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Vanishing conharmonic tensor of normal locally conformal almost cosymplectic manifold

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Abstract. The main purpose of the present paper is to study the geometric properties of the conharmonic curvature tensor of normal locally conformal almost cosymplectic manifolds (normal LCAC-manifold). In particular, three conhoronic invariants are distinguished with regard to the vanishing conharmonic tensor. Subsequentaly, three classes of normal LCAC-manifolds are established. Moreover, it is proved that the manifolds of these classes are η-Einstein manifolds of type (α, β). Furthermore, we have determined α and β for each class.

Keywords: normal locally conformal almost cosymplectic manifold; conharmonic curvature tensor; constant curvature; η-Einstein manifold

Classification: 53C55, 53B35

1. Introduction

It is worth mentioning, that many researchers have focused on the study of conformal mappings of manifolds that admits conformal transformations of special types. The conformal mapping and conformal structure found their application not only in the geometry, but also in the theory of potential, the theory of functions and widely in cartography. The properties of the locally conformal almost cosymplectic manifold that admits conformal transformations were studied by many researchers including, Z. Olszak in [11], H. M. Abood and F. H. Al-Hussini in [1]. Moreover, Z. Olszak and R. Rosca in [12] described the local structure of the normal locally conformal almost cosymplectic manifolds. On the other hand, D. Chinea and J. C. Marrero in [3], studied the conformal transformation of almost cosymplectic manifolds. In particular, they obtained certain characteristics of the locally conformal almost cosymplectic manifolds and locally conformal cosymplectic manifolds. Lastly, V. F. Kirichenko and S. V. Khartinova in [9] established the structure equations of the normal locally conformal almost cosymplectic manifolds and calculated the components of the Riemannian curvature tensor and Ricci tensor.

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2. Preliminaries

This section contains some concepts and facts related to the content of this paper. In particular, the structure equations and the components of the Riemannian curvature tensor of the normal locally conformal almost cosymplectic manifold have been established.

Definition 2.1 ([2])**.** Suppose that M is smooth manifold of odd dimension, an almost contact metric structure ($\mathcal{AC}$-structure) on a manifold M is a quadruple (η, ξ, Φ, g) of tensors, where η is 1-form called a contact form, ξ is a vector field called a characteristic, Φ is a tensor of type $(1,1)$ called a structure endomorphism of a module of vector fields $\chi(M)$, and $g = \langle \cdot, \cdot \rangle$ is a Riemannian metric, moreover the following conditions hold:

$$\begin{gathered}(1)\ \eta(\xi) = 1, \quad (2)\ \Phi(\xi) = 0, \quad (3)\ \eta \circ \Phi = 0, \quad (4)\ \Phi^2 = -\mathrm{id} + \eta \otimes \xi, \\ (5)\ \langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y), \qquad X, Y \in X(M).\end{gathered} \tag{2.1}$$

In this case, the manifold M endowed with this structure is called an almost contact metric manifold ($\mathcal{AC}$-manifold).

It is easy to verify that the tensor $\Omega(X, Y) = g(X, \Phi Y)$ is skew-symmetric, i.e. is a 2-form on M. It is called a fundamental form of the $\mathcal{AC}$-structure.

Let (η, ξ, Φ, g) be an $\mathcal{AC}$-structure on the manifold $M^{(2n+1)}$. In the $C^\infty(M)$-module $\chi(M)$ of smooth vector fields on the $\mathcal{AC}$-manifold M there are two complementary projections m, l, where $m = \eta \otimes \xi$ and $l = -\Phi^2$. Thus $\chi(M) = L \oplus \aleph$, where $L = \operatorname{Im} \Phi = \ker \eta$, $\dim L = 2n$ and $\mathcal{M} = \operatorname{Im} m = \ker \Phi$, $\dim \mathcal{M} = 1$. A contact distribution or the first fundamental distribution is called L. A second fundamental distribution is called $\mathcal{M}$. Obviously, the distributions of $\aleph$ and L are invariant with respect to Φ and are orthogonal. It is also obvious that $\widetilde{\Phi}^2 = -\mathrm{id}$, $\langle \widetilde{\Phi} X, \widetilde{\Phi} X \rangle = \langle X, X \rangle$, $X, Y \in \chi(M)$, where $\widetilde{\Phi} = \Phi|_L$. Therefore, if $p \in M$, then in the tangent space $T_p(M)$ one can construct an orthonormal frame $(p, e_0, e_1, \dots, e_n, \Phi e_1, \dots, \Phi e_n)$, where $e_0 = \xi_p$. Such a frame is called a real adapted frame, see [10]. On the other hand, let $\mathfrak{L}^c = L \otimes \mathbb{C}$ be the complexification of the distribution L. In it, there are two complementary projectors $\sigma = \frac{1}{2}(\mathrm{id} - \sqrt{-1}\Phi^c)$ and $\overline{\sigma} = \frac{1}{2}(\mathrm{id} + \sqrt{-1}\Phi^c)$, are intrinsically defined on the proper submodules $D_\Phi^{\sqrt{-1}}$ and $D_\Phi^{-\sqrt{-1}}$ of the endomorphism Φ corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. That is, the module L^c decomposes into a direct sum of proper submodules $L^c = D_\Phi^{\sqrt{-1}} \oplus D_\Phi^{-\sqrt{-1}}$.

Thus, the complexification of the module $\chi(M)$ decomposes into a direct sum of proper submodules of the endomorphism $\chi^c(M) = D_\Phi^{\sqrt{-1}} \oplus D_\Phi^{-\sqrt{-1}} \oplus D_\Phi^0$, where $D_\Phi^0 = \mathcal{M}^c$. The projections on the terms of this direct sum are respectively endomorphisms $\Pi = -\frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi)$, $\overline{\Pi} = \frac{1}{2}(-\Phi^2 + \sqrt{-1}\Phi)$ and $m = \mathrm{id} + \Phi^2$.

Consequently, we can construct a frame $(p, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{\hat{1}}, \ldots, \varepsilon_{\hat{n}})$ of the complexification of the space $T_P(M)$, where $\varepsilon_0 = \xi_p$, $\varepsilon_a = \sqrt{2}\,\sigma(e_a)$ and $\varepsilon_{\hat{a}} = \sqrt{2}\,\overline{\sigma}(e_a)$, consisting of the eigenvectors of the operator Φ. Such a frame is called an A-frame, see [8], [10]. It is easy to see that the matrices of the components of the tensors Φ_p and g_p in the A-frame have the following forms respectively:

$$(2.2)\qquad (\Phi^i_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & 0 \\ 0 & 0 & -\sqrt{-1}I_n \end{pmatrix}, \qquad (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & I_n & 0 \end{pmatrix},$$

where I_n is the unit matrix of order n. It is well known, see [7], [10], that the collection of such frames defines a G-structure on M with the structure group $1 \times U(n)$, represented by matrices of the form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}$, where $A \in U(n)$. This G-structure is called associated.

Remark 2.1. An associated G-structure space consists of complex frames, i.e. frames of complexification of corresponding tangent spaces. Therefore, even when dealing with real tensors, when we speak of their components on the associated G-structure space, we mean the components of complex extensions of these tensors. In turn, the complex tensor is a complex extension of the real tensor if and only if it is invariant under the complex conjugation operator. Following the generally accepted tradition, we will call such a tensor real. In particular, the sum of the pure complex tensor and the complex conjugate tensor is a real tensor.

Throughout this paper we will assume that the indices $i, j, k, \ldots$ range from 0 to $2n$, while the indices $a, b, c, d, f, g, \ldots$ are values from 1 to n, and put $\hat{a} = a+n$, $\hat{\hat{a}} = a$, $\hat{0} = 0$.

The structural endomorphism Φ and the metric structure g are tensors of type $(1,1)$ and $(2,0)$, respectively, on the manifold M^{2n+1}. Therefore, according to the basic theorem of tensor analysis, and also because of the covariant constancy of the metric tensor of the Riemannian connection, their components, as systems of functions on the space of the principal bundle of frames $B(M^{2n+1})$ satisfy the relations:

$$(2.3)\qquad (1)\ \mathrm{d}\Phi^i_j - \Phi^i_k\omega^k_j + \Phi^k_j\omega^i_k = \Phi^i_{j,k}\omega^k, \qquad (2)\ \mathrm{d}g_{ij} - g_{ik}\omega^k_j - g_{kj}\omega^k_i = 0,$$

where $\{\omega^i\}$,$\{\omega^i_j\}$ are the components of the displacement forms and the Riemannian connection ∇, respectively, and $\Phi^i_{j,k}$ are the components of the covariant differential Φ in this connection.

Taking (2.2) into account, the relations (2.3) on the associated G-structure space can be rewritten in the following form, see [8], [10],

$$
(2.4)\quad
\begin{array}{lll}
1)\ \Phi^a_{b,k}=0, & 5)\ \omega^a_{\hat b}=\dfrac{\sqrt{-1}}{2}\Phi^a_{\hat b,k}\omega^k, & 9)\ \omega^0_{\hat a}=\sqrt{-1}\Phi^0_{\hat a,k}\omega^k,\\
2)\ \Phi^{\hat a}_{\hat b,k}=0, & 6)\ \omega^{\hat a}_{b}=-\dfrac{\sqrt{-1}}{2}\Phi^{\hat a}_{b,k}\omega^k, & 10)\ \omega^0_{a}=-\sqrt{-1}\Phi^0_{a,k}\omega^k,\\
3)\ \Phi^0_{0,k}=0, & 7)\ \omega^a_{0}=\sqrt{-1}\Phi^a_{0,k}\omega^k, & 11)\ \omega^i_j+\omega^{\hat j}_{\hat i}=0.\\
4)\ \ \omega^0_0=0, & 8)\ \omega^{\hat a}_{0}=-\sqrt{-1}\Phi^{\hat a}_{0,k}\omega^k, &
\end{array}
$$

In addition, we note that, because of the fact that the corresponding forms and tensors are real, $\overline{\omega}^{\,i}=\omega^{\hat i}$, $\overline{\omega}^{\,i}_j=\omega^{\hat i}_{\hat j}$, $\nabla\overline{\Phi}^{\,i}_{j,k}=\nabla\Phi^{\hat i}_{\hat j,\hat k}$ where $t\to\bar t$ is the complex conjugation operator. Also, from (2.4) it follows that the system of functions $\{\Phi^i_{j,k}\}$ is skew-symmetric with respect to the indices i and j, that is $\Phi^i_{j,k}=-\Phi^j_{i,k}$.

Taking these relations into account, the first group of structural equations of the Riemannian connection $d\omega^i=-\omega^i_j\ \Lambda\omega^j$ on the associated G-structure space of an $\mathcal{AC}$-manifold can be written in the following form, called the first group of structural equations of an almost contact metric manifold, see [8], [10]:

$$
(2.5)\quad
\begin{aligned}
&1)\ \mathrm{d}\omega^a=-\omega^a_b\wedge\omega^b+B^{ab}_c\omega^c\wedge\omega_b+B^{abc}\omega_b\wedge\omega_c+B^a_b\omega\wedge\omega^b+B^{ab}\omega\wedge\omega_b,\\
&2)\ \mathrm{d}\omega_a=\omega^b_a\wedge\omega_b+B^c_{ab}\omega_c\wedge\omega^b+B_{abc}\omega^b\wedge\omega^c+B^b_a\omega\wedge\omega_b+B_{ab}\omega\wedge\omega^b,\\
&3)\ \ \mathrm{d}\omega=C_{bc}\omega^b\Lambda\omega^c+C^{bc}\omega_b\Lambda\omega_c+C^b_c\omega^c\Lambda\omega_b+C_b\omega\Lambda\omega^b+C^b\omega\Lambda\omega_b,
\end{aligned}
$$

where $\omega=\omega^0=\omega_0=\pi^*(\eta)$, π is the natural projection of the the associated G-structure space onto the manifold M where,

$$
(2.6)\quad
\begin{array}{ll}
B^{abc}=\dfrac{1}{2}\sqrt{-1}\Phi^a_{[\hat b,\hat c]}, & B_{abc}=\dfrac{1}{2}\sqrt{-1}\Phi^{\hat a}_{[b,c]},\\
B^{ab}_c=-\dfrac{1}{2}\sqrt{-1}\Phi^a_{\hat b,c}, & B^c_{ab}=\dfrac{1}{2}\sqrt{-1}\Phi^{\hat a}_{b,\hat c},\\
B^{ab}=-\sqrt{-1}\Big(\dfrac{1}{2}\Phi^a_{\hat b,0}-\Phi^a_{0,\hat b}\Big), & B_{ab}=\sqrt{-1}\Big(\dfrac{1}{2}\Phi^{\hat a}_{b,0}-\Phi^{\hat a}_{0,b}\Big),\\
B^a_b=\sqrt{-1}\Phi^a_{0,b}, & B^b_a=\sqrt{-1}\Phi^{\hat a}_{0,\hat b},\\
C^{ab}=\sqrt{-1}\Phi^0_{[\hat a,\hat b]}, & C_{ab}=-\sqrt{-1}\Phi^0_{[a,b]},\\
C^a=-\sqrt{-1}\Phi^0_{\hat a,0}, & C_a=\sqrt{-1}\Phi^0_{a,0},\\
C^a_b=-\sqrt{-1}(\Phi^0_{b,\hat a}+\Phi^0_{\hat a,b})=B^a_b-B^b_a. &
\end{array}
$$

Taking these relations into account and the skew-symmetry of the system of functions $\{\Phi^i_{j,k}\}$, we note that

$$
\begin{gathered}
B^{abc}=-B^{acb}, \qquad B_{abc}=-B_{acb}, \qquad \overline{B^{abc}}=B_{abc}, \qquad B_c^{ab}=-B_c^{ba}, \\
B_{ab}^c=-B_{ba}^c, \qquad \overline{B_c^{ab}}=B_{ab}^c, \qquad \overline{B^{ab}}=B_{ab}, \qquad C^{ab}=-C^{ba}, \\
C_{ab}=-C_{ba}, \qquad \overline{C^{ab}}=C_{ab}, \qquad \overline{C^a}=C_a, \qquad \overline{\omega_b^a}=-\omega_a^b.
\end{gathered} \tag{2.7}
$$

The Nijenhuis tensor of an endomorphism Φ is a tensor N_Φ of type (2.1), defined by

$$
N_\Phi(X,Y)=\frac{1}{4}(\Phi^2[X,Y]+[\Phi X,\Phi Y]-\Phi[\Phi X,Y]-\Phi[X,\Phi Y]), \qquad X,Y\in X(M).
$$

Its vanishing is equivalent to the integrability of the structure in [4]. A direct calculation, taking into account the identity $[X,Y]=\nabla_x Y-\nabla_Y X$, shows, see [8], [10], that

$$
N_\Phi(X,Y)=\frac{1}{4}\{\nabla_{\Phi X}(\Phi)Y-\Phi\nabla_X(\Phi)Y-\nabla_{\Phi Y}(\Phi)X+\Phi\nabla_Y(\Phi)X\}.
$$

Taking (2.4) into account, we obtain that on the associated G-structure space, the components of the tensor N_Φ are determined by the identities:

$$
\begin{aligned}
&1)\ \ N_{ab}^0=-\frac{\sqrt{-1}}{2}\Phi_{[a,b]}^0, \\
&2)\ \ N_{\hat{a}b}^0=-N_{b\hat{a}}^0=-\frac{\sqrt{-1}}{2}\Phi_{(\hat{a},b)}^0, \\
&3)\ \ N_{\hat{a}\hat{b}}^0=\frac{\sqrt{-1}}{2}\Phi_{[\hat{a},\hat{b}]}^0, \\
&4)\ \ N_{\hat{b}0}^a=-N_{0\hat{b}}^0=\frac{\sqrt{-1}}{4}\Phi_{\hat{b},0}^a-\frac{\sqrt{-1}}{2}\Phi_{0\hat{b}}^a, \\
&5)\ \ N_{\hat{b}\hat{c}}^a=\sqrt{-1}\Phi_{[\hat{b},\hat{c}]}^a, \\
&6)\ \ N_{b0}^{\hat{a}}=-N_{0b}^{\hat{a}}=\frac{\sqrt{-1}}{2}\Phi_{0b}^{\hat{a}}-\frac{\sqrt{-1}}{4}\Phi_{b0}^{\hat{a}}, \\
&7)\ \ N_{bc}^{\hat{a}}=-\sqrt{-1}\Phi_{[b,c]}^{\hat{a}}.
\end{aligned} \tag{2.8}
$$

The remaining components of the Nijenhuis tensor are identically equal to zero.

Definition 2.2 ([8], [10]). An almost contact metric structure is called a normal if $N_\Phi+2\,d\eta\otimes\xi=0$, where N_Φ is the Nijenhuis tensor of the operator Φ.

The notion of normality was introduced by S. Sasaki and Y. Hatakeuyama in 1961 in [13], it is one of the most fundamental concepts of contact geometry, closely related to the notion of integrability.

The following proposition immediately follows from (2.8) and Definition 2.2.

Proposition 2.1 ([8], [10]). *An almost contact metric structure is normal if and only if on the space of the associated G-structure, we have*

$$
\Phi_{b,c}^{\hat{a}}=\Phi_{\hat{b},\hat{c}}^a=\Phi_{b,0}^{\hat{a}}=\Phi_{\hat{b},0}^a=\Phi_{a,b}^0=\Phi_{\hat{a},\hat{b}}^0=\Phi_{a,0}^0=\Phi_{\hat{a},0}^0=0.
$$

In virtue of which the first group of structure equations of the normal structure of the associated G-structure space will be:

(1) $\mathrm{d}\omega^a = -\omega^a_b \wedge \omega^b + B^{ab}_c \omega^c \wedge \omega_b + B^a_b \omega \wedge \omega^b$;
(2) $\mathrm{d}\omega_a = \omega^b_a \wedge \omega_b + B^c_{ab} \omega_c \wedge \omega^b + B^b_a \omega \wedge \omega_b$;
(3) $\mathrm{d}\omega = C^b_c \omega^c \wedge \omega_b$.

Definition 2.3 ([4]). An almost contact metric structure $S = (\eta, \xi, \Phi, g)$ is called an almost cosymplectic structure ($\mathcal{AC}_f$-structure), if the conditions below hold:

(1) $\mathrm{d}\eta = 0$;
(2) $\mathrm{d}\Omega = 0$.

The following proposition holds.

Proposition 2.2 ([8]). *Let Ω be the fundamental form of an $\mathcal{AC}$-structure. Then on the associated G-structure space we have*

(1) $\pi^*\Omega = -\sqrt{-1}\omega^a \wedge \omega_a$;
(2) $\pi^* d\Omega = \sqrt{-1}\{B^{ab}_c \omega_a \wedge \omega_b \wedge \omega^c - B^{abc}\omega_a \wedge \omega_b \wedge \omega_c + B^a_b \omega_a \wedge \omega^b \wedge \omega + B^{ab}\omega_a \wedge \omega_b \wedge \omega - B^c_{ab}\omega^a \wedge \omega^b \wedge \omega_c + B_{abc}\omega^a \wedge \omega^b \wedge \omega^c - B^b_a \omega^a \wedge \omega_b \wedge \omega + B_{ab}\omega^a \wedge \omega^b \wedge \omega\}$.

According to the conditions of the Definition 2.3 and taking the Proposition 2.2 into account, we find on the associated G-structure space that the first group of structural equations of the $\mathcal{AC}_f$-structure takes the form:

$$\begin{aligned} &1)\ \mathrm{d}\omega^a = -\theta^a_b \wedge \omega^b + B^{abc}\omega_b \wedge \omega_c + F^{ab}\omega_b \wedge \omega, \\ &2)\ \mathrm{d}\omega_a = \omega^b_a \wedge \omega_b + B_{abc}\omega^b \wedge \omega^c + F_{ab}\omega^b \wedge \omega, \\ &3)\ \ \mathrm{d}\omega = 0. \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} &B^{abc} = \frac{1}{2}\sqrt{-1}\Phi^a_{[\hat{b},\hat{c}]}, \qquad B_{abc} = \frac{1}{2}\sqrt{-1}\Phi^{\hat{a}}_{[b,c]}, \qquad F^{ab} = \sqrt{-1}\Phi^0_{\hat{a},\hat{b}}, \\ &F_{ab} = -\sqrt{-1}\Phi^0_{a,b}, \qquad B^{[abc]} = B_{[abc]} = 0, \qquad \overline{B^{abc}} = B_{abc}, \\ &F^{[ab]} = F_{[ab]} = 0, \qquad \overline{F^{ab}} = F_{ab}. \end{aligned} \tag{2.10}$$

Definition 2.4 ([2]). A normal almost cosymplectic structure is called a cosymplectic structure.

The first group of the structural equations of the cosymplectic structure on the associated G-structure space takes the form:

$$1)\ \mathrm{d}\omega^a = -\theta^a_b \wedge \omega^b, \qquad 2)\ \mathrm{d}\omega_a = \omega^b_a \wedge \omega_b, \qquad 3)\ \mathrm{d}\omega = 0. \tag{2.11}$$

A conformal transformation of an $\mathcal{AC}$-structure $S = (\eta, \xi, \Phi, g)$ on a manifold M is a transition from S to an $\mathcal{AC}$-structure $\widetilde{S} = (\widetilde{\eta}, \tilde{\xi}, \widetilde{\Phi}, \widetilde{g})$, in this case

$\widetilde{\eta} = \mathrm{e}^{-\sigma}\eta$, $\tilde{\xi} = \mathrm{e}^{\sigma}\xi$, $\widetilde{\Phi} = \Phi$, $\widetilde{g} = \mathrm{e}^{-2\sigma}g$, where σ is an arbitrary smooth function on M, called the defining function of the transformation, see [8], [11]. If $\sigma = \mathrm{const}$, a conformal transformation is said to be trivial.

Definition 2.5 ([9], [11])**.** An $\mathcal{AC}$-structure $S = (\eta, \xi, \Phi, g)$ on M is said to be a locally conformally almost cosymplectic structure (LCAC-structure), if the restriction of this structure on some neighborhood U of an arbitrary point $p \in M$ admits a conformal transformation into almost cosymplectic structure. We call this transformation a locally conformal. A manifold M equipped with a LCAC-structure is called, a LCAC-manifold.

We note that for $\sigma = \mathrm{const}$, we obtain an $\mathcal{AC}_f$-manifold.

Recall [6] that for normal LCAC-manifolds, the nonzero components of the Riemannian curvature tensor on the associated G-structure space have the forms:
(2.12)

$$1)\ R^a_{bc\hat{d}} = A^{ad}_{bc} - \delta^a_c\delta^d_b\sigma^2_0, \quad 2)\ R^a_{\hat{b}cd} = -2\delta^{ab}_{cd}\sigma^2_0, \quad 3)\ R^a_{0b0} = -(\sigma_{00} - \sigma^2_0)\delta^a_b,$$

where $\delta^{ab}_{cd} = \delta^a_c\delta^b_d - \delta^b_c\delta^a_d$. Plus the relations obtained from those above given with allowance for the classical symmetric properties and reality of the Riemann-Christoffel tensor.

The nonzero components of the Ricci tensor on the associated G-structure space are given by the following relations, see [6]:

$$1)\ S_{oo} = -2n(\sigma_{00} + \sigma^2_0), \qquad 2)\ S_{\hat{a}b} = A^{ac}_{bc} - 2n\delta^a_b\sigma^2_0 - \delta^a_b\sigma_{00}. \tag{2.13}$$

3. Some classes of normal LCAC-manifolds

Let M^{2n+1} be a LCAC-manifold. One of the subclasses of the conformal transformations is conharmonic transformations, these transformations preserve the harmonicity of functions.

A tensor, see [5], which is invariant under conharmonic transformations has the form:

$$\begin{aligned} T(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{2n-1}\{&g(X,W)S(Y,Z) - g(X,Z)S(Y,W) \\ &+ g(Y,Z)S(X,W) - g(Y,W)S(X,Z)\}, \qquad X,Y,Z,W \in X(M) \end{aligned}$$

where R, S and g are respectively the Riemannian curvature tensor, Ricci tensor and Riemannian metric.

The conharmonic curvature tensor of an almost contact metric structure on the bundle space of all frames is calculated by the formula below, see [5]:

$$T^i_{jkl} = R^i_{jkl} - \frac{1}{2n-1}(\delta^i_k S_{jl} - \delta^i_l S_{jk} + g_{jl}S^i_k - g_{jk}S^i_l), \tag{3.1}$$

where R^i_{jkl}, S_{ij} and g_{ij} are respectively the components of the Riemannian curvature tensor, Ricci tensor and Riemannian metric. It is easy to show from (3.1), that the conharmonic curvature tensor possesses all the classical symmetric properties of the Riemann-Christoffel tensor.

On the associated G-structure space, the conharmonic curvature tensor of a normal LCAC-manifold has the following nonzero components, see [6]:

(3.2)

$$1)\ T^a_{obo} = R^a_{obo} - \frac{1}{2n-1}(\delta^a_b S_{oo} + g_{00} S^a_b)$$
$$= \frac{1}{2n-1}\Big[2\delta^a_b\Big\{\Big(n+\frac{1}{2}\Big)\sigma_0^2 + \sigma_{00}\Big\} - A^{ac}_{bc}\Big],$$

$$2)\ T^a_{bc\hat{d}} = R^a_{bc\hat{d}} - \frac{1}{2n-1}(\delta^a_c S_{b\hat{d}} + g_{b\hat{d}} S^a_c)$$
$$= \frac{1}{2n-1}\Big[2\delta^a_c\delta^d_b\Big\{\Big(n+\frac{1}{2}\Big)\sigma_0^2 + \sigma_{00}\Big\} + A^{ad}_{bc}(2n-1) - \delta^d_b A^{ah}_{ch} - \delta^a_c A^{ah}_{bh}\Big],$$

$$3)\ T^a_{\hat{b}cd} = R^a_{\hat{b}cd} - \frac{1}{2n-1}(\delta^a_c S_{\hat{b}d} - \delta^a_d S_{\hat{b}c} + g_{\hat{b}d} S^a_c - g_{\hat{b}c} S^a_d)$$
$$= \frac{1}{2n-1}\Big[2\delta^{ab}_{cd}\Big\{\Big(n+\frac{1}{2}\Big)\sigma_0^2 + \sigma_{00}\Big\} + \delta^b_c A^{ah}_{dh} + \delta^a_d A^{ah}_{ch} - \delta^b_d A^{ah}_{ch} - \delta^a_c A^{bh}_{dh}\Big]$$

plus the components obtained with allowance for the realness and symmetric properties of this tensor as an algebraic curvature tensor. The remaining components of aformentioned tensor are equal to zero.

An almost contact metric manifold is called a conharmonically flat if the conharmonic curvature tensor of such manifold is identically equal to zero.

In [6], it was proved that a conharmonically flat normal LCAC-manifold is a flat cosymplectic manifold. It is known the cosymplectic manifold is locally equivalent to the product of the Kähler manifold by the real line, see [8]. Moreover, using the Hawley and Igusa classification of complete simply connected Kähler manifolds of dimension greater than two of constant holomorphic sectional curvatures, the previous assertion can be formulated as the following theorem.

Theorem 3.1. *A conharmonic plane normal* LCAC-*manifold of constant curvature is locally equivalent to the product of the complex Euclidean space* $\mathbb{C}^n$ *equipped with the standard Hermitian metric* $\langle\langle .,.\rangle\rangle = \mathrm{d}s^2$ *in the canonical atlas given by the relation* $\mathrm{d}s^2 = \sum_{a=2}^n = \mathrm{d}Z^a\,\mathrm{d}\overline{Z}^a$ *by the real line.*

Suppose that M is a conharmonicly flat normal LCAC-manifold. According to the Theorem 1 in [6] , we have $A^{ad}_{bc} = 0$ and $\sigma_{00} = -\big(n+\frac{1}{2}\big)\sigma_0^2$, taking into account (2.13), we get $S_{\hat{a}b} = -\frac{2n-1}{2}\sigma_0^2\delta^a_b$ where $\alpha = -\frac{2n-1}{2}\sigma_0^2$ and $S_{00} = (2n^2-n)\sigma_0^2$, where $\beta = \frac{4n^2-1}{2}\sigma_0^2$. Hence, a manifold is a η-Einstein manifold of type (α,β).

Thus, the following assertion is valid.

Theorem 3.2. *Let M be a conharmonicly normal flat* LCAC*-manifold, then it is an η-Einstein manifold of type (α, β), where $\alpha = -\frac{2n-1}{2}\sigma_0^2$, $\beta = -\frac{4n^2-1}{2}\sigma_0^2$.*

Let us clarify the geometric meaning of the vanishing of the individual components of the conharmonic curvature tensor.

1) Let $T^a_{0b0} = 0$, then applying the procedure of restoring identity in [8], [10] to the relations $T^a_{0b0} = 0$, $T^{\hat{a}}_{0b0} = 0$, $T^0_{0b0} = 0$, i.e. $T^i_{0b0} = 0$, we obtain $T(\Phi X, \xi)\xi = 0$, for all $X \in X(M)$. By virtue of the identity (2.1) (4), the last equation can be written in the form:

$$T(X, \xi)\xi = 0, \qquad \forall\, X \in X(M). \tag{3.3}$$

Conversely, if (3.3) holds, then the relation $T^a_{0b0} = 0$ holds.

Thus, the vanishing of the component $T^a_{0b0} = 0$ is equivalent to the fulfillment of the identity (3.3). This conclusion justifies the introduction of the following definition.

Definition 3.1. A normal LCAC-manifold whose conharmonic curvature tensor satisfies the identity (3.3) is called a normal LCAC-manifold of class T_1.

Let M be a normal LCAC-manifold of class T_1, then the identity (3.3) holds, that is, $T^a_{0b0} = 0$. Taking into account (3.2) 1), (2.12) 3), and (2.13) 1), so we have $S_{\hat{a}b} = (\sigma_0^2 + \sigma_{00})\delta^a_b = \alpha\delta^a_b = A^{ac}_{bc} - 2n\delta^a_b\sigma_0^2 - \delta^a_b\sigma_{00}$, where $\alpha = (\sigma_0^2 + \sigma_{00})$. Furthermore, $S_{00} = -2n(\sigma_0^2 + \sigma_{00}) = \alpha + \beta$, where $\beta = -(2n+1)(\sigma_0^2 + \sigma_{00})$. Thus, a manifold M is η-Einstein manifold of type (α, β).

Conversely, let M be η-Einstein normal LCAC-manifold. Then on the associated G-structure space, the following relations, see [6], hold: $\alpha = \frac{1}{n}A^{ab}_{ab} - 2n\sigma_0^2 - \sigma_{00}$, $\beta = -\frac{1}{n}A^{ab}_{ab} - (2n-1)\sigma_{00}$ i.e. $\alpha\delta^a_b = A^{ac}_{bc} - 2n\delta^a_b\sigma_0^2 - \delta^a_b\sigma_{00}$, $\alpha + \beta = -2n(\sigma_0^2 + \sigma_{00})$. In this case $T^a_{0b0} = -\delta^a_b(\sigma_0^2 + \sigma_{00}) - \frac{1}{2n-1}\{-2n(\sigma_0^2 + \sigma_{00})\delta^a_b + (\sigma_0^2 + \sigma_{00})\} = 0$.

Thus, the following theorem is proved.

Theorem 3.3. *If a normal* LCAC*-manifold of class T_1 is a manifold of constant curvature, then it is locally equivalent to the product of the complex Euclidean space $\mathbb{C}^n$ equipped with the standard Hermitian metric $\langle\langle ., . \rangle\rangle = \mathrm{d}s^2$ in the canonical atlas given by the relation $\mathrm{d}s^2 = \sum_{a=2}^{n} = \mathrm{d}Z^a\, \mathrm{d}\overline{Z}^a$ by the real line.*

2) Suppose that $T^a_{bcd} = 0$. Applying the identity renewal procedure, see [8], [10], to the relations $T^a_{bc\hat{d}} = 0$, $T^{\hat{a}}_{bc\hat{d}} = 0$, $T^0_{bc\hat{d}} = 0$, i.e. to the relations $T^i_{bc\hat{d}} = 0$, we get

$$\begin{aligned} T(\Phi^2 X, \Phi^2 Y)\Phi^2 Z + T(\Phi^2 X, \Phi Y)\Phi Z - T(\Phi X, \Phi^2 Y)\Phi Z \\ + T(\Phi X, \Phi Y)\Phi^2 Z = 0, \qquad \forall\, X, Y, Z \in X(M). \end{aligned} \tag{3.4}$$

Suppose that the conharmonic curvature tensor of a normal LCAC-manifold satisfies the identity (3.4). On the space of the principal bundle of frames $B(M^{2n+1})$, the identity (3.4) can be written in the form:

$$T^i_{jkl}\Phi^j_h\Phi^h_r\Phi^k_m\Phi^m_p\Phi^l_s\Phi^s_q+T^i_{jkl}\Phi^j_r\Phi^k_m\times\Phi^m_p\Phi^l_q-T^i_{jkl}\Phi^j_r\Phi^k_P\Phi^l_s\Phi^s_q+T^i_{jkl}\Phi^j_h\Phi^h_r\Phi^k_p\Phi^{l^s_q}_q=0.$$

Taking into account (2.2), (3.2) on on the associated G-structure space, so that the last relation can be written in the form:

$$4T^a_{bc\hat{d}}+4T^{\hat{a}}_{\hat{b}\hat{c}d}=0, \qquad \text{i.e. } T^a_{bc\hat{d}}=0,\ T^{\hat{a}}_{\hat{b}\hat{c}d}=0.$$

Thus, the identity (3.4) is equivalent to the fulfillment of the relation $T^a_{bc\hat{d}}=0$ on the space of the associated G-structure, which gives grounds to introduce the following definition.

Definition 3.2. A normal LCAC-manifold whose conharmonic curvature tensor satisfies the identity (3.4) is called a normal LCAC-manifold of class T_2.

Let M be a normal LCAC-manifold of class T_2, then the identity (3.4) holds, that is, $T^a_{bc\hat{d}}=0$. Taking into account (3.2) 2), (2.12) 1), and (2.13) 2), we have:

$$\begin{aligned}A^{ad}_{bc}-\delta^a_c\delta^d_b\sigma^2_0-\frac{1}{2n-1}\{&\delta^a_c(A^{dh}_{bh}-2n\delta^d_b\sigma^2_0-\delta^d_b\sigma_{00})\\&+\delta^d_b(A^{ah}_{ch}-2n\delta^a_c\sigma^2_0-\delta^a_c\sigma_{00})\}=0.\end{aligned} \tag{3.5}$$

We reduce equality (3.5) with respect to the indices a and b, then we obtain:

$$A^{ad}_{ac}-\delta^d_c\sigma^2_0-\frac{2}{2n-1}(A^{dh}_{ch}-2n\delta^d_c\sigma^2_0-\delta^d_c\sigma_{00})=0,$$

which is equivalent to

$$A^{bc}_{ac}=-\frac{2}{2n-3}\delta^a_b\sigma_{00}-\frac{2n-1}{2n-3}\delta^a_b\sigma^2_0. \tag{3.6}$$

As in Theorem 3.2, the following theorem is proved.

Theorem 3.4. *A normal* LCAC*-manifold is a manifold of class T_2 if and only if it is an η-Einstein manifold of type (α,β), where $\alpha=-\frac{2n-1}{2n-3}\sigma_{00}-\frac{(2n+1)^2}{2n-3}\sigma^2_0$, $\beta=-\frac{4n2-8n+1}{2n-3}\sigma_{00}+\frac{2n+1}{2n-3}\sigma^2_0$.*

We reduce equality (3.6) with respect to the indices a and b, then we obtain $A^{ab}_{ab}=-\frac{2n}{2n-3}\sigma_{00}-\frac{(2n+1)n}{2n-3}\sigma^2_0$. The scalar curvature is $\chi=-\frac{8n(n-1)}{2n-3}\sigma_{00}+\frac{4n(2n^2-5n+1)n}{(2n-3)}\sigma^2_0$.

Substituting (3.6) into (3.5), we obtain, that the components of the structure tensor of the second kind on the associated G-structure space have the form:

$$A^{ad}_{bc} = -\frac{4n^2-1}{(2n-1)(2n-3)}\delta^a_c\delta^d_b\sigma^2_0 - \frac{4(n-1)}{(2n-1)(2n-3)}\delta^a_c\delta^d_b\sigma_{00}. \tag{3.7}$$

3) Analogously, considering the equalities $T^a_{\hat{b}cd} = 0$, $T^{\hat{a}}_{\hat{b}cd} = 0$, $T^0_{\hat{b}cd} = 0$, we obtain

$$\begin{aligned} T(\Phi^2X,\Phi^2Y)\Phi^2Z + T(\Phi^2X,\Phi Y)\Phi Z - T(\Phi X,\Phi^2Y)\Phi Z \\ + T(\Phi X,\Phi Y)\Phi^2 Z = 0, \qquad \forall\, X,Y,Z\in X(M). \end{aligned} \tag{3.8}$$

As above, we show that the identity (3.8) on the space of the associated G-structure is equivalent to the relations $T^a_{\hat{b}cd} = 0$.

Therefore, we introduce the following definition.

Definition 3.3. A normal LCAC-manifold whose conharmonic curvature tensor satisfies the identity (3.8) is called a normal LCAC-manifold of class T_3.

Remark 3.1. By virtue of the symmetric properties of the conharmonic curvature tensor as an algebraic tensor, the normal LCAC-manifold of class T_2 is a normal LCACs-manifold of class T_3. In fact, if $T_2 = 0$, that is, $T^a_{bc\hat{d}} = 0$, then $T^a_{\hat{b}cd} = T^a_{cd\hat{b}} - T^a_{d\hat{b}c} = -T^a_{cd\hat{b}} + T^a_{dc\hat{b}}0$, which means, $T_3 = 0$.

Thus, consideration of class T_3 is of independent interest.

Let M be a normal LCAC-manifold of class T_3. Then the identity (3.8) is satisfied, which is equivalent to the relation

$$R^a_{\hat{b}cd} = \frac{1}{2n-1}(\delta^a_c S_{\hat{b}d} - \delta^a_d S_{\hat{b}c} + g_{\hat{b}d}S^a_c - g_{\hat{b}c}S^a_d).$$

The last equality in view of (2.12) 2) and (2.13) can be written as:

$$S^{\hat{a}}_{\hat{b}} = \Big(\frac{3n-1}{n-2}\sigma^2_0 + \frac{n}{n-2}\sigma_{00} - \frac{1}{n-2}A^{cd}_{cd}\Big)\delta^a_b.$$

Then, arguing as in the proof of Theorem 3.2, we can prove the following theorem.

Theorem 3.5. *A normal* LCAC-*manifold is a manifold of class* T_3 *if and only if it is an* η*-Einstein manifold of type* (α,β), *where* $\alpha = \frac{3n-1}{n-2}\sigma^2_0 + \frac{n}{n-2}\sigma_{00} - \frac{1}{n-2}A^{cd}_{cd}$, $\beta = \frac{1}{n-2}A^{ab}_{ab} - \frac{n(2n-3)}{n-2}\sigma_{00} - \frac{(2n-1)(n-1)}{n-2}\sigma^2_0$.

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