma_h - u_h'\|_K &\leqslant \frac{1}{2}(|[u_h'](a_L)|\|r_L\|_K + |[u_h'](a_R)|\|r_R\|_K)\\ &= \frac{1}{4}\frac{\sqrt{h_K}}{\sqrt{2}}(|[u_h'](a_L)| + |[u_h'](a_R)|)\|\widehat{\chi}_{p+1}\|_{(-1,1)}\\ &= \frac{1}{4}\frac{\sqrt{h_K(p+1)}}{\sqrt{4(p+1)^2-1}}(|[u_h'](a_L)| + |[u_h'](a_R)|)\\ &\lesssim \frac{\sqrt{h_K}}{\sqrt{p}}(|[u_h'](a_L)| + |[u_h'](a_R)|).\end{aligned} \tag{5.7}$$

Now, let us show the relation between $|[u_h'](a)|$ for $a = a_L, a_R$ and $\mathrm{Err}_{\omega_K}(u_h)$. The case $a = a_R$ is very similar to the case $a = a_L$. Therefore, we discuss only the version with $a = a_L$. Let φ_{a_L} be piece-wise linear function associated with vertex a_L such that $\varphi_{a_L}(a_L) = 1$ and $\varphi_{a_L}(a) = 0$ for other vertices a. Let us define ϕ_{a_L} a piece-wise polynomial function of degree at most $p+2$ satisfying $\mathrm{supp}(\phi_{a_L}) \subset \omega_{a_L}$, $\phi_{a_L}(a_L) = 1$ and ϕ_{a_L} be orthogonal to piece-wise polynomials up to degree $p+1$. Now, we are able to design a suitable test function $w_{a_L} \in V_{\omega_{a_L}}$

$$w_{a_L} = -\mathrm{sgn}([u_h'](a_L))\varphi_{a_L}\phi_{a_L}. \tag{5.8}$$

Then

$$\begin{aligned}\mathrm{Err}_{\omega_{a_L}}(u_h) &= \sup_{0\neq v\in V_{\omega_{a_L}}} \frac{(f,v) - (u_h', v') - (bu_h' + cu_h, v)}{\|v'\|}\\ &\geqslant \frac{(f, w_{a_L}) - (u_h', w_{a_L}') - (bu_h' + cu_h, w_{a_L})}{\|w_{a_L}'\|}\\ &= \frac{\sum_K (f + u_h'' - bu_h' - cu_h, w_{a_L})_K - \sum_a [u_h'](a) w_{a_L}(a)}{\|w_{a_L}'\|}\\ &= \frac{|[u_h'](a_L)|}{\|w_{a_L}'\|}.\end{aligned} \tag{5.9}$$

We shall investigate $\|w'_{a_L}\|^2 = \|w'_{a_L}\|_K^2 + \|w'_{a_L}\|_{K'}^2$, where $K' \subset \omega_a$ is the neighbouring element of K. The forthcoming analysis is very similar for both elements. Therefore, we focus only on $\|w'_{a_L}\|_K^2$. From (5.8) it follows that

$$\begin{aligned}\|w'_{a_L}\|_K^2 &= \int_{a_L}^{a_R} (w'_{a_L})^2 \,\mathrm{d}x = \int_{a_L}^{a_R} (\varphi'_{a_L}\phi_{a_L} + \varphi_{a_L}\phi'_{a_L})^2 \,\mathrm{d}x \\ &\lesssim \int_{a_L}^{a_R} (\varphi'_{a_L})^2 \phi_{a_L}^2 \,\mathrm{d}x + \int_{a_L}^{a_R} \varphi_{a_L}^2 (\phi'_{a_L})^2 \,\mathrm{d}x.\end{aligned} \tag{5.10}$$

We estimate the final integrals individually. Since $(\varphi'_{a_L})^2|_K = 1/h_K^2$, we obtain by Lemma 3.1 and by scaling between $[-1,1]$ and K

$$\int_{a_L}^{a_R} (\varphi'_{a_L})^2 \phi_{a_L}^2 \,\mathrm{d}x = \frac{1}{h_K^2}\int_{a_L}^{a_R} \phi_{a_L}^2 \,\mathrm{d}x = \frac{1}{2h_K}\|\widehat{\phi}_{p+2}\|_{(-1,1)}^2 = \frac{1}{h_K(2p+5)}. \tag{5.11}$$

Since $0 \leqslant \varphi_{a_L} \leqslant 1$, we get

$$\begin{aligned}\int_{a_L}^{a_R} \varphi_{a_L}^2 (\phi'_{a_L})^2 \,\mathrm{d}x &\leqslant \int_{a_L}^{a_R} \varphi_{a_L}(x)(\phi'_{a_L})^2 \,\mathrm{d}x \\ &= \varphi_{a_L}(a_R)\phi'_{a_L}(a_R)\phi_{a_L}(a_R) - \varphi_{a_L}(a_L)\phi'_{a_L}(a_L)\phi_{a_L}(a_L) \\ &\quad - \int_{a_L}^{a_R} (\varphi'_{a_L}\phi'_{a_L} + \varphi_{a_L}\phi''_{a_L})\phi_{a_L} \,\mathrm{d}x \\ &= -\,\phi'_{a_L}(a_L).\end{aligned} \tag{5.12}$$

We get by Lemma 3.1 and by scaling between $[-1,1]$ and K

$$-\phi'_{a_L}(a_L) = \frac{2}{h_K}\widehat{\phi}'_{p+2}(1) = \frac{(p+2)(p+3)}{h_K}. \tag{5.13}$$

Putting these individual estimates together and applying the local quasi-uniformity of the mesh, we obtain

$$\|w'_{a_L}\|^2 = \|w'_{a_L}\|_K^2 + \|w'_{a_L}\|_{K'}^2 \lesssim \frac{p^2}{h_K} + \frac{p^2}{h_{K'}} \lesssim \frac{p^2}{h_K}. \tag{5.14}$$

Then estimates (5.7), (5.9), and (5.14) give

$$\begin{aligned}\|\sigma_h - u'_h\|_K^2 &\lesssim \frac{h_K}{p}(|[u'_h](a_L)|^2 + |[u'_h](a_R)|^2) \\ &\leqslant \frac{h_K}{p}\,\mathrm{Err}_{\omega_K}(u_h)^2(\|w'_{a_L}\|^2 + \|w'_{a_R}\|^2) \lesssim p\,\mathrm{Err}_{\omega_K}(u_h)^2.\end{aligned} \tag{5.15}$$

□

Lemma 5.3. *Let $d = 1$, $f \in V_h$, $u_h \in V_h$ and let σ_h be the reconstruction obtained from u_h by (4.1). Then*

$$\eta_{R,K} = \frac{h_K}{\sqrt{(2p+3)(2p-1)}} \|f + \sigma_h' - bu_h' - cu_h\|_K \lesssim p^{1/2} \operatorname{Err}_{\omega_K}(u_h). \tag{5.16}$$

P r o o f. Let us denote $w = f + \sigma_h' - bu_h' - cu_h$. Let us represent $v \in V_K$ as

$$v = \sum_{s=2}^{\infty} \alpha_s \psi_s, \tag{5.17}$$

where ψ_s are Lobatto polynomials defined by (3.2) and transformed from the reference element $[-1,1]$ to K and $\{\alpha_s\}_{s=2}^{\infty} \subset \mathbb{R}$ are the corresponding coefficients. Let us show that

$$\sum_{s=2}^{\infty} \alpha_s^2 \|\psi_s\|_K^2 \lesssim \Big\|\sum_{s=2}^{\infty} \alpha_s \psi_s\Big\|_K^2 = \|v\|_K^2. \tag{5.18}$$

It is possible to show it equivalently on the reference element $[-1,1]$ instead of K. Applying Lemma 3.1, we can see that

$$\begin{aligned}
\Big\|\sum_{s=2}^{\infty} \alpha_s \widehat{\psi}_s\Big\|_{(-1,1)}^2 &= \sum_{s=2}^{\infty} \alpha_s^2 \|\widehat{\psi}_s\|_{(-1,1)}^2 - \sum_{s=4}^{\infty} \alpha_s \alpha_{s-2} \|\widehat{\phi}_{s-2}\|_{(-1,1)}^2 \\
&\geqslant \sum_{s=2}^{\infty} \alpha_s^2 \|\widehat{\psi}_s\|_{(-1,1)}^2 - \frac{1}{2} \sum_{s=4}^{\infty} (\alpha_s^2 + \alpha_{s-2}^2) \|\widehat{\phi}_{s-2}\|_{(-1,1)}^2 \\
&\geqslant \sum_{s=2}^{\infty} \alpha_s^2 (\|\widehat{\phi}_s\|_{(-1,1)}^2 + \|\widehat{\phi}_{s-2}\|_{(-1,1)}^2) \\
&\quad - \frac{1}{2} \sum_{s=2}^{\infty} \alpha_s^2 (\|\widehat{\phi}_s\|_{(-1,1)}^2 + \|\widehat{\phi}_{s-2}\|_{(-1,1)}^2) = \frac{1}{2} \sum_{s=2}^{\infty} \alpha_s^2 \|\widehat{\psi}_s\|_{(-1,1)}^2.
\end{aligned} \tag{5.19}$$

Using density of $H_0^1(K)$ in $L^2(K)$ and (5.18), we get

$$\|w\|_K^2 = \sup_{v \in V_K} \frac{(w,v)^2}{\|v\|^2} \lesssim \sup_{v \in V_K} \frac{(w,v)^2}{\sum\limits_{s=2}^{\infty} \alpha_s^2 \|\psi_s\|_K^2}. \tag{5.20}$$

Since $\psi_s \perp P_{s-3}(K)$ and $w \in P_p(K)$ and since $w \perp \psi_s$, $s = 2, \ldots, p$ according to (4.2), we can see that it is possible to take supremum in (5.20) over $v \in V_{K,p+1}$ only, where

$$V_{K,p+1} = \operatorname{span}\{\psi_{p+1}, \psi_{p+2}\} \subset V_K. \tag{5.21}$$

From this follows

$$
\frac{h_K^2}{p^2}\|w\|_K^2 \lesssim \sup_{v\in V_{K,p+1}} \frac{(w,v)^2}{\alpha_{p+1}^2\|\psi_{p+1}\|_K^2+\alpha_{p+2}^2\|\psi_{p+2}\|_K^2}\frac{h_K^2}{p^2} = \sup_{v\in V_{K,p+1}} \frac{(w,v)^2}{\|v'\|^2}\frac{h_K^2}{p^2}\frac{\|v'\|^2}{\alpha_{p+1}^2\|\psi_{p+1}\|_K^2+\alpha_{p+2}^2\|\psi_{p+2}\|_K^2}. \tag{5.22}
$$

According to Lemma 5.2,

$$
\begin{aligned}
\sup_{v\in V_{K,p+1}} \frac{(w,v)}{\|v'\|} &= \sup_{v\in V_{K,p+1}} \frac{(f+\sigma_h'-bu_h'-cu_h,v)}{\|v'\|}\\
&\leqslant \sup_{v\in V_{K,p+1}} \frac{(f-bu_h'-cu_h,v)-(u_h',v')}{\|v'\|}\\
&\quad+\sup_{v\in V_{K,p+1}} \frac{(u_h'-\sigma_h,v')}{\|v'\|}\\
&\leqslant \mathrm{Err}_{\omega_K}(u_h)+\|u_h'-\sigma_h\|_K \lesssim p^{1/2}\,\mathrm{Err}_{\omega_K}(u_h).
\end{aligned} \tag{5.23}
$$

Then it is sufficient to prove that

$$
h_K^2\|v'\|_K^2 \lesssim p^2(\alpha_{p+1}^2\|\psi_{p+1}\|_K^2+\alpha_{p+2}^2\|\psi_{p+2}\|_K^2) \quad \forall\, v\in V_{K,p+1} \tag{5.24}
$$

to finish the proof. We can show (3.18) in the same way as in the proof of Lemma 3.3. Since ψ_s' are othogonal, see Lemma 3.1, we get with the aid of (3.18)

$$
\begin{aligned}
h_K^2\|v'\|_K^2 &= h_K^2(\alpha_{p+1}^2\|\psi_{p+1}'\|_K^2+\alpha_{p+2}^2\|\psi_{p+2}'\|_K^2)\\
&= h_K^2\Big(\alpha_{p+1}^2\frac{2(2p+3)(2p-1)}{h_K^2}\|\psi_{p+1}\|_K^2\\
&\qquad+\alpha_{p+2}^2\frac{2(2p+5)(2p+1)}{h_K^2}\|\psi_{p+2}\|_K^2\Big)\\
&\lesssim p^2(\alpha_{p+1}^2\|\psi_{p+1}\|_K^2+\alpha_{p+2}^2\|\psi_{p+2}\|_K^2).
\end{aligned} \tag{5.25}
$$

□

We summarize the results from Lemma 5.2 and Lemma 5.3 in the following theorem.

Theorem 5.1. *Let $d=1$, $f\in V_h$, $u_h\in V_h$, and let σ_h be the reconstruction obtained from u_h by (4.1). Then*

$$
\begin{aligned}
\eta_{R,K} &\lesssim p^{1/2}\,\mathrm{Err}_{\omega_K}(u_h),\\
\eta_{F,K} &\lesssim p^{1/2}\,\mathrm{Err}_{\omega_K}(u_h).
\end{aligned} \tag{5.26}
$$

Global efficiency estimate is a direct consequence of Theorem 5.1 and (5.4).

Theorem 5.2. *Let $d=1$, $f\in V_h$, $u_h\in V_h$, and let σ_h be the reconstruction obtained from u_h by* (4.1). *Then*

$$\sum_K (\eta_{R,K}+\eta_{F,K})^2 \lesssim p\,\mathrm{Err}(u_h)^2. \tag{5.27}$$

6. Numerical experiments

The aim of this section is to show the reliability, robustness and efficiency of the estimate from Theorem 4.1 for $d=1$.

The computation of the individual a posteriori error estimators can be made directly according to (4.9) or (4.11). On the other hand, the computation of the error measures $\mathrm{Err}(u_h)$ or $\mathrm{Err}_{\omega_K}(u_h)$ is difficult even if the exact solution is known, since these error measures are defined as suprema over infinite dimensional spaces. We approximate these error measures by computing these suprema over space $V_h^+\subset V$ that is richer than the original FEM space V_h, but still finite dimensional. We use four times denser mesh than V_h and polynomial degree $p+2$ instead of p for the construction of V_h^+. We construct spaces $V_{h,M}^+\subset V_M$ as subspaces of V_h^+ containing functions with supports restricted to $M\subset\overline{\Omega}$. We compute the approximation of the Riesz representative of residual $z\in V_h^+$ satisfying

$$(z,v_h)=(f-b\cdot\nabla u_h-cu_h,v_h)-(\nabla u_h,\nabla v_h)\quad\forall\,v_h\in V_h^+. \tag{6.1}$$

Then $\mathrm{Err}(u_h)\approx\mathrm{Err}_h^+(u_h)=\|\nabla z\|$. The localized versions $\mathrm{Err}_M(u_h)$ are approximated analogically with the aid of $V_{h,M}^+$ instead of V_h^+.

Let us denote approximate effectivity index

$$\mathrm{Eff}=\frac{\eta}{\mathrm{Err}_h^+(u_h)} \tag{6.2}$$

and its local counterparts for element K

$$\mathrm{Eff}_{R,K}=\frac{\eta_{R,K}}{\mathrm{Err}_{h,\omega_K}^+(u_h)},\quad \mathrm{Eff}_{F,K}=\frac{\eta_{F,K}}{\mathrm{Err}_{h,\omega_K}^+(u_h)}. \tag{6.3}$$

6.1. Problem settings. We restrict ourselves to $d=1$ and $\Omega=(0,1)$. We assume two problems: purely elliptical problem (PEP), where $b,c=0$, and convection-diffusion-reaction problem (CDRP), where $b=2$ and $c=1$. We set the right-hand side $f=\pi^2\sin(\pi x)$ for PEP and $f=1$ for CDRP.

6.2. Global h-performance. We test the error estimate (4.10) with respect to the mesh refinement. The polynomial degree is set as $p = 3$. We assume a sequence of successively refined equidistant meshes started with $h = 1/10$ and halved in each step.

We can see from Table 1 that the effectivity indices are tending to one for decreasing h.

		PEP			CDRP	
$1/h$	$\mathrm{Err}_h^+(u_h)$	η	Eff	$\mathrm{Err}_h^+(u_h)$	η	Eff
10	$2.1672-4$	$2.6869-4$	1.24	$1.7478-5$	$2.9540-5$	1.69
20	$2.7111-5$	$3.0187-5$	1.11	$2.1903-6$	$3.1610-3$	1.44
40	$3.3896-7$	$3.5760-6$	1.06	$2.7397-7$	$3.4520-7$	1.26
80	$4.2372-7$	$4.3520-7$	1.03	$3.4251-8$	$3.9150-8$	1.14
160	$5.2966-8$	$5.3678-8$	1.01	$4.2816-9$	$4.6046-9$	1.08
320	$6.6214-9$	$6.6650-9$	1.01	$5.3521-10$	$5.5598-10$	1.04

Table 1. Global h-performance for PEP and CDRP, $p = 3$.

6.3. Global p-performance. We test the error estimate (4.10) with respect to the changing polynomial degree p. We assume equidistant mesh with $h = 1/10$ and $p = 1, \dots, 7$.

We can observe from Table 2 that two regimes for odd and even polynomial degrees appear. For both regimes the efficiency indices very mildly (sublinearly) increase with increasing p.

		PEP			CDRP	
p	$\mathrm{Err}_h^+(u_h)$	η	Eff	$\mathrm{Err}_h^+(u_h)$	η	Eff
1	$2.0113-1$	$2.4015-1$	1.19	$3.0604-2$	$4.9461-2$	1.62
2	$8.1594-3$	$1.4489-2$	1.78	$8.3845-4$	$1.4924-3$	1.78
3	$2.1669-4$	$2.6883-4$	1.24	$1.7478-5$	$2.9540-5$	1.69
4	$4.2891-6$	$9.6339-6$	2.25	$2.6469-7$	$5.9576-7$	2.25
5	$6.7722-8$	$8.7754-8$	1.30	$3.2125-9$	$5.9543-9$	1.85
6	$8.8966-10$	$2.3607-9$	2.65	$3.2419-11$	$8.6252-11$	2.66
7	$9.9930-12$	$1.3472-11$	1.35	$3.4397-13$	$5.6761-13$	1.65

Table 2. Global p-performance for PEP and CDRP, $h = 1/10$.

6.4. Local efficiency, h-performance. We test the robustness of efficiency estimates (5.26) with respect to decreasing h. The polynomial degree is set as $p = 3$. We assume a sequence of successively refined equidistant meshes started with $h = 1/10$ and halved in each step. For each mesh we take element $K = [0.4, 0.4 + h]$ and we investigate local efficiency on this element.

We can see that the efficiency indices in Table 3 and Table 4 are uniformly bounded for decreasing h.

$1/h$	$\mathrm{Err}^{+}_{h,\omega_K}(u_h)$	$\eta_{R,K}$	$\mathrm{Eff}_{R,K}$	$\eta_{F,K}$	$\mathrm{Eff}_{F,K}$
10	$1.6053-4$	$6.3399-6$	0.04	$9.3672-5$	0.58
20	$1.4327-5$	$3.4539-7$	0.02	$8.2921-6$	0.58
40	$1.2611-6$	$1.7357-8$	0.01	$7.2852-7$	0.58
80	$1.1099-7$	$8.1726-10$	0.01	$6.4090-8$	0.58
160	$9.7842-9$	$3.7263-11$	0.00	$5.6491-9$	0.58
320	$8.6359-10$	$1.6483-12$	0.00	$4.9857-10$	0.58

Table 3. Local h-performance for PEP, $p=3$, $K=[0.4, 0.4+h]$.

$1/h$	$\mathrm{Err}^{+}_{h,\omega_K}(u_h)$	$\eta_{R,K}$	$\mathrm{Eff}_{R,K}$	$\eta_{F,K}$	$\mathrm{Eff}_{F,K}$
10	$5.8645-6$	$9.6390-7$	0.16	$3.3251-6$	0.57
20	$4.7421-7$	$3.9647-8$	0.08	$2.7252-7$	0.58
40	$4.0379-8$	$1.6952-9$	0.04	$2.3286-8$	0.58
80	$3.5096-9$	$7.3741-11$	0.02	$2.0257-9$	0.58
160	$3.0776-10$	$3.2330-12$	0.01	$1.7767-10$	0.58
320	$2.7096-11$	$1.4142-13$	0.01	$1.5645-11$	0.58

Table 4. Local h-performance for CDRP, $p=3$, $K=[0.4, 0.4+h]$.

6.5. Local efficiency, p-performance. We test the robustness of efficiency estimates (5.26) with respect to the changing polynomial degree p. We assume equidistant mesh with $h=1/10$ and $p=1,\ldots,7$. Similarly as in the previous tests, we take $K=[0.4, 0.5]$ and we investigate local efficiency on this element.

We can observe again in Table 5 and Table 6 two regimes for odd and even polynomial degrees, where the dominating estimator is $\eta_{F,K}$ for odd degrees and $\eta_{R,K}$ for even degrees. The efficiency indices stagnates or very mildly (sublinearly) increase with increasing p.

p	$\mathrm{Err}^{+}_{h,\omega_K}(u_h)$	$\eta_{R,K}$	$\mathrm{Eff}_{R,K}$	$\eta_{F,K}$	$\mathrm{Eff}_{F,K}$
1	$1.4891-1$	$4.5229-3$	0.03	$8.7188-2$	0.59
2	$1.8492-3$	$1.6367-3$	0.89	$3.7698-4$	0.20
3	$1.6053-4$	$6.3399-6$	0.04	$9.3672-5$	0.58
4	$9.6990-7$	$1.0032-6$	1.03	$2.3560-7$	0.24
5	$5.0174-8$	$2.4500-9$	0.05	$2.9248-8$	0.58
6	$2.0106-10$	$2.3903-10$	1.19	$5.2231-11$	0.26
7	$7.4029-12$	$4.2582-13$	0.06	$4.3191-12$	0.58

Table 5. Local p-performance for PEP, $h=1/10$, $K=[0.4, 0.5]$.

p	$\mathrm{Err}^{+}_{h,\omega_K}(u_h)$	$\eta_{R,K}$	$\mathrm{Eff}_{R,K}$	$\eta_{F,K}$	$\mathrm{Eff}_{F,K}$
1	1.1293 − 2	1.6445 − 3	0.15	6.4324 − 3	0.57
2	2.7437 − 4	2.6495 − 4	0.97	1.5933 − 5	0.06
3	5.8645 − 6	9.6390 − 7	0.16	3.3251 − 6	0.57
4	8.8444 − 8	1.0897 − 7	1.23	5.3687 − 9	0.06
5	1.0742 − 9	2.0228 − 10	0.19	6.0930 − 10	0.57
6	1.0838 − 11	1.5873 − 11	1.46	6.8052 − 13	0.06
7	1.1321 − 13	2.4244 − 14	0.21	5.4811 − 14	0.48

Table 6. Local p-performance for CDRP, $h = 1/10$, $K = [0.4, 0.5]$.

7. Conclusion

We derived a posteriori upper bound for not necessarily self-adjoint elliptic problems based on the cheap direct evaluation. We showed that this reconstruction is efficient up to $p^{1/2}$ for one-dimensional problems, where p is the underlying polynomial degree given by the finite element approximation. The robustness with respect to the mesh-size h and to the polynomial degree p was verified by numerical experiments.

Since the majority of the techniques applied in the efficiency proofs in this paper are extendable to multi-dimensional problems, the author hopes that the proof of the efficiency up to $p^{1/2}$ of this direct reconstruction for multi-dimensional problems will be possible in the future.

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