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# ON HYBRID CONSENSUS-BASED EXTENDED KALMAN FILTERING WITH RANDOM LINK FAILURES OVER SENSOR NETWORKS

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This paper is concerned with the distributed filtering problem for nonlinear time-varying systems over wireless sensor networks under random link failures. To achieve consensus estimation, each sensor node is allowed to communicate with its neighboring nodes according to a prescribed communication topology. Firstly, a new hybrid consensus-based filtering algorithm under random link failures, which affect the information exchange between sensors and are modeled by a set of independent Bernoulli processes, is designed via redefining the interaction weights. Second, a novel observability condition, called parameterized jointly uniform observability, is proposed to ensure the stochastic boundedness of the error covariances of the hybrid consensus-based filtering algorithm. Finally, an example is given to demonstrate the effectiveness of the derived theoretical results.

*Keywords:* extended Kalman filter, hybrid consensus filter, sensor network, distributed state estimation, random link failure

## 1. INTRODUCTION

In the past few decades, research on sensor networks has drawn extensive attention in the control and signal processing societies, owing to their widespread applications in military, aerospace, environmental, and medical fields, see, e. g., [44, 42, 15, 41, 43]. A sensor network is referred to a network composed of a large number of stationary or mobile sensor nodes in a self-organizing and multi-hop way. Each sensor node possesses the capabilities of sensing, collecting, processing and transmitting information, all of which together complete certain tasks in a collaborative manner ultimately.

It is known that the filtering (or state estimation) techniques have been widely applied to a wide range of fields including wireless camera networks [19], localization and mapping [36], and attack detection [16]. Depending on whether there is a fusion center or not, the problem of filtering can be classified into two categories: centralized filtering and distributed filtering. When the fusion center exists, this problem can be regarded as a centralized data fusion that aims to obtain improved estimation through the combination of multiple measurements [38]. In this way, it is possible to obtain the global optimal value with a high estimation accuracy. However, the large amount of data processed by

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the fusion center is likely to cause huge energy consumption and the failure in the fusion center will lead to the defects of the whole system, consequently limiting its application scopes in practical systems. On the other hand, the method of distributed filtering has recently received extensive research attention, see, e. g., [24, 35, 13, 39, 9]. To be more specific, in terms of the distributed filtering algorithm, each estimator only makes use of local information and the messages from neighboring sensors to generate local estimates. Such a distributed framework is highly preferable when resource constraints become a problem and/or the global knowledge of the network is no longer available for designing the distributed estimators.

Several information processing techniques have been reported to deal with the problem of the distributed filtering in sensor networks, see, e. g., [16, 24, 35, 13, 39, 9, 14, 8, 26, 22, 7, 6]. In general, the distributed filtering algorithm can be divided into two steps. The first step is that each estimator makes use of the observed information of every intelligent sensor to generate initial local estimation. The second step is that each sensor node exchanges information with the neighbor nodes according to certain communication topology to produce the final local estimation. As for sensor networks, in order to fuse limited information from scattered nodes for the purpose of improving the estimation accuracy, a suitable multi-sensor fusion algorithm has become critical. Additionally, the requirement of scalability, lack of fusion centers, and robustness of sensor networks call upon suitable consensus approaches [14, 28, 29, 12, 9, 34], aiming to iteratively fuse local estimates and further achieve a common fusion estimate. Arguably, there are two widely adopted consensus-based strategies for distributed filtering algorithms: consensus on measurements (CM) and consensus on information (CI). In terms of CM [27], the developed distributed filtering algorithms aim to achieve consensus among local measurement information. More specifically, local innovation pairs in a distributed manner so as to achieve updated values of the centralized filter. It is worth noting that to achieve the stability of the CM-based filter, it is of great necessity to execute a sufficient large number of consensus steps for each sampling period. Besides, such a consensus strategy relies on the assumption that the measurement errors coming from different sensors are mutually independent, and this strategy is limited to the Kalman-like filters. In the past few decades, the CM-based strategy has been widely used in the field of signal processing and control, see, e. g., [27, 17].

Note that only one or few consensus iterations per time can be afforded in order to reduce the communication overhead for higher energy efficiency in particular wireless sensor network environment. Furthermore, there may not be enough time to wait for CM to convergence [2]. As a result, the so-called CI strategy [1] comes into play. Concerning the characteristic of the distributed filtering algorithm, CI conducts the local average of the inverse variance (information) matrices and information vectors, aiming to guarantee the stability under any number of consensus steps (even a single step). For example, such a CI-based distributed filtering problem was studied in [4]. Later, it was rigorously mathematically processed in [1], where CI was interpreted as a consensus on the probability density function in the Kullback-Leibler average sense. However, it should be pointed out that the employed fusion rules are conservative in the sense that the correlations between local estimates are supposed to be completely unknown. The authors in [3] jointly considered the characteristics of the respective algorithms of CI and CM, and thus presented a new class of hybrid CMCI (HCMCI) filters.

It is well recognized that in practical applications temporary failures often occur to sensor communication links due to various factors such as signal attenuation, multipath fading, background noise, external blocking and so on [30, 40]. Therefore, significant efforts have been made in order to address how link failures affect filter performance. For example, in [25], the filter performance under communication link failures was studied. The convergence problem of the infinite product of random matrices has also been well explored. As mentioned in [29], it is convergent in the infinite product of primitive random matrices. Unfortunately, when it comes to distributed filtering of time-varying systems which experience random link failures over sensor networks, there are relatively few results due to lack of adequate analysis methods.

Motivated by the discussion above, in this paper, the distributed filtering problem over sensor networks in the presence of random link failures is studied. Since nonlinearity is not uncommon in real-world dynamical systems [10, 11], the plant to be monitored over the sensor network is described by a nonlinear time-varying stochastic system. It has been shown in the literature that the extended Kalman filter (EKF) serves as an effective estimation method to address the problem of filtering for nonlinear systems, see, e. g., [32, 33]. In [3], a two-stage hybrid consensus Kalman filter algorithm was put forward, including an update based on EKF and a consensus update of information matrices/vectors as well as the innovation matrices/vectors. Different from [3], in the process of the proposed consensus algorithm, we consider that the communication link failures occur randomly in terms of independent non-uniform probability.

**Notation.** The notations used throughout the paper are fairly standard unless otherwise stated. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the $n$ dimensional Euclidean space and the set of all $n \times m$ real matrices. $I$ denotes the identity matrix of compatible dimension. $\mathbf{1}$ is a column vector with all entries equal to one. The notation $X \geq Y$ (respectively, $X > Y$) where $X$ and $Y$ are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). $M^T$ represents the transpose of matrix $M$. $||P||$ describes the Euclidean norm of a matrix $P$. $\mathbb{M}_+^n$ denotes the set of all real $n \times n$ positive definite matrices. The primitive matrix means that if matrix $A$ is non-negative and there is a positive integer $m$ such that each element of its $m$th power is positive, then matrix $A$ is the primitive matrix.

## 2. STATEMENT OF THE PROBLEM

The communication topology of the sensor network is modeled as an undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where $\mathcal{N} = \{1, 2, 3, \cdots, N\}$ is the set of sensor nodes. $N$ is the number of all sensor nodes. $\mathcal{E}$ represents the set of connections between nodes. An edge $(i, j) \in \mathcal{E}$ indicates that node $j$ can receive information from node $i$. Furthermore, for each node $i \in \mathcal{N}$, $\mathcal{N}_i = \{j|(j, i) \in \mathcal{E}\}$ represents the set of its neighbors.

### 2.1. System model and extended Kalman filtering

In this paper, we consider a nonlinear time-varying stochastic system described by the following state-space model:

$$x_k = f_{k-1}(x_{k-1}) + w_{k-1}, \tag{1}$$

and the measurement equation of the $i$th sensor is described by

$$z_k^i = h_k^i(x_k) + v_k^i, \quad i = 1, 2, \ldots, N, \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the system state vector, $w_{k-1} \in \mathbb{R}^n$ and $v_k^i \in \mathbb{R}^m$ are mutually independent white Gaussian random variables with zero mean values and bounded covariances $Q_{k-1} > 0$ and $R_k^i > 0$. $f_k(\cdot)$ and $h_k^i(\cdot)$ are known nonlinear functions which are twice continuously differentiable, and they are bounded for any $x_k$.

The local state estimation for the system (1) – (2) can be generated by EKF, which consists of the prediction process

$$\begin{aligned} \hat{x}_{k|k-1}^i &= f_{k-1}(\hat{x}_{k-1}^i), \\ P_{k|k-1}^i &= A_{k-1}^i P_{k-1}^i (A_{k-1}^i)^{\mathrm{T}} + Q_{k-1} \end{aligned} \tag{3}$$

and the local update process

$$\begin{aligned} \hat{x}_k^i &= \hat{x}_{k|k-1}^i + K_k^i (z_k^i - h_k^i(\hat{x}_{k|k-1}^i)), \\ P_k^i &= (I_n - K_k^i C_k^i) P_{k|k-1}^i \\ &= [(P_{k|k-1}^i)^{-1} + (C_k^i)^T (R_k^i)^{-1} C_k^i]^{-1}, \end{aligned} \tag{4}$$

where $\hat{x}_{k|k-1}^i$ and $\hat{x}_k^i$ represents the one-step prediction and the update estimation of the plant state, respectively, and their corresponding error covariances are $P_{k|k-1}^i$ and $P_k^i$. $A_{k-1}^i$ and $C_k^i$ are given by

$$A_{k-1}^i = \frac{\partial f_{k-1}}{\partial x_{k-1}} |_{x_{k-1} = \hat{x}_{k-1}^i} \tag{5}$$

and

$$C_k^i = \frac{\partial h_k^i}{x_k} |_{x_k = \hat{x}_{k|k-1}^i}. \tag{6}$$

In addition, the Kalman gain matrix is given as follows:

$$K_k^i = P_{k|k-1}^i (C_k^i)^T [C_k^i P_{k|k-1}^i (C_k^i)^T + R_k^i]^{-1}. \tag{7}$$

From the above analysis, we can get the following equations:

$$(P_k^i)^{-1} = (P_{k|k-1}^i)^{-1} + (C_k^i)^T (R_k^i)^{-1} C_k^i, \tag{8}$$

$$(P_k^i)^{-1} \hat{x}_k^i = (P_{k|k-1}^i)^{-1} \hat{x}_{k|k-1}^i + (C_k^i)^T (R_k^i)^{-1} \bar{z}_k^i, \tag{9}$$

where $\bar{z}_k^i = z_k^i - h_k^i(\hat{x}_{k|k-1}^i) + C_k^i \hat{x}_{k|k-1}^i$ is known as the virtual measurement, $(C_k^i)^T (R_k^i)^{-1} C_k^i$ and $(C_k^i)^T (R_k^i)^{-1} \bar{z}_k^i$ are usually called the innovation matrix and the innovation vector.

### 2.2. CI and CM algorithms

The local estimation method is given in the above subsection. In this subsection, we will briefly introduce two existing solutions to solve the distributed state estimation problem, i. e., the well-known CI and CM algorithms.

This paper adopts an information form of CI algorithm, under which the information needs to be exchanged via a predetermined communication topology in order to realize the consensus filtering. Specifically, denote the information matrices

$$\Omega^i_{ci,k|k-1} \triangleq (P^i_{k|k-1})^{-1}, \tag{10}$$

$$\Omega^i_{ci,k}(0) \triangleq (P^i_k)^{-1}, \tag{11}$$

and the information vectors

$$q^i_{ci,k|k-1} \triangleq (P^i_{k|k-1})^{-1}\hat{x}^i_{k|k-1}, \tag{12}$$

$$q^i_{ci,k}(0) \triangleq (P^i_k)^{-1}\hat{x}^i_k. \tag{13}$$

It should be pointed out that $(q^i_{ci,k}(0), \Omega^i_{ci,k}(0))$ $(i \in \mathcal{N})$ is the initial information pair at time-step $k$ which will be broadcast to sensor $i$'s neighbors. Furthermore, the number of consensus steps is denoted as $L$ and the CI algorithm is essentially to implement the following iterations

$$q^i_{ci,k}(l+1) = \sum\nolimits_{j\in\mathcal{N}_i} \pi_{i,j} q^j_{ci,k}(l), \tag{14}$$

$$\Omega^i_{ci,k}(l+1) = \sum\nolimits_{j\in\mathcal{N}_i} \pi_{i,j} \Omega^j_{ci,k}(l), \tag{15}$$

where $l = 0, 1, \dots, L-1$.

**Remark 2.1.** For each consensus iteration, each node $i$ calculates the regional mean, i. e., the combination of the values from its neighboring $\mathcal{N}_i$ with the appropriate consensus weight $\pi_{i,j}$ $(j \in \mathcal{N}_i)$. In this paper, we use linear convex combination by supposing $\pi_{i,j} \geq 0$ and $\sum_{j\in\mathcal{N}_i} \pi_{i,j} = 1$, $\forall i \in \mathcal{N}$. Furthermore, the CI algorithm is degenerated into the so-called covariance intersection [18] under a single consensus step. In other words, such an algorithm is the generalization of the covariance intersection to multiple consensus steps. It is worth noting that the larger the step size $L$, the greater the computational and communication burden although the consensus performance is improved. Therefore, $L$ should be chosen appropriately as a trade-off between the calculation and communication cost, and the consensus performance.

The mechanism of CM algorithm is to exploit both the innovation vector and the innovation matrix

$$q^i_{cm,k} = (C^i_k)^T (R^i_k)^{-1} \bar{z}^i_k, \tag{16}$$

$$\Omega^i_{cm,k} = (C^i_k)^T (R^i_k)^{-1} C^i_k. \tag{17}$$

At time $k$, for each node $i \in \mathcal{N}$, $q^i_{cm,k} = q^i_{cm,k}(0)$, $\Omega^i_{cm,k} = \Omega^i_{cm,k}(0)$, consensus steps can be described as follows

$$q^i_{cm,k}(l+1) = \sum\nolimits_{j\in\mathcal{N}_i} \pi_{i,j} q^j_{cm,k}(l), \tag{18}$$

$$\Omega_{cm,k}^{i}(l+1) = \sum\nolimits_{j\in\mathcal{N}_i} \pi_{i,j}\Omega_{cm,k}^{j}(l). \tag{19}$$

where $l = 0, 1, \ldots, L-1$.

### 2.3. The objective of this paper

Due to the limited bandwidth, inter-sensor communication could be subject to random link failures, which inevitably affect the consensus performance and the convergence of consensus algorithms. For the convenience of later analysis, we redefine the interaction weights as follows:

$$\tilde{\pi}_{i,j}^{k,l} = \begin{cases} \gamma_{i,j}^{k,l}\pi_{i,j}, & i \neq j \\ \pi_{i,i}, & i = j \end{cases} \tag{20}$$

where the stochastic variables $\gamma_{i,j}^{k,l}(k \geq 0, l \geq 1, j, i \in \mathcal{N})$ follow the well-known Bernoulli distribution with the probabilities

$$\text{Prob}\{\gamma_{i,j}^{k,l} = 1\} = \bar{\gamma}_{i,j}, \quad \text{Prob}\{\gamma_{i,j}^{k,l} = 0\} = 1 - \bar{\gamma}_{i,j} \tag{21}$$

where $0 < \bar{\gamma}_{i,j} \leq 1$ are known scalars. For $j \neq i$, $k \geq 0$ and $0 \leq l \leq L-1$, the communication between nodes $i$ and $j$ is successful if $\gamma_{i,j}^{k,l} = \gamma_{j,i}^{k,l} = 1$, otherwise it fails. Furthermore, the nodes access to the local information regardless of the network conditions for the case $i = j$. It is assumed that the stochastic variables $\gamma_{i,j}^{k,l}$ $(i \neq j)$ are independent with the process noise $w_k$, measurement noise $v_k$ and the initial state $x_0$.

**Remark 2.2.** The stochastic variables $\gamma_{i,j}^{k,l}(k \geq 0, l \geq 1, j, i \in \mathcal{N})$ follow the well-known Bernoulli distribution and are not known. They are randomly generated sequences based on link failure probabilities.

It is worth mentioning that CI and CM algorithms have their own advantages and disadvantages. On the one hand, the boundedness of the estimation error can be guaranteed for the CI algorithm with any $L$ consensus steps, but for CM requiring $L$ to be large enough. On the other hand, the performance of the CM algorithm is close to the one of the centralized filter when $L$ is large enough but CI is not. In order to prolong the life of sensor networks in the actual situation, only a limited number of consensus steps can be provided to reduce both the communication and calculation burden. As such, it is very interesting to develop a new algorithm to achieve a trade-off between the communication and calculation burden and the consensus performance while thoroughly addressing the impact from random link failures. Therefore, the purpose of this paper is to

R1) design an improved consensus-based filtering algorithm so as to achieve a trade-off between the communication and calculation burden and the consensus performance;

R2) disclose the condition to guarantee the boundedness of the error covariance of developed algorithm under random link failures.

## 3. MAIN RESULTS

Random link failures over sensor networks make it difficult to design the desired distributed filters and analyze the convergence of the consensus filtering algorithm. Since sensor network link failures occur randomly, a suitable model needs to be found to characterize it. (20) has given a portrayal model of link failures, which facilitates the design of incomplete information hybrid consensus filtering. The observability condition is an important condition for guaranteeing the boundedness of the corresponding error covariance. The inevitable requirement in distributed extended Kalman filtering requires linearization of nonlinear functions to produce multiple system matrices and measurement matrices, which poses challenges for designing observability conditions.

### 3.1. Hybrid consensus-based EKF

In this subsection, a hybrid consensus-based approach, named as the hybrid consensus-based EKF, is designed via the complementary advantages of both CM and CI algorithms. Taking the random link failures into consideration, we first introduce a modified consensus matrix. Specifically, denote the traditional consensus matrix as $\mathcal{M}$, whose elements are the consensus weights $\pi_{i,j}$ $(i,j \in \mathcal{N})$. When link failures occur, the element of such a matrix is changed and we denote the practical consensus matrix at step $l$ and time $k$ as $\tilde{\mathcal{M}}_{k,l}$, whose elements are the consensus weights $\tilde{\pi}^{k,l}_{i,j}$ defined in (20).

Now, the pseudocode of the hybrid consensus-based EKF with random link failures is shown in Table 1.

| | |
|---|---|
| Step 1. | For $i \in \mathcal{N}$, $i$th sensor utilizes the EKF (3) and (4) to compute the local estimate. |
| Step 2. | For $l = 0, 1, \cdots, L-1$, perform the consensus of $L$ steps as (14) - (15) and (18) - (19) |
| Step 3. | Correction: $$\Omega^i_k = \sum_{j\in\mathcal{N}} \tilde{\pi}^{k,L}_{i,j} \Omega^j_{ci,k\vert k-1} + \vert\mathcal{N}\vert \sum_{j\in\mathcal{N}} \tilde{\pi}^{k,L}_{i,j} \Omega^j_{cm,k}$$ $$q^i_k = \sum_{j\in\mathcal{N}} \tilde{\pi}^{k,L}_{i,j} q^j_{ci,k\vert k-1} + \vert\mathcal{N}\vert \sum_{j\in\mathcal{N}} \tilde{\pi}^{k,L}_{i,j} q^j_{cm,k}$$ where $\vert\mathcal{N}\vert$ is the cardinality of the set $\mathcal{N}$. |
| Step 4. | Compute $P^i_k = (\Omega^i_k)^{-1}$ and $x^i_k = P^i_k q^i_k$. |
| Step 5. | Set $k = k+1$, prediction step described as $\hat{x}^i_{k+1\vert k} = f_k(\hat{x}^i_k), \quad A^i_k = \frac{\partial f_k}{x_k}\vert_{\hat{x}^i_k}$, $P^i_{k+1\vert k} = A^i_k P^i_k (A^i_k)^{\mathrm{T}} + Q_k$. |

**Tab. 1.** Hybrid consensus-based EKF algorithm.

**Remark 3.1.** In Step 2, each node $i \in \mathcal{N}$ needs to broadcast messages $(q^i_{cm,k}, q^i_{ci,k|k-1})$ and $(\Omega^i_{cm,k}, \Omega^i_{ci,k|k-1})$ to its neighbor nodes, where the information on predicted covariance matrices is employed in the final consensus step (i. e. Step 3). Obviously, using $q^i_{ci,k|k-1}$ and $\Omega^i_{ci,k|k-1}$ as information to implement consensus steps is different from the traditional CI algorithm (i. e. (14) and (15)).

**Remark 3.2.** In this paper, $|\mathcal{N}|$ represents the cardinality of the set $\mathcal{N}$ which makes hybrid consensus-based EKF algorithm converging to the centralized EKF. Different from [3], $|\mathcal{N}|$ is a time-invariant parameter.

Next, we will discuss the boundedness of the error covariance under random link failures in the cases of the finite consensus step and the infinite consensus step, respectively.

### 3.2. Boundedness analysis

In this subsection, we aim to analyze the stochastic boundedness of error covariance matrices of hybrid consensus-based filtering scheme with random link failures, which include two cases, i. e. the finite and infinite consensus steps.

In order to analyze the stability of the hybrid consensus-based EKF algorithm, inspired by [45, 37, 5], we propose a definition of parameterized jointly uniform observability, and give some assumptions and lemmas.

**Assumption 3.3.** The system is reconfigurable, i. e., suppose $A^i_k$ is a bounded nonsingular matrix, there exist positive real numbers $\chi_1$ and $\chi_2$ so that for all $k \geq m$ and for some finite $m > 0$ we have

$$\chi_1 I_n \leq O^T_\epsilon(k-m,k)\mathfrak{R}(k-m,k)O_\epsilon(k-m,k) \leq \chi_2 I_n$$

with

$$O_\epsilon(k-m,k) = \begin{bmatrix} C^i_{k-m}(A^i_{k-m})^{-1}(A^i_{k-m+1})^{-1}\cdots(A^i_{k-1})^{-1} \\ C^i_{k-m+1}(A^i_{k-m+1})^{-1}(A^i_{k-m+2})^{-1}\cdots(A^i_{k-1})^{-1} \\ \cdot \\ \cdot \\ C^i_k \end{bmatrix}$$

$$\mathfrak{R}(k-m,k) = \mathrm{Diag}((R^i_{k-m})^{-1},\cdots,(R^i_k)^{-1})$$

where $A^i_k$ and $C^i_k$ are defined by (5)and (6), respectively, then the system (and its associated EKF for $\hat{x}^i_{k|k-1}$ and $\hat{x}^i_k$ sufficiently close to the true state $x_k$) is said to be reconstructible.

**Definition 3.4.** (Parameterized Jointly Uniform Observability) The system (1) with measurements (2) is said to be parameterized jointly uniformly observability, if there are parameters $m \geq 1$, $0 < \omega < 1$ and constants $\underline{\lambda}$ and $\bar{\lambda}$ satisfing $0 < \underline{\lambda} \leq \bar{\lambda} < \infty$ such that the Grammian matrix satisfies

$$\underline{\lambda} I_n \leq \sum_{t=k-m}^{k} \mathcal{O}(t,k) \leq \bar{\lambda} I_n$$

where

$$\mathcal{O}(k,k) = |\mathcal{N}| \sum_{j\in\mathcal{N}} \pi_{i,j}^L (C_k^j)^T (R_k^j)^{-1} C_k^j$$

$$\mathcal{O}(k-1,k) = |\mathcal{N}|\omega \sum_{j\in\mathcal{N}} \sum_{s\in\mathcal{N}} \pi_{i,j}^L \pi_{j,s}^L (A_{k-1}^j)^{-T} \times (C_k^s)^T (R_{k-1}^s)^{-1} C_k^s (A_{k-1}^j)^{-1}$$

$$\vdots$$

$$\mathcal{O}(k-m,k) = |\mathcal{N}|\omega^m \underbrace{\sum_{j\in\mathcal{N}} \sum_{s\in\mathcal{N}} \cdots \sum_{t\in\mathcal{N}}}_{m+1} \pi_{i,j}^L \pi_{j,s}^L \cdots \pi_{u,t}^L (A_{k-1}^j)^{-T} \times (A_{k-2}^s)^{-T} \cdots (A_{k-m}^t)^{-T} (C_{k-m}^t)^T (R_{k-m}^t)^{-1} \times C_{k-m}^t (A_{k-m}^t)^{-1} \cdots (A_{k-2}^s)^{-1} (A_{k-1}^j)^{-1}.$$

**Remark 3.5.** To perform EKF stability analysis, it is shown in [33] that the uniform observability condition is a necessary condition to guarantee the boundedness of the corresponding error covariance, and the relationship between the nonlinear observability matrix and the observability Gramian matrix has also been given. The proposed distributed EKF in this paper needs to design new observability conditions. According to Assumption 3.3, there are constants $\underline{\lambda}$ and $\bar{\lambda}$ that satisfy the definition of the parameterized jointly uniform observability.

**Remark 3.6.** Different from existing uniform observability, scalars $\omega$ and $|\mathcal{N}|$ are introduced into Grammian matrix to get a tighter bound of the error covariance, where $\omega$ can be any number in interval $(0, 1)$. In this paper, $\omega$ is determined by the Proposition 3.14.

**Assumption 3.7.** For each node $i \in \mathcal{N}$, there exist real numbers $\underline{f}$, $\bar{f}$, $\underline{h}$, $\bar{h} \neq 0$ and positive real constants $\underline{q}$, $\overline{q}$, $\underline{r}$, $\overline{r} > 0$, such that the following bounds on various matrices are fulfilled for every $k \geq 0$

$$\underline{f}^2 I_n \leq A_k^i (A_k^i)^T \leq \overline{f}^2 I_n \tag{22}$$

$$\underline{h}^2 I_m \leq C_k^i (C_k^i)^T \leq \overline{h}^2 I_m \tag{23}$$

$$\underline{q} I_n \leq Q_k \leq \overline{q} I_n \tag{24}$$

$$\underline{r} I_m \leq R_k^i \leq \overline{r} I_m. \tag{25}$$

**Remark 3.8.** Due to the nonlinear function $f_k(\cdot)$ of this paper is bounded for any $x_k$, it is reasonable to assume that the spectral norm of $A_k^i$ is bounded. Note that such a bounded assumption is conservative, but it is nevertheless serving as an effective way to derive bounds for the error covariance matrices produced by EKF [20, 5, 31].

**Assumption 3.9.** The initial error covariances $P_0^i$ $(i \in \mathcal{N})$ are positive semi-definite.

**Lemma 3.10.** For $A$, $B \in \mathbb{R}^{n\times n}$, if $A > 0$ and $B \geq 0$, then

$$A^{-1} \geq (A+B)^{-1}. \tag{26}$$

**Lemma 3.11.** (Li et al. [21]) Consider the local algorithm (4) and Assumption 3.7. If there is a positive real number $\underline{p}$ such that, for all $k > 0$ and $i \in \mathcal{N}$, the error covariance $P_k^i$ satisfies $P_k^i \geq \underline{p} I_n$, then there is always a positive real number $\alpha$ satisfying $0 < \alpha < 1$ such that

$$(P_{k+1|k}^i)^{-1} \geq \alpha (A_k^i)^{-T} (P_k^i)^{-1} (A_k^i)^{-1} \tag{27}$$

is true.

**Lemma 3.12.** (Olfati-Saber et al. [29]) Let $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ be a finite set of primitive stochastic matrices. For any sequence of matrices $S_{p_1}, S_{p_2}, \ldots, S_{p_k} \in \mathcal{S}$ with $k \geq 1$, the product $S_{p_k} \cdots S_{p_2} S_{p_1}$ is also a primitive matrix. Furthermore, there exists a row vector such that

$$\lim_{k \to \infty} \{S_{p_k} \cdots S_{p_2} S_{p_1}\} = \mathbf{1} d.$$

Many stability analysis criteria of distributed Kalman filter are limited to time-invariant systems, while systems in real-world applications are usually time-varying. Therefore, it is necessary to establish a systematic method to study the stability of distributed Kalman filter for general time-varying systems.

Let $\pi_{i,j}^l$ be the $(i,j)$th element of $\mathcal{M}^l$, i. e., the $l$th power of the consensus matrix $\mathcal{M}$.

**Remark 3.13.** The matrix $P_k^i$ obtained in the proposed algorithm is not the actual error covariance matrix. The actual estimation error covariance after multi-step consensus update is $E[(x_k - \hat{x}_k^i)(x_k - \hat{x}_k^i)']$. The hybrid consensus-based EKF algorithm subject to stochastic link failures yields a consistent fused estimate. The phrase "consistent" here indicates that the actual covariance is always bounded by the estimated covariance i. e., $E[(x_k - \hat{x}_k^i)(x_k - \hat{x}_k^i)'] \leq P_k^i$, $\forall i = 1, \cdots, N$. By imitating the proof of Theorem 3 in [1], consistent proof can be obtained. We can see that, for arbitrary cross correlations, the proposed hybrid consensus-based EKF algorithm still yields the estimate consistency, which is essential for establishing the conditions for stochastic boundedness. In the following bounded analysis, we only need to prove that $P_k^i$ is bounded.

**Case 1: Finite number of consensus steps**

In this case, the finite number of consensus steps is considered. Before analyzing the stochastic boundedness of the error covariance under the random link failures, the uniform boundedness of the error covariance matrix $P_k^i$ without link failures can be shown in the following proposition.

**Proposition 3.14.** Under Definition 3.4 and Assumption 3.7, there exists an instant $m$ satisfying $m \leq k$, such that when no communication failure occurs during the interval $[k-m, k]$, then the error covariance matrix $P_k^i$ $(i \in \mathcal{N})$ given by Table 1 is uniformly bounded. Specifically, there exist positive scalars $\underline{p}$ and $\bar{p}$ such that

$$\underline{p} I_n \leq P_k^i \leq \bar{p} I_n, \forall P_k^1, P_k^2, \cdots, P_k^N \in \mathbb{M}_+^n \tag{28}$$

where

$$\begin{aligned} \underline{p} &= [(1/\underline{q}) + (|\mathcal{N}|\overline{h}^2)/\underline{r}]^{-1} \\ \overline{p} &= (\underline{\lambda})^{-1}. \end{aligned}$$

Proof. We first prove that there is a lower bound, and then give the proof of the upper bound.

*A Lower Bound:* According to Step 4 of the hybrid consensus-based EKF algorithm, we can get the following equation

$$(P_k^i)^{-1} = \sum_{j\in\mathcal{N}} \pi_{i,j}^L (P_{k|k-1}^j)^{-1} + |\mathcal{N}| \sum_{j\in\mathcal{N}} \pi_{i,j}^L (C_k^j)^T (R_k^j)^{-1} C_k^j. \tag{29}$$

According to Lemma 3.10, the following inequality is obtained

$$(Q_{k-1})^{-1} \geq (A_{k-1}^j P_k^i (A_{k-1}^j)^{\mathrm{T}} + Q_{k-1})^{-1} = (P_{k|k-1}^j)^{-1}. \tag{30}$$

Then, due to Assumption 3.7, (29) and (30), we have

$$(P_k^i)^{-1} \leq (Q_{k-1})^{-1} + |\mathcal{N}| \sum_{j\in\mathcal{N}} \pi_{i,j}^L (C_k^j)^T (R_k^j)^{-1} C_k^j \tag{31}$$

$$\leq (\frac{1}{\underline{q}} + |\mathcal{N}|\frac{\overline{h}^2}{\underline{r}}) I_n. \tag{32}$$

Finally, from above analysis we can conclude

$$P_k^i \geq (\frac{1}{\underline{q}} + |\mathcal{N}|\frac{\overline{h}^2}{\underline{r}})^{-1} I_n \triangleq \underline{p} I_n. \tag{33}$$

*A Upper Bound:* Apply Lemma 3.11 to rewrite (33) as

$$\begin{aligned} &(P_k^i)^{-1} \\ &\geq \alpha \sum_{j\in\mathcal{N}} \pi_{i,j}^L (A_{k-1}^j)^{-T} (P_{k-1}^j)^{-1} (A_{k-1}^j)^{-1} \\ &\quad + |\mathcal{N}| \sum_{j\in\mathcal{N}} \pi_{i,j}^L (C_k^j)^T (R_k^j)^{-1} C_k^j. \end{aligned} \tag{34}$$

Once again, we utilize the Lemma 3.11 with respect to the above inequality and have

$$\begin{aligned} &(P_k^i)^{-1} \\ &\geq \alpha[\sum_{j\in\mathcal{N}} \pi_{i,j}^L (A_{k-1}^j)^{-T} (\alpha[\sum_{s\in\mathcal{N}} \pi_{j,s}^L (A_{k-2}^s)^{-T} (P_{k-2}^s)^{-1} \\ &\quad \times (A_{k-2}^s)^{-1}] + |\mathcal{N}| \sum_{s\in\mathcal{N}} \pi_{j,s}^L (C_{k-1}^s)^T (R_{k-1}^s)^{-1} C_{k-1}^s) \end{aligned}$$

$$\times (A_{k-1}^j)^{-1}] + |\mathcal{N}| \sum_{j\in\mathcal{N}} \pi_{i,j}^L (C_k^j)^T (R_k^j)^{-1} C_k^j. \tag{35}$$

The above inequality is equivalent to

$$\begin{aligned}
&(P_k^i)^{-1} \\
&\geq \alpha^2 \sum_{j\in\mathcal{N}} \sum_{s\in\mathcal{N}} \pi_{i,j}^L \pi_{j,s}^L (A_{k-1}^j)^{-T} (A_{k-2}^s)^{-T} (P_{k-2}^s)^{-1} \\
&\quad \times (A_{k-2}^s)^{-1} (A_{k-1}^j)^{-1} \\
&\quad + \alpha|\mathcal{N}| \sum_{j\in\mathcal{N}} \sum_{s\in\mathcal{N}} \pi_{i,j}^L \pi_{j,s}^L (A_{k-1}^j)^{-T} \\
&\quad \times (C_{k-1}^s)^T (R_{k-1}^s)^{-1} (C_{k-1}^s)^{-1} (A_{k-1}^j)^{-1} \\
&\quad + |\mathcal{N}| \sum_{j\in\mathcal{N}} \pi_{i,j}^L (C_k^j)^T (R_k^j)^{-1} C_k^j.
\end{aligned} \tag{36}$$

Then we continue to recurse to $k-m$ step for (35), we arrive at

$$\begin{aligned}
&(P_k^i)^{-1} \\
&\geq \alpha^{m+1} \underbrace{\sum_{j\in\mathcal{N}} \sum_{s\in\mathcal{N}} \cdots \sum_{t\in\mathcal{N}}}_{m+1} \pi_{i,j}^L \pi_{j,s}^L \cdots \pi_{u,t}^L (A_{k-1}^j)^{-T} \\
&\quad \times (A_{k-2}^s)^{-T} \cdots (A_{k-m-1}^t)^{-T} (P_{k-m-1}^t)^{-1} \\
&\quad \times (A_{k-m-1}^t)^{-1} \dots (A_{k-2}^s)^{-1} (A_{k-1}^j)^{-1} \\
&\quad + |\mathcal{N}|\alpha^m \underbrace{\sum_{j\in\mathcal{N}} \sum_{s\in\mathcal{N}} \cdots \sum_{t\in\mathcal{N}}}_{m+1} \pi_{i,j}^L \pi_{j,s}^L \cdots \pi_{u,t}^L (A_{k-1}^j)^{-T} \\
&\quad \times (A_{k-2}^s)^{-T} \cdots (A_{k-m}^t)^{-T} (C_{k-m}^t)^T (R_{k-m}^t)^{-1} \\
&\quad \times C_{k-m}^t (A_{k-m}^t)^{-1} \cdots (A_{k-2}^s)^{-1} (A_{k-1}^j)^{-1} \\
&\quad + \cdots + |\mathcal{N}|\alpha^2 \sum_{j\in\mathcal{N}} \sum_{s\in\mathcal{N}} \sum_{u\in\mathcal{N}} \pi_{i,j}^L \pi_{j,s}^L \pi_{s,u}^L (A_{k-1}^j)^{-T} \\
&\quad \times (A_{k-2}^s)^{-T} (C_{k-2}^u)^T (R_{k-m}^u)^{-1} C_{k-2}^u (A_{k-2}^s)^{-1} (A_{k-1}^j)^{-1} \\
&\quad + \alpha|\mathcal{N}| \sum_{j\in\mathcal{N}} \sum_{s\in\mathcal{N}} \pi_{i,j}^L \pi_{j,s}^L (A_{k-1}^j)^{-T} (C_{k-1}^s)^T (R_{k-1}^s)^{-1} C_{k-1}^s \\
&\quad \times (A_{k-1}^j)^{-1} + |\mathcal{N}| \sum_{j\in\mathcal{N}} \pi_{i,j}^L (C_k^j)^T (R_k^j)^{-1} C_k^j.
\end{aligned} \tag{37}$$

Since there is a positive real number $\alpha$ satisfying $0 < \alpha < 1$, $\omega$ can be an arbitrary number in the interval $(0, 1)$. Therefore, the value of $\omega$ can be taken as $\alpha$. The above formula uses the parameterized jointly uniform observability, we get the following conclusion

$$
\begin{aligned}
&(P_k^i)^{-1} \\
&\geq \alpha^{m+1} \underbrace{\sum_{j\in\mathcal{N}} \sum_{s\in\mathcal{N}} \cdots \sum_{t\in\mathcal{N}}}_{m+1} \pi_{i,j}^L \pi_{j,s}^L \cdots \pi_{u,t}^L \\
&\quad \times (A_{k-1}^j)^{-T} (A_{k-2}^s)^{-T} \cdots (A_{k-m-1}^t)^{-T} (P_{k-m-1}^t)^{-1} \\
&\quad \times (A_{k-m-1}^t)^{-1} \dots (A_{k-2}^s)^{-1} (A_{k-1}^j)^{-1} \\
&\quad + \underline{\lambda} I_n.
\end{aligned}
$$

At this point, we get an upper bound on $P_k^i$ at the interval $[k-m, k]$

$$P_k^i \leq \frac{1}{\underline{\lambda}} I_n \triangleq \bar{p} I_n. \tag{38}$$

Combining the results in (33) and (38), at the end of the interval $[k-m, k]$, we obtain

$$\underline{p} I_n \leq P_k^i \leq \bar{p} I_n. \tag{39}$$

□

So far, we have given uniformly bounded results of error covariance under perfect communication link conditions. In the case of communication links suffering from random failures, covariance stochastic boundedness will be proved in the following analysis.

Link failures in sensor networks occur randomly, for presentation convenience, we denote

$$\mathcal{O}_k = \{\Theta^{k,l} = [\gamma_{i,j}^{k,l}]_{N\times N}, 0 \leq l \leq L-1\}$$

where $\Theta^{k,l} \in R^{N\times N}$ is the matrix of all link condition indicator variables as entries. Moreover, during the interval $[k-m+1, k]$, a set describing the communication statuses is denoted as follows:

$$\mathcal{Q}_k = \{\mathcal{O}_{k-m+1}, \mathcal{O}_{k-m}, \cdots, \mathcal{O}_k\}.$$

Furthermore, we denote a set to represent the event that no communication failure occurs by

$$\bar{\mathcal{Q}} = \{\bar{\mathcal{O}}_1, \bar{\mathcal{O}}_2, \cdots, \bar{\mathcal{O}}_m\}$$

where $\bar{\mathcal{O}}_k = \{\bar{\Theta}^{k,l} = [1]_{N\times N}, 0 \leq l \leq L-1\}$. Finally, we introduce a finite set $\Psi$, let $\mathcal{Q}_k \in \Psi$, and obviously, $\bar{\mathcal{Q}} \in \Psi$.

By stochastic analysis, we get the following result.

**Theorem 3.15.** Under the Definition 3.4 and Assumption 3.7, the prediction error covariance of the sensor network is stochastically bounded, i. e.,

$$\lim_{\epsilon\to\infty} \sup_{k\in\mathbb{N}} \mathbb{P}(\|P_{k+1|k}^i\| > \epsilon) = 0. \tag{40}$$

Proof. From the Lemma 3.11, we can get

$$(P^i_{k+1|k})^{-1} \geq \alpha(A^i_k)^{-T}(P^i_k)^{-1}(A^i_k)^{-1}. \tag{41}$$

Substituting Step 3 of Table 1 into the above inequality and have

$$\begin{aligned}(P^i_{k+1|k})^{-1} \geq \alpha(A^i_k)^{-T}(&\sum_{j\in\mathcal{N}} \tilde{\pi}^{k,L}_{i,j}(P^j_{k|k-1})^{-1} \\ &+ |\mathcal{N}| \sum_{j\in\mathcal{N}} \tilde{\pi}^{k,L}_{i,j}(C^j_k)^T(R^j_k)^{-1}C^j_k)(A^i_k)^{-1}.\end{aligned} \tag{42}$$

Each sensor receives local information without a link failure, for $\forall i \in \mathcal{N}$, $\gamma^{k,l}_{i,i} = 1$, $\pi_{i,i} > 0$, and obvious $\tilde{\pi}^{k,L}_{i,i} > 0$, so there exists a positive scalar $\mu$ satisfying $\tilde{\pi}^{k,L}_{i,i} \geq \mu$. Also because $\tilde{\pi}^{k,L}_{i,j} \geq 0$, $\Sigma_{j\in\mathcal{N}}\tilde{\pi}^{k,L}_{i,j} > \mu$ can obviously be obtained. We can find a positive scalar $\tilde{\alpha}$ such that the following inequality holds

$$\begin{aligned}(P^i_{k+1|k})^{-1} &\geq \tilde{\alpha}(A^i_k)^{-T}(P^i_{k|k-1})^{-1}(A^i_k)^{-1} \\ &\quad + |\mathcal{N}|\alpha(A^i_k)^{-T} \sum_{j\in\mathcal{N}} \tilde{\pi}^{k,L}_{i,j}(C^j_k)^T(R^j_k)^{-1}C^j_k(A^i_k)^{-1} \\ &\geq \tilde{\alpha}(A^i_k)^{-T}(P^i_{k|k-1})^{-1}(A^i_k)^{-1} + \frac{|\mathcal{N}|\alpha\mu\underline{h}^2}{\bar{r}\bar{f}^2} I_n.\end{aligned} \tag{43}$$

For convenience of representation, we define $\mathcal{P}_{k+1} \triangleq P^i_{k+1|k}$. Given Lemma 3.10, there exists a positive parameter $\nu$ such that the following inequality

$$\mathcal{P}_{k+1} < \tilde{\alpha}^{-1}A^i_k(P^i_{k|k-1})^{-1}{A^i_k}^T + \nu I_n. \tag{44}$$

Let

$$\kappa(\epsilon) = \max_{k\in\mathbb{N}}\{k \in \mathbb{N}|\tilde{p}\zeta^{2k} + \frac{\zeta^{2k}-1}{\zeta^2-1}\nu \leq \epsilon\} \tag{45}$$

where $\tilde{p} = \bar{p}\overline{f}^2 + \overline{q}$, and $\zeta = \tilde{\alpha}^{-1\backslash 2}|\overline{f}|$. Under the formula (43), we choose a small $\tilde{\alpha}$ to ensure $\zeta > 1$.

Next, we will prove that the following inequality holds:

$$\begin{aligned}&\mathbb{P}(\|\mathcal{P}_{k_0+1}\| \leq \epsilon) \\ &\quad > \mathbb{P}(\text{ event } \bar{\mathcal{Q}} \text{ occurs in } k \in [k_0 - \kappa(\epsilon), k_0]).\end{aligned} \tag{46}$$

The event $\bar{\mathcal{Q}}$ occurs in the time interval $[k_0 - \kappa(\epsilon), k_0]$, this means that there is a constant $k^{'} \in [k_0 - \kappa(\epsilon), k_0]$ satisfying $\mathcal{O}_{k'-m+l} = \bar{\mathcal{O}}_l$, $\forall l = 1, 2, \cdots, m$. Therefore, from the Proposition 3.14 we can obtain $P^i_{k'} \leq \bar{p}I_n$, then from the prediction equation (3) and Assumption 3.7, we get

$$\begin{aligned}\mathcal{P}_{k'+1} &= A^i_{k'}P^i_{k'}(A^i_{k'})^{\mathrm{T}} + Q_{k'} \\ &\leq \bar{p}\|A^i_{k'}\|^2 I_n + \|Q_{k'}\| I_n \\ &\leq \bar{p}\overline{f}^2 I_n + \overline{q}I_n \\ &= \tilde{p}I_n.\end{aligned} \tag{47}$$

Take the norm of (44)

$$\|\mathcal{P}_{k+1}\| = \tilde{\alpha}^{-1}\|A_k^i\|^2\|\mathcal{P}_k\| + \nu. \tag{48}$$

Accordingly, it follows from (47) and (48) that

$$\begin{aligned}\|\mathcal{P}_{k_0+1}\| &\leq \tilde{p}\zeta^{2(k_0-k^{'})} + \frac{\zeta^{2(k_0-k^{'})}-1}{\zeta^2-1}\nu \\ &\leq \tilde{p}\zeta^{2\kappa(\epsilon)} + \frac{\zeta^{2\kappa(\epsilon)}-1}{\zeta^2-1}\nu \\ &\leq \epsilon,\end{aligned} \tag{49}$$

by (46) and $k_0 - k^{'} \leq \kappa(\epsilon)$, the above inequality is true. According to the above explanation, we can infer when the event $\bar{\mathcal{Q}}$ occurs in the interval $[k_0 - \kappa(\epsilon), k_0]$, the inequality $\|\mathcal{P}_{k_0+1}\| \leq \epsilon$ is always true, so the inequality (46) holds and we can conclude

$$\{\text{ event } \bar{\mathcal{Q}} \text{ occurs in } k \in [k_0 - \kappa(\epsilon), k_0]\} \subset \{\|\mathcal{P}_{k_0+1}\| \leq \epsilon\}$$

Counter-proposition through the above conclusion, we can get

$$\begin{aligned}&\mathbb{P}(\|\mathcal{P}_{k_0+1}\| > \epsilon) \\ &\quad\leq \mathbb{P}(\text{no event } \bar{\mathcal{Q}} \text{ occurs in } k \in [k_0 - \kappa(\epsilon), k_0]).\end{aligned} \tag{50}$$

In order to continue the analysis, we also define two time constants $k_1$ and $k_2$

$$k_1 \triangleq \min_k\{k \geq m | \mathcal{Q}_k = \bar{\mathcal{Q}}\}$$

$$k_2 \triangleq \min_k\{k \geq k_0 - \kappa(\epsilon) | \mathcal{Q}_k = \bar{\mathcal{Q}}\}$$

By the total probability formula and inequality (46), we have

$$\begin{aligned}&\mathbb{P}(\|\mathcal{P}_{k_0+1}\| > \epsilon) \leq \mathbb{P}(k_2 > k_0) \\ &\quad= \sum_{\tau\in\Psi} \mathbb{P}(k_2 > k_0 | \mathcal{Q}_{k_0-\kappa(\epsilon)-1=\tau})\mathbb{P}(\mathcal{Q}_{k_0-\kappa(\epsilon)-1=\tau})\end{aligned} \tag{51}$$

Recalling

$$\mathbb{P}(k_2 > k_0 | \mathcal{Q}_{k_0-\kappa(\epsilon)-1=\tau}) = \mathbb{P}(k_1 > \kappa(\epsilon) + 1 | \mathcal{Q}_m = \tau) \tag{52}$$

it is easy to get from (45) that when $\epsilon \to \infty$, $\kappa(\epsilon) \to \infty$. Obviously,

$$\lim_{\epsilon\to\infty} \mathbb{P}(k_1 > \kappa(\epsilon) + 1 | \mathcal{Q}_m = \tau) = 0 \tag{53}$$

Finally, combining (52) – (53), we have

$$\lim_{\epsilon\to\infty} \sup_{k\in\mathbb{N}} \mathbb{P}(\|\mathcal{P}_{k+1}\| > \epsilon) = 0$$

Further, due to $\mathcal{P}_{k+1} \triangleq P^i_{k+1|k}$, one has

$$\lim_{\epsilon\to\infty} \sup_{k\in\mathbb{N}} \mathbb{P}(\|P^i_{k+1|k}\| > \epsilon) = 0$$

This completes the proof. □

**Case 2:** **Infinite number of consensus steps**

In the previous case, we have analyzed the stochastic boundedness of error covariance matrix with finite number of consensus steps. This case, we analyze the case of the hybrid consensus-based EKF algorithm when the number of consensus steps is large enough.

**Theorem 3.16.** Under Definition 3.4 and Assumption 3.7, if the number of consensus steps is large enough, there must be a positive scalar $\hat{p}$ such that the error covariance matrix satisfies the following uniform boundedness condition

$$P_k^i \leq \hat{p} I_n, \quad i = 1, 2, \ldots, N.$$

Proof. We first define $\Omega_k = col(\Omega_k^i)_{i \in \mathcal{N}}$, $\Omega_{ci,k|k-1} = col(\Omega_{ci,k|k-1}^i)_{i \in \mathcal{N}}$, $\Omega_{cm,k} = col(\Omega_{cm,k}^i)_{i \in \mathcal{N}}$ and $\tilde{\mathcal{M}}_k^l = \tilde{\mathcal{M}}_{k,l} \cdots \tilde{\mathcal{M}}_{k,1} \tilde{\mathcal{M}}_{k,0}, 0 \leq l \leq L-1$, then according to Step 3 of Table 1, we have

$$\Omega_k = (\tilde{\mathcal{M}}_k^L \otimes I_n) \Omega_{ci,k|k-1} + |\mathcal{N}| (\tilde{\mathcal{M}}_k^L \otimes I_n) \Omega_{cm,k} \tag{54}$$

Note that $\tilde{\mathcal{M}}_{k,l}, \ldots, \tilde{\mathcal{M}}_{k,1}$ and $\tilde{\mathcal{M}}_{k,0}$ are non-negative matrices. Recalling Lemma 3.12, the following form is easy to get

$$\lim_{l \to \infty} \tilde{\mathcal{M}}_{k,l}, \cdots \tilde{\mathcal{M}}_{k,1} \tilde{\mathcal{M}}_{k,0} = \mathbf{1} d,$$

where $d = [d_1, \cdots, d_N] \in \mathbb{R}^N$. Then, when $1 \leq k \leq m-1$ and $l \to \infty$, (54) can be approximated as follows

$$\Omega_k = (\mathbf{1} d \otimes I_n) \Omega_{ci,k|k-1} + |\mathcal{N}| (\mathbf{1} d \otimes I_n) \Omega_{cm,k}$$

or equivalently,

$$\begin{aligned} \Omega_k^i =& d_1 \Omega_{ci,k|k-1}^1 + d_1 |\mathcal{N}| \Omega_{cm,k}^1 + d_2 \Omega_{ci,k|k-1}^2 + d_2 |\mathcal{N}| \Omega_{cm,k}^2 \\ &+ \cdots + d_N \Omega_{ci,k|k-1}^N + d_N |\mathcal{N}| \Omega_{cm,k}^N \\ =& \Omega_k^*. \end{aligned}$$

As can be seen from the above, $l \to \infty$, the information matrix tends to reach consensus. Therefore, when $1 \leq k \leq m-1$, we can always find a positive scalar $\hat{p}_1$ in a limited time interval to satisfy $P_k^i \leq \hat{p}_1 I_n$. As for the case $k \geq m$, from (37), we can verify that

$$
\begin{aligned}
&(P_k^i)^{-1} \\
&\geq |\mathcal{N}|\alpha^m \underbrace{\sum_{j\in\mathcal{N}}\sum_{s\in\mathcal{N}}\cdots\sum_{t\in\mathcal{N}}}_{m+1} \tilde{\pi}_{i,j}^{k,\infty}\tilde{\pi}_{j,s}^{k-1,\infty}\cdots\tilde{\pi}_{u,t}^{k-m,\infty} \\
&\quad\times (A_{k-1}^j)^{-T}(A_{k-2}^s)^{-T}\cdots(A_{k-m}^t)^{-T}(C_{k-m}^t)^T \\
&\quad\times (R_{k-m}^t)^{-1}C_{k-m}^t(A_{k-m}^t)^{-1}\cdots(A_{k-2}^s)^{-1}(A_{k-1}^j)^{-1} \\
&\quad+\cdots+|\mathcal{N}|\alpha^2\sum_{j\in\mathcal{N}}\sum_{s\in\mathcal{N}}\sum_{u\in\mathcal{N}}\tilde{\pi}_{i,j}^{k,\infty}\tilde{\pi}_{j,s}^{k-1,\infty}\tilde{\pi}_{s,u}^{k-2,\infty}(A_{k-1}^j)^{-T} \\
&\quad\times (A_{k-2}^s)^{-T}(C_{k-2}^u)^T(R_{k-m}^u)^{-1}C_{k-2}^u(A_{k-2}^s)^{-1}(A_{k-1}^j)^{-1} \\
&\quad+\alpha|\mathcal{N}|\sum_{j\in\mathcal{N}}\sum_{s\in\mathcal{N}}\tilde{\pi}_{i,j}^{k,\infty}\tilde{\pi}_{j,s}^{k-1,\infty}(A_{k-1}^j)^{-T}(C_{k-1}^s)^T(R_{k-1}^s)^{-1} \\
&\quad\times C_{k-1}^s(A_{k-1}^j)^{-1}+|\mathcal{N}|\sum_{j\in\mathcal{N}}\tilde{\pi}_{i,j}^{k,\infty}(C_k^j)^T(R_k^j)^{-1}C_k^j.
\end{aligned}
$$

Next, we denote $\tilde{\pi}_{i,j}^{k,\infty}\tilde{\pi}_{j,s}^{k-1,\infty}\cdots\tilde{\pi}_{u,t}^{k-m,\infty} = \tilde{\pi}^{(m+1)\infty}$ and $\pi_{i,j}^L\pi_{j,s}^L\cdots\pi_{u,t}^L = \pi^{(m+1)L}$. Since $0 \leq \pi_{i,j}^L \leq 1$ and $\tilde{\pi}_{i,j}^{k,L} \leq \pi_{i,j}^L$, when the number of consensus steps is infinite, $\tilde{\pi}^{(m+1)\infty} \leq \pi^{(m+1)\infty}$ $(\forall m \geq 0, \forall i,j \in \mathcal{N})$ holds almost surely. There is a number $0 < c < 1$ that satisfies $c \leq \underline{\lambda}$, then by utilizing parameterized jointly uniform observability, we have

$$(P_k^i)^{-1} \geq cI_n,$$

therefore

$$P_k^i \leq \frac{1}{c}I_n.$$

Choosing $\hat{p} = \max\{\hat{p}_1, \frac{1}{c}\}$, we can conclude that $P_k^i \leq \hat{p}I_n$. This completes the proof. □

## 4. SIMULATION RESULTS

In this section, a kinematic nearly constant velocity model [1] is adopted to verify the results established in this paper. For the sake of simplicity, we choose $f_k(x_k) = A_kx_k$, here $x_k = [p_{x_k}, v_{x_k}, p_{y_k}, v_{y_k}]$ is an unknown state vector, $p_{x_k}, p_{y_k}$ and $v_{x_k}$, $v_{y_k}$ are the position and velocity components on the coordinate axes, respectively. The system model is described as follows:

$$x_{k+1} = \begin{bmatrix} 1+0.01sink & \Delta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1+0.01sink & \Delta \\ 0 & 0 & 0 & 1 \end{bmatrix} x_k + w_k$$

where $\Delta = 4$ is the sampling interval and the covariance matrix of the process noise $w_k$ is $Q = Gq^2$ with

$$G = \begin{bmatrix} \frac{\Delta^3}{3} & \frac{\Delta^2}{2} & 0 & 0 \\ \frac{\Delta^2}{2} & \Delta & 0 & 0 \\ 0 & 0 & \frac{\Delta^3}{3} & \frac{\Delta^2}{2} \\ 0 & 0 & \frac{\Delta^2}{2} & \Delta \end{bmatrix}$$

and $q^2 = 5$.

The state trajectory is estimated to pass through the sensor network $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ of 10 sensor nodes in Figure 1.

Each sensor measures the position of the target in Cartesian coordinates, that is,

$$y_k^i = \begin{bmatrix} 1 + 0.01sink & 0 & 0 & 0 \\ 0 & 0 & 1 + 0.01sink & 0 \end{bmatrix} x_k + v_k^i, \qquad i \in \mathcal{N}$$

where $v_k$ is an Gaussian measurement noise with zero mean and the covariance $R^i = 6.25I$.

For consensus steps $L = 1$, consensus weights are selected according to Metropolis's weighting rules [23]. In the case of random link failures of the sensor network, the probabilities of successfully transmitting informations are given as follows

$$\shortmid(\gamma_{i,j}^{k,l} = 1) = 0.75, \qquad (i, j) \in \mathcal{E}.$$

To verify the stochastic boundedness of the error covariance in Theorem 3.15, we perform the simulation and then confirm that the event $\|P_k^i\| > \epsilon$, where $\epsilon = 100N$ and $i = 1, 2, \cdots, N$. The result shows that $\mathbb{P}(\|P_k^i\| > \epsilon) = 0$, and thus the error covariance is bounded. In order to obtain an objective result, a 50-step independent Monte Carlo experiment is performed here, and all experiments use the same initialization conditions. The average root mean-square error (ARMSE) of the position is used as a performance indicator. First define the root mean-square error (RMSE) at time instant $k$ as

$$\text{RMSE}_k^i = [\frac{1}{50} \sum_{m=1}^{50} ((p_{x,k}^m - \hat{p}_{x,k}^{i,m})^2 + (p_{y,k}^m - \hat{p}_{y,k}^{i,m})^2)]^{\frac{1}{2}}$$

where $(p_{x,k}^m, p_{y,k}^m)$ and $(\hat{p}_{x,k}^{i,m}, \hat{p}_{y,k}^{i,m})$ represent the true and estimated position at the $m$th Monte Carlo run. Then ARMSE is expressed as follows

$$\text{ARMSE}_k = \frac{1}{N} \sum_{i=1}^{N} \text{RMSE}_k^i.$$

The ARMSE of the hybrid consensus-based filtering is shown in Figure 3, from which we can see that the HCMCI algorithm can reach a reliable filtering performance under no link failure rates and larger noise covariance $R^i = 400$. In Figure 4, we can clearly see that filtering performance mainly depends on the link failure rate. In addition, we also compare the ARMSE of the HCMCI filter with different measured noise variances in

Figure 5, show that the greater the noise covariance, the worse the performance. Finally, for a sufficiently large number of consensus steps (L=100), let $\mathbb{P}(\gamma_{i,j}^{k,l} = 1) = 0.95$ and run 10 independent simulation experiments, by obtaining the eigenvalues of the matrix $\tilde{\mathcal{M}}_k^L$ in the results, it is shown that $\tilde{\mathcal{M}}_k^L$ is a non-negative matrix, and each element of the 10th power of $\tilde{\mathcal{M}}_k^L$ is positive. Therefore, we conclude that the error covariance is bounded.

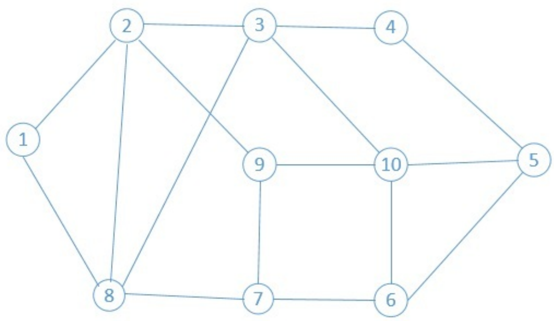


**Fig. 1.** Communication topology of 10 sensor nodes.

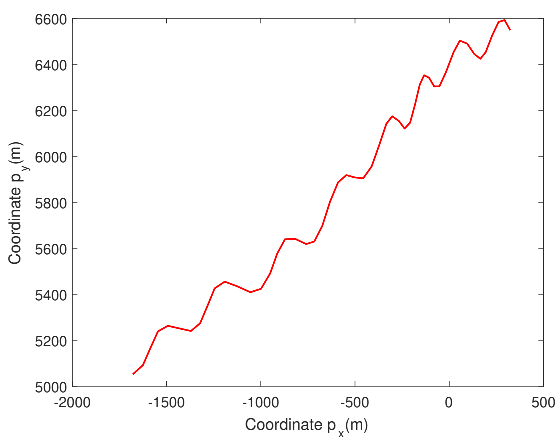


**Fig. 2.** State trajectory of the system.

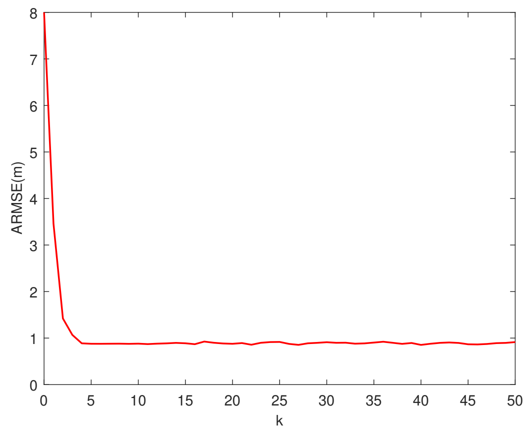


**Fig. 3.** Position ARMSE of the HCMCI filters under no link failure rates. The number of consensus steps is chosen as L = 1 and the covariance $R^i = 400$.

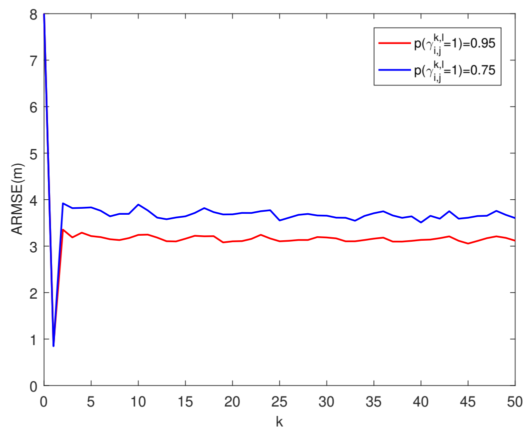


**Fig. 4.** Position ARMSE of the HCMCI filters under difference link failure rates. The number of consensus steps is chosen as L = 1.

## 5. CONCLUSIONS

This paper has investigated the distributed state estimation problems of sensor networks under random communication link failures. According to two existing consensus approaches, i. e., CI and CM, a hybrid consensus-based approach, consisting of a stan-

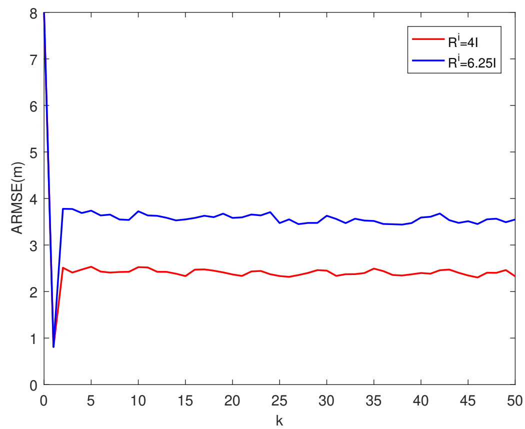


**Fig. 5.** Position ARMSE of the HCMCI filters under difference measurement noise variances. The number of consensus steps is chosen as L = 1.

dard extend Kalman filter update and a consensus update, has been introduced. The analysis of stochastic boundedness has been carried out in the cases of finite and infinite consensus steps, respectively. The results have shown that the system observability and the characteristics of some matrices satisfy certain conditions, and the error covariance of HCMCI filtering is stochastically bounded in distribution. Furthermore, if the number of consensus steps is infinite, it has been indicated that uniform boundedness can be almost certainly achieved. Finally, the effectiveness of the proposed HCMCI filter has been evaluated through a numerical example.

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## REFERENCES

[1] G. Battistelli and L. Chisci: Kullback-Leibler average, consensus on probability densities, and distributed state estimation with guaranteed stability. Automatica *50* (2014), 3, 707–718. DOI:10.1016/j.automatica.2013.11.042

[2] G. Battistelli, L. Chisci, and C. Fantacci: Parallel consensus on likelihoods and priors for networked nonlinear fitering. IEEE Signal Process. Lett. *21* (2014), 7, 787–791. DOI:10.1109/lsp.2014.2316258

[3] G. Battistelli, L. Chisci, G. Mugnai, A. Farina, and A. Graziano: Consensus-based linear and nonlinear fitering. IEEE Trans. Automat. Control *60* (2015), 5, 1410–1415. DOI:10.1109/tac.2014.2357135

[4] G. Battistelli, L. Chisci, M. Stefano, and P. Francesco: An information-theoretic approach to distributed state estimation. In: Proc. 18th IFAC World Congress 2011, 12477–12482. DOI:10.3182/20110828-6-it-1002.01998

[5] M. Boutayeb, H. Rafaralahy, and M. Darouach: Convergence analysis of the extended Kalman filter used as an observer for nonlinear deterministic discrete-time systems. IEEE Trans. Automat. Control *42* (1997), 4, 581–586. DOI:10.1109/9.566674

[6] W. Chen, D. Ding, H. Dong, and G. Wei: Distributed resilient filtering for power systems subject to denial-of-service attacks. IEEE Trans. Systems Man Cybernet.: Systems *49* (2019), 8, 1688–1697. DOI:10.1109/tsmc.2019.2905253

[7] W. Chen, D. Ding, X. Ge, Q.-L. Han, and G. Wei: $\mathcal{H}_\infty$ containment control of multiagent systems under event-triggered communication scheduling: The finite-horizon case. IEEE Trans. Cybernet. (2019), 1–11. DOI:10.1109/tcyb.2018.2885567

[8] D. Ding, Q.-L. Han, Z. Wang, and X. Ge: A survey on model-based distributed control and filtering for industrial cyber-physical systems. IEEE Trans. Industr. Inform. *17* (2019), 5, 2483–2499. DOI:10.1109/tii.2019.2905295

[9] D. Ding, Z. Wang, and Q.-L. Han: A set-membership approach to event-triggered filtering for general nonlinear systems over sensor networks. IEEE Trans. Automat. Control (2019), 1–1.

[10] H. Dong, Z. Wang, S. X. Ding, and H. Gao: Finite-horizon estimation of randomly occurring faults for a class of nonlinear time-varying systems. Automatica *50* (2014), 12, 3182–3189. DOI:10.1016/j.automatica.2014.10.026

[11] H. Dong, Z. Wang, D. W. C. Ho, and H. Gao: Variance-constrained filtering for a class of nonlinear time-varying systems with multiple missing measurements: the finite-horizon case. IEEE Trans. Signal Process. *58* (2010), 5, 2534–2543. DOI:10.1109/tsp.2010.2042489

[12] X. Ge and Q.-L. Han: Consensus of multiagent systems subject to partially accessible and overlapping Markovian network topologies. IEEE Trans. Cybernet. *47* (2017), 8, 1807–1819. DOI:10.1109/tcyb.2016.2570860

[13] X. Ge, Q.-L. Han, and Z. Wang: A dynamic event-triggered transmission scheme for distributed set-membership estimation over wireless sensor networks. IEEE Trans. Cybernet. *49* (2019), 1, 171–183. DOI:10.1109/tcyb.2017.2769722

[14] X. Ge, Q.-L. Han, and Z. Wang: A threshold-parameter-dependent approach to designing distributed event-triggered $\mathcal{H}_\infty$ consensus filters over sensor networks. IEEE Trans. Cybernet. *49* (2019), 4, 1148–1159. DOI:10.1109/tcyb.2017.2789296

[15] X. Ge, Q.-L. Han, X.-M. Zhang, L. Ding, and F. Yang: Distributed event-triggered estimation over sensor networks: A survey. newblockIEEE Trans. Cybernet. *50* (2019), 3, 1306–1320.

[16] X. Ge, Q.-L. Han, M. Zhong, and X.-M. Zhang: Distributed Krein space-based attack detection over sensor networks under deception attacks. Automatica *109* (2019), 108557. DOI:10.1016/j.automatica.2019.108557

[17] O. Hlinka, O. Sluciak, F. Hlawatsch, P. M. Djuric, and M. Rupp: Likelihood consensus and its application to distributed particle fitering. IEEE Transs. Signal Process. *60* (2012), 8, 4334–4349. DOI:10.1109/tsp.2012.2196697

[18] S. J. Julier and J. K. Uhlmann: A non-divergent estimation algorithm in the presence of unknown correlations. In: Proc. American Control Conference, Albuquerque 1997, pp. 2369–2373. DOI:10.1109/acc.1997.609105

[19] A. T. Kamal, J. A. Farrell, and A. K. Roy-Chowdhury: Information weighted consensus fiters and their application in distributed camera networks. IEEE Trans. Automat. Control *58* (2013), 12, 3112–3125. DOI:10.1109/tac.2013.2277621

[20] S. Kluge, K. Reif, and M. Brokate: Stochastic stability of the extended Kalman filter with intermittent observations. IEEE Trans. Automat. Control *55* (2010), 2, 514–518. DOI:10.1109/tac.2009.2037467

[21] W. Li, G. Wei, D. Ding, Y. Liu, and F. E. Alsaadi: A new look at boundedness of error covariance of Kalman filtering. IEEE Trans. Systems Man Cybernet.: Systems *48* (2018), 2, 309–314. DOI:10.1109/tsmc.2016.2598845

[22] J. Liang, Z. Wang, T. Hayat, and A. Alsaedi: Distributed $\mathcal{H}_\infty$ state estimation for stochastic delayed 2-D systems with randomly varying nonlinearities over saturated sensor networks. Information Sciences *370* (2016), 708–724. DOI:10.1016/j.ins.2015.11.020

[23] X. Lin , S. Boyd , and S. Lall: A scheme for robust distributed sensor fusion based on average consensus. In: Proc. 4th International Symposium on Information Processing in Sensor Networks *25* (2005), 63–70. DOI:10.1109/ipsn.2005.1440896

[24] Q. Liu, Z. Wang, X. He, G. Ghinea, and F. E. Alsaadi: A resilient approach to distributed filter design for time-varying systems under stochastic nonlinearities and sensor degradation. IEEE Trans. Signal Process. *65* (2017), 5, 1300–1309. DOI:10.1109/tsp.2016.2634541

[25] Q. Liu, Z. Wang, X. He, and D. Zhou: On Kalman-consensus filtering with random link failures over sensor networks. IEEE Trans. Automat. Control *63* (2018), 8, 2701–2708. DOI:10.1109/tac.2017.2774601

[26] L. Ma, Z. Wang, H. K. Lam, and N. Kyriakoulis: Distributed event based set-membership filtering for a class of nonlinear systems with sensor saturations over sensor networks. IEEE Trans. Cybernet. *47* (2017), 11, 3772–3783. DOI:10.1109/tcyb.2016.2582081

[27] R. Olfati-Saber: Distributed Kalman filter with embedded consensus filters. In: Proc. Conference on Decision and Control, and the European Control Conference, Seville 2005, pp. 8179–8184. DOI:10.1109/cdc.2005.1583486

[28] R. Olfati-Saber, and R. M. Murray: Consensus problems in networks of agents with switching topology and time-delays. IEEE Trans. Automat. Control *49* (2004), 9, 1520–1533. DOI:10.1109/tac.2004.834113

[29] R. Olfati-Saber, J. A. Fax, and R. M. Murray: Consensus and cooperation in networked multi-agent systems. Proc. IEEE *95* (2007), 1, 215–233. DOI:10.1109/jproc.2006.887293

[30] L. Paradis and Q. Han: A survey of fault management in wireless sensor networks. J. Network Systems Mgt *5* (2007), 2, 171–190. DOI:10.1007/s10922-007-9062-0

[31] K. Rapp and P. O. Nyman: Stability properties of the discrete-time extended Kalman filter. IFAC Proc. *37* (2004), 13, 1377–1382. DOI:10.1016/s1474-6670(17)31420-9

[32] K. Reif and R. Unbehauen: The extended Kalman filter as an exponential observer for nonlinear systems. IEEE Trans. Signal Process. *47* (1999), 8, 2324–2328. DOI:10.1109/78.774779

[33] K. Reif et al: Stochastic stability of the discrete-time extended Kalman filter. IEEE Trans. Automat. Control *44* (1999), 4, 714–728. DOI:10.1109/9.754809

[34] W. Ren, R.W. Beard, and E.M. Atkins: Information consensus in multivehicle cooperative control. IEEE Control Systems Magazine *27* (2007), 11, 71–82. DOI:10.1109/mcs.2007.338264

[35] A. Simonetto, T. Keviczky, and R. Babuska: Distributed nonlinear estimation for robot localization using weighted consensus. In: IEEE International Conference Robotics and Automation, Alaska 2010, pp. 3026–3031. DOI:10.1109/robot.2010.5509143

[36] V. Ugrinovskii: Conditions for detectability in distributed consensus-based observer networks. IEEE Trans. Automat. Control *58* (2013), 10, 2659–2664. DOI:10.1109/tac.2013.2256675

[37] G. Wei, W. Li, D. Ding, and Y. Liu: Stability analysis of covariance intersection-based Kalman consensus filtering for time-varying systems. IEEE Trans. Systems Man Cybernet.: Systems (2018), 1–12. DOI:10.1109/tsmc.2018.2855741

[38] D. Willner, C. Chang, and K. Dunn: Kalman filter algorithms for a multi-sensor system. In: IEEE Conference on Decision and Control Including the Symposium on Adaptive Processes 1976, pp. 570–574. DOI:10.1109/cdc.1976.267794

[39] S. Xiao, Q.-L. Han, X. Ge, and Y. Zhang: Secure distributed finite-time filtering for positive systems over sensor networks under deception attacks. IEEE Trans. Cybernet. *50* (2019), 3, 1220–1229. DOI:10.1109/tcyb.2019.2900478

[40] Y. Yuan, H. Yuan, L. Guo, H. Yang, and S. Sun: Resilient control of networked control system under DoS attacks: a unified game approach. IEEE Tran. Industr. Inform. *12* (2016), 2, 1786–1794. DOI:10.1109/tii.2016.2542208

[41] X.-M. Zhang and Q.-L. Han: A decentralized event-triggered dissipative control scheme for systems with multiple sensors to sample the system outputs,. IEEE Trans. Cybernet. *46* (2016), 12, 2745–2757. DOI:10.1109/tcyb.2015.2487420

[42] X.-M. Zhang, Q.-L. Han, X. Ge, D. Ding, L. Ding, D. Yue, and C. Peng: Networked control systems: a survey of trends and techniques. IEEE/CAA J. Automat. Sinica. (2019), 1–17.

[43] Z. Zuo, Q.-L. Han, B. Ning, X. Ge, and X.-M. Zhang: An overview of recent advances in fixed-time cooperative control of multi-agent systems. IEEE Trans. Industr. Inform. *14* (2018), 6, 2322–2334. DOI:10.1109/tii.2018.2817248

[44] L. Zou, Z. Wang, Q.-L. Han, and D. H. Zhou: Ultimate boundedness control for networked systems with Try-Once-Discard protocol and uniform quantization effects. IEEE Trans. Automat. Control *62* (2017), 12, 6582–6588. DOI:10.1109/tac.2017.2713353

[45] L. Zou, Z. Wang, Q.-L. Han, and D. H. Zhou: Recursive filtering for time-varying systems with random access protocol. IEEE Trans. Automat. Control *60* (2019), 2, 720–727. DOI:10.1109/tac.2018.2833154

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