Mathematica Bohemica

Saeide Zahiri; Arsham Borumand Saeid
An investigation on the n-fold IVRL-filters in triangle algebras

Mathematica Bohemica, Vol. 145 (2020), No. 1, 75–91

Persistent URL: `http://dml.cz/dmlcz/148066`

AN INVESTIGATION ON THE n-FOLD IVRL-FILTERS IN TRIANGLE ALGEBRAS

Saeide Zahiri, Eghlid, Arsham Borumand Saeid, Kerman

Received September 13, 2017. Published online March 25, 2019.
Communicated by Radomír Halaš

Abstract. The present study aimed to introduce n-fold interval valued residuated lattice (IVRL for short) filters in triangle algebras. Initially, the notions of n-fold (positive) implicative IVRL-extended filters and n-fold (positive) implicative triangle algebras were defined. Afterwards, several characterizations of the algebras were presented, and the correlations between the n-fold IVRL-extended filters, n-fold (positive) implicative algebras, and the Gödel triangle algebra were discussed.

Keywords: interval-valued structure; triangle algebra; interval valued residuated lattice filter; n-fold interval valued residuated lattice extended filter

MSC 2010: 08A72, 08A30, 03B50, 03B52

1. Introduction

Various logical algebras have been proposed as semantic systems of non-classical logic systems. Such examples are residuated lattices, BL-algebras, Gödel algebras, and MTL-algebras. Among these logical algebras, residuated lattices are especially basic and contain important algebraic structures since the other logical algebras are all particular cases of residuated lattices. Therefore, residuated lattices are considered to be a fundamental concept of algebraic structures, which have been investigated by Dilworth, and Ward et al. (see [2], [6]).

Triangle algebras are remarkable examples of residuated lattice structures. In a study, Van Gasse et al. introduced the notion of triangle algebras as a variety of residuated lattices equipped with two modals or the approximation operators ν and μ, together with u as a third angular point, which differs from 0 (false) and 1 (true). In the mentioned research (Theorem 26), the findings indicated that these algebras serve as an equational representation of interval-valued residuated lattices

DOI: 10.21136/MB.2019.0104-17

(IVRLs). Based on the definition and properties of triangle algebras, the authors defined triangle logic (TL), demonstrating that this logic is sound and complete considering the variety of triangle algebras (see [4]). The same researchers introduced the notion of IVRL-filters in triangle algebras, defining a few types of IVRL-filters (e.g. Boolean and prime filters) and discussing their notable properties (see [5]).

The n-fold filter theory of residuated lattices and BL-algebras has been previously investigated, yielding important finding. Triangle algebras and BL-algebras are among the foremost logical algebras, which motivated us to study the notion of n-fold filters in triange algebras.

The present study aimed to assess the properties of triangle algebras. Triangle algebras differ from other algebraic structures. Triangle algebras play a key role in fuzzy logics and the associated algebraic structures. Furthermore, the filter theory is of paramount importance in studying this type of algebra. Since n-fold IVRL-extended filters enable the classification of triangle algebras, they are of particular importance. Due to the operations ν and μ, triangle algebras are remarkable algebraic structures. The differences between triangle algebras and other algebra types have caused the definition of n-fold IVRL-extended filters in this algebra to differ from the definition of n-fold filters in other algebraic structures. For instance, the role of operation ν that causes the IVRL-filter conditions is essential to the definition of n-fold IVRL-extended (positive) implicative filters. Continuing our study on algebraic structures, especially triangle algebras, we defined n-fold (positive) implicative IVRL-extended filters and n-fold (positive) implicative triangle algebras. Afterward, some of their properties were provided. Additionally, we considered the correlations between the mentioned n-fold IVRL-extended filters. In Section 2 of this paper, we present some of the definitions, lemmas and theorems required for the sequel. In Section 3, we define n-fold IVRL-extended positive implicative filters in triangle algebras as well as some of their properties. Moreover, the concept of n-fold positive implicative triangle algebras is introduced, and some of the related results are proved as well. In this regard, it is demonstrated that F is an n-fold IVRL-extended positive implicative filter if and only if $\nu x \vee \neg(\nu x^n) \in F$. In Section 4, the notion of n-fold IVRL-extended implicative filters and n-fold implicative triangle algebras are introduced. It will be proven that if A is associated with Gödel triangle algebra, $\{1\}$ is an n-fold IVRL-extended implicative IVRL-filter of A. Furthermore, it will be denoted that they are equivalent under specific cirumstances. Therefore, the aforementioned finding will be exploited to determine the classification of this structure.

Definition 2.1 ([1]). A residuated lattice is an algebra $\mathcal{L} = (L, \vee, \wedge, *, \to, 0, 1)$ with four binary operations and two constants 0, 1 such that:

▷ $(L, \vee, \wedge, 0, 1)$ is a bounded lattice with 0 as the smallest and 1 as the greatest element,
▷ $*$ is commutative and associative with 1 as neutral element, and
▷ $x * y \leqslant z$ if and only if $x \leqslant y \to z$ for all x, y and z in L (residuation principle).

The ordering $\leqslant$ and negation $\neg$ in a residuated lattice $\mathcal{L} = (L, \vee, \wedge, *, \to, 0, 1)$ are defined as follows: for all x and y in L: $x \leqslant y$ if and only if $x \wedge y = x$ (or equivalently, if and only if $x \vee y = y$; or, also equivalently, if and only if $x \to y = 1$), $\neg x = x \to 0$, $x \leftrightarrow y = (x \to y) \wedge (y \to x)$ and $x^n = \underbrace{x * \ldots * x}_{n \text{ times}}$.

Lemma 2.2 ([5], [2]). *Let $(L, \vee, \wedge, *, \to, 0, 1)$ be a residuated lattice. Then the following properties are valid for all x, y and z in L:*
(1) $x * y \leqslant x \wedge y$, $x \leqslant y \to x$,
(2) $x \vee y \leqslant (y \to x) \to x$,
(3) $x \to (y \to z) = y \to (x \to z)$,
(4) $x \leqslant y$ *implies* $x * z \leqslant y * z$, $z \to x \leqslant z \to y$, $y \to z \leqslant x \to z$,
(5) $x \to (y \to z) = (x * y) \to z = y \to (x \to z)$,
(6) $x * \neg x = 0$,
(7) $(y_1 \to x_1) * (x_2 \to y_2) \leqslant (x_1 \to x_2) \to (y_1 \to y_2)$,
(8) $(x_1 \to y_1) * (x_2 \to y_2) \leqslant (x_1 * x_2) \to (y_1 * y_2)$,
(9) $(x \to y) * (y \to z) \leqslant x \to z$,
(10) $x \leqslant y \to (x * y), x * (x \to y) \leqslant y$.

Definition 2.3 ([4]). Given a lattice $\mathcal{A} = (A, \vee, \wedge)$, its triangularization $\mathbb{T}(\mathcal{A})$ is the structure $\mathbb{T}(\mathcal{A}) = (\mathrm{Int}(\mathcal{A}), \vee, \wedge)$ defined by

▷ $\mathrm{Int}(\mathcal{A}) = \{[x_1, x_2] \colon (x_1, x_2) \in A^2 \text{ and } x_1 \leqslant x_2\}$,
▷ $[x_1, x_2] \wedge [y_1, y_2] = [x_1 \wedge y_1, x_2 \wedge y_2]$,
▷ $[x_1, x_2] \vee [y_1, y_2] = [x_1 \vee y_1, x_2 \vee y_2]$.

The set $D_{\mathcal{A}} = \{[x, x] \colon x \in A\}$ is called the diagonal of $\mathbb{T}(\mathcal{A})$.

Definition 2.4 ([4]). An interval-valued residuated lattice (IVRL) is a residuated lattice $(\mathrm{Int}(\mathcal{A}), \vee, \wedge, \odot, \to_\odot, [0, 0], [1, 1])$ on the triangularization $\mathbb{T}(\mathcal{A})$ of a bounded lattice $\mathcal{A}$, in which the diagonal $D_{\mathcal{A}}$ is closed under $\odot$ and $\to_\odot$, i.e. $[x, x] \odot [y, y] \in D_{\mathcal{A}}$ and $[x, x] \to_\odot [y, y] \in D_{\mathcal{A}}$ for all x, y in A.

In triangle algebra $\mathcal{A} = (A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$, operators ν ('necessity') and μ ('possibility') are modal operators, and u ('uncertainty', $u \neq 0$, $u \neq 1$) is a new constant. It turns out that triangle algebras are the equational representations of interval-valued residuated lattices (IVRLs).

Definition 2.5 ([4]). A triangle algebra is a structure $\mathcal{A} = (A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$ in which $(A, \vee, \wedge, *, \to, 0, 1)$ is a residuated lattice, ν and μ are unary operators on A, u is a constant, and which satisfies the following conditions:

(T.1) $\nu x \leqslant x$,
(T.2) $\nu x \leqslant \nu\nu x$,
(T.3) $\nu(x \wedge y) = \nu x \wedge \nu y$,
(T.4) $\nu(x \vee y) = \nu x \vee \nu y$,
(T.5) $\nu u = 0$,
(T.6) $\nu\mu x = \mu x$,
(T.7) $\nu(x \to y) \leqslant \nu x \to \nu y$,
(T.8) $(\nu x \leftrightarrow \nu y) * (\mu x \leftrightarrow \mu y) \leqslant (x \leftrightarrow y)$,
(T.9) $\nu x \to \nu y \leqslant \nu(\nu x \to \nu y)$.

(T.1′) $x \leqslant \mu x$,
(T.2′) $\mu\mu x \leqslant \mu x$,
(T.3′) $\mu(x \wedge y) \leqslant \mu x \wedge \mu y$,
(T.4′) $\mu(x \vee y) \leqslant \mu x \vee \mu y$,
(T.5′) $\mu u = 1$,
(T.6′) $\mu\nu x = \nu x$,

Theorem 2.6 ([4]). *There is a one-to-one correspondence between the class of IVRLs and the class of triangle algebras. Every extended IVRL is a triangle algebra and conversely, every triangle algebra is isomorphic to an extended IVRL.*

Definition 2.7 ([7]). A triangle algebra A is called a Gödel-triangle algebra (G-triangle algebra) if $x^2 = x$ for all $x \in A$.

Definition 2.8 ([5]). Let $\mathcal{A} = (A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$ be a triangle algebra. An element x in A is called exact if $\nu x = x$. The set of exact elements of $\mathcal{A}$ is denoted by $E(\mathcal{A})$.

It was proved in [4] that $E(\mathcal{A})$ is closed under all defined operations on $\mathcal{A}$. We denote the subalgebra $(E(\mathcal{A}), \vee, \wedge, *, \to, 0, 1)$ which is a residuated lattice by $\mathcal{E}(\mathcal{A})$.

Proposition 2.9 ([3]). *In a triangle algebra $(A, \vee, \wedge, \to, *, \nu, \mu, 0, u, 1)$, the following identity and inequality hold for every x, y and z in A:*
(i) $\nu(x * y) = \nu x * \nu y$.
(ii) $\mu(x * y) \leqslant \mu x * \mu y$.

By the above proposition we have $\nu(x^n) = (\nu x)^n = \nu x^n$.

Definition 2.10 ([5]). An IVRL-filter (IF) of triangle algebra $\mathcal{A} = (A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$ is a nonempty subset F of A satisfying:
(F.1) if $x \in F$, $y \in A$ and $x \leqslant y$, then $y \in F$,

(F.2) if $x, y \in F$, then $x * y \in F$,

(F.3) if $x \in F$, then $\nu x \in F$.

An alternative definition for an IVRL-filter F of a triangle algebra $\mathcal{A} = (A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$ is the following:

▷ $1 \in F$,

▷ for all x and y in A: if $x \in F$ and $x \to y \in F$, then $y \in F$,

▷ if $x \in F$, then $\nu x \in F$.

Notice that, because of (F.1), (F.3) and (T.1), we have

(F.3′) $x \in F$ if and only if $\nu x \in F$.

There is an obvious connection between the notion "IVRL-filter of triangle algebra" and "filter of a residuated lattice", which is given in the next proposition.

Proposition 2.11 ([5]). *Let $\mathcal{A}$ be a triangle algebra, $\mathcal{E}(A) = (E(A), \vee, \wedge, *, \to, 0, 1)$ be its subalgebra of exact elements and $F \subseteq A$. Then F is a filter of the triangle algebra $\mathcal{A}$ if and only if* (F.3′) *holds and $F \cap E(A)$ is a filter of the residuated lattice $\mathcal{E}(A)$.*

Proposition 2.11 suggests two different ways to define specific kinds of IVRL-filters of triangle algebras. The first is to impose a property on a filter of the subalgebra of exact elements and extend this filter to the whole triangle algebra, using (F.3′). We call these IVRL-extended filters. For example, an IVRL-extended implicative filter of triangle algebra $\mathcal{A} = (A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$ is a subset F of A such that $F \cap E(\mathcal{A})$ is an implicative filter of $\mathcal{E}(\mathcal{A})$ and $x \in F$ if and only if $\nu x \in F \cap \mathcal{E}(\mathcal{A})$.

The second way is to impose a property on the whole IVRL-filter. For example, an implicative IVRL-filter of a triangle algebra $\mathcal{A} = (A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$ is an IVRL-filter of A such that F is an implicative filter of $(A, \vee, \wedge, *, \to, 0, 1)$, see [5].

Definition 2.12. A triangle algebra A is called a divisible triangle algebra if $x \wedge y = x * (x \to y)$ for all $x, y \in A$.

Lemma 2.13. $\neg((x \vee \neg(x^n))^n) \to \neg(x^n) = 1$ *for all x in triangle algebra $(A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$.*

P r o o f. By Lemma 2.2, we have

$$\begin{aligned}\neg((x \vee \neg(x^n))^n) \to \neg(x^n) &\geqslant x^n \to (x \vee \neg(x^n))^n \\ &\geqslant [x \to (x \vee \neg(x^n))]^n = 1.\end{aligned}$$

So $\neg((x \vee \neg(x^n))^n) \to \neg(x^n) = 1$. □

From now on, $(A, \vee, \wedge, *, \rightarrow, \nu, \mu, 0, u, 1)$ or simply A is a triangle algebra and n is a natural number.

Definition 3.1. An IVRL-filter F of A is called n-fold IVRL-extended positive implicative filter of A if it satisfies: $1 \in F$ and $\nu x \rightarrow ((\nu y^n \rightarrow \nu z) \rightarrow \nu y)$, $\nu x \in F$, implies $\nu y \in F$ for all $x, y, z \in A$.

The IVRL-filter condition is essential in the above definition as the following example shows:

Example 3.2. Let $A = \{0, u, 1\}$ be a chain. We define operations ν, μ, $*$, $\rightarrow$ as follows:

x	νx
0	0
u	0
1	1

x	μx
0	0
u	1
1	1

$*$	0	u	1
0	0	0	0
u	0	u	u
1	0	u	1

$\rightarrow$	0	u	1
0	1	1	1
u	0	1	1
1	0	u	1

Then $\mathcal{A} = (A, \vee, \wedge, *, \rightarrow, \nu, \mu, 0, u, 1)$ is a triangle algebra. Clearly, $F = \{u, 1\}$ meets the condition of n-fold IVRL-extended positive implicative filter of A. Since $\nu u = 0 \notin F$, F is not an IVRL-filter of A.

Theorem 3.3. *For all $x, y \in A$, the following conditions are equivalent:*

(i) *F is an n-fold IVRL-extended positive implicative filter,*

(ii) *$(\nu x^n \rightarrow 0) \rightarrow \nu x \in F$ implies $\nu x \in F$,*

(iii) *$(\nu x^n \rightarrow \nu y) \rightarrow \nu x \in F$ implies $\nu x \in F$.*

Proof. (i) $\Rightarrow$ (iii): Let F be an n-fold IVRL-extended positive implicative filter of A and $(\nu x^n \rightarrow \nu y) \rightarrow \nu x \in F$. Since $1 \rightarrow ((\nu x^n \rightarrow \nu y) \rightarrow \nu x) = ((\nu x^n \rightarrow \nu y) \rightarrow \nu x)$, one has $1 \rightarrow ((\nu x^n \rightarrow \nu y) \rightarrow \nu x) \in F$. Since $1 \in F$, we infer $\nu x \in F$.

(iii) $\Rightarrow$ (ii): It is clear.

(ii) $\Rightarrow$ (i): Let $\nu x \rightarrow ((\nu y^n \rightarrow \nu z) \rightarrow \nu y) \in F$ and $\nu x \in F$. Since F is an IVRL-filter, $(\nu y^n \rightarrow \nu z) \rightarrow \nu y \in F$. Lemma 2.2 gives $\nu y^n \rightarrow 0 \leqslant \nu y^n \rightarrow \nu z$. So, $(\nu y^n \rightarrow \nu z) \rightarrow \nu y \leqslant (\nu y^n \rightarrow 0) \rightarrow \nu y$. Hence $(\nu y^n \rightarrow 0) \rightarrow \nu y \in F$ and thus $\nu y \in F$. Therefore F is an n-fold IVRL-extended positive implicative filter of A. □

Proposition 3.4. *If F is an n-fold IVRL-extended positive implicative filter, then F is an $(n+1)$-fold IVRL-extended positive implicative filter.*

P r o o f. Let F be an n-fold IVRL-extended positive implicative filter and $x \in A$ be such that $(\nu x^{n+1} \to 0) \to \nu x \in F$. Since $\nu x^{n+1} \leqslant \nu x^n$, we have $\nu x^n \to 0 \leqslant \nu x^{n+1} \to 0$ and so $(\nu x^{n+1} \to 0) \to \nu x \leqslant (\nu x^n \to 0) \to \nu x$. Since F is an IVRL-filter, $(\nu x^n \to 0) \to \nu x \in F$. Since F is an n-fold IVRL-extended positive implicative filter, $\nu x \in F$. Thus, F is an $(n+1)$-fold IVRL-extended positive implicative filter. $\square$

In the following example we show that the converse of the above proposition is not true.

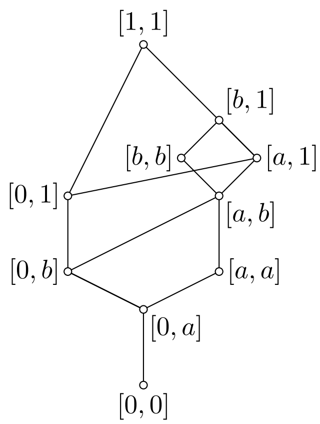
E x a m p l e 3.5. Let $A = \{[0,0],[0,a],[0,b],[a,a],[a,b],[b,b],[0,1],[a,1],[b,1],[1,1]\}$. Define $\odot$ and $\Rightarrow$ as:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

$\Rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Now, we define ν, μ, $*$ and $\to$ one as:

$$\nu[x_1,x_2]=[x_1,x_1],\quad \mu[x_1,x_2]=[x_2,x_2],\quad [x_1,x_2]*[y_1,y_2]=[x_1\odot y_1, x_2\odot y_2],$$
$$[x_1,x_2]\to[y_1,y_2]=[(x_1\Rightarrow y_1)\wedge(x_2\Rightarrow y_2), x_2\Rightarrow y_2].$$



Then $(A,\vee,\wedge,*,\to,\nu,\mu,[0,0],[0,1],[1,1])$ is a triangle algebra with $[0,0]$ as the smallest element and $[1,1]$ as the greatest element. It is clear that $\{[1,1]\}$ is a 3-fold IVRL-extended positive implicative filter of A, but $(\nu[b,1]^2 \to [0,0]) \to \nu[b,1] = [1,1] \neq [b,b]$. Thus, $\{[1,1]\}$ is not a 2-fold IVRL-extended positive implicative filter of A.

Definition 3.6. A is called n-fold positive implicative triangle algebra if $(\nu x^n \to 0) \to \nu x = \nu x$ for all $x \in A$.

Example 3.7. Let $A = \{0, u, 1\}$ be a chain. We define operations ν, μ, $*$, $\to$ as:

x	νx
0	0
u	0
1	1

x	μx
0	0
u	1
1	1

$*$	0	u	1
0	0	0	0
u	0	0	u
1	0	u	1

$\to$	0	u	1
0	1	1	1
u	u	1	1
1	0	u	1

$\mathcal{A} = (A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$ is a triangle algebra. It is clear that A is an n-fold positive implicative triangle algebra for all $n \in \mathbb{N}$.

Example 3.8. In Example 3.5, A is not a 2-fold positive implicative triangle algebra.

Proposition 3.9. *Let F be an IVRL-filter of A. Then the following conditions are equivalent:*

(i) *A is an n-fold positive implicative triangle algebra,*

(ii) *$\{1\}$ is an n-fold IVRL-extended positive implicative filter of A.*

Proof. (i) $\Rightarrow$ (ii): By the definition of the n-fold positive implicative triangle algebra and Theorem 3.3, $\{1\}$ is an n-fold IVRL-extended positive implicative filter.

(ii) $\Rightarrow$ (i): Consider $x \in A$ and $z = ((\nu x^n \to 0) \to \nu x) \to \nu x$. By Lemma 2.2 we have

$$\begin{aligned}(z^n \to 0) \to z &= (z^n \to 0) \to (((\nu x^n \to 0) \to \nu x) \to \nu x)\\ &= ((\nu x^n \to 0) \to \nu x) \to ((z^n \to 0) \to \nu x)\\ &\geqslant (z^n \to 0) \to (\nu x^n \to 0) \geqslant \nu x^n \to z^n.\end{aligned}$$

Since $\nu x \leqslant ((\nu x^n \to 0) \to \nu x) \to \nu x = z$, one has $\nu x^n \leqslant z^n$. So, $\nu x^n \to z^n = 1$. Hence $(z^n \to 0) \to z = 1 \in \{1\}$, since $\{1\}$ is an n-fold IVRL-extended positive implicative filter, $z = ((\nu x^n \to 0) \to \nu x) \to \nu x = 1$, so $(\nu x^n \to 0) \to \nu x \leqslant \nu x$. By Lemma 2.2, we have $(\nu x^n \to 0) \to \nu x \geqslant \nu x$. Hence $(\nu x^n \to 0) \to \nu x = \nu x$ for all $x \in A$. That is, A is an n-fold IVRL-extended positive implicative triangle algebra. $\square$

Theorem 3.10. *Let F be an IVRL-filter of A. Then A/F is an n-fold IVRL-extended positive implicative triangle algebra if and only if F is an n-fold IVRL-extended positive implicative filter.*

Proof. Let F be an n-fold IVRL-extended positive implicative filter and $x \in A$ be such that $([\nu x]^n \to [0]) \to [\nu x] = [1]$. Then $[(\nu x^n \to 0) \to \nu x] = ([\nu x]^n \to [0]) \to [\nu x] = [1]$. Thus $(\nu x^n \to 0) \to \nu x \in F$. Since F is an n-fold IVRL-extended positive implicative filter by Theorem 3.3, $\nu x \in F$. Hence $[\nu x] = [1]$. So $\{[1]\}$ is an n-fold IVRL-extended positive implicative filter of A/F. Proposition 3.9 gives that A/F is an n-fold positive implicative triangle algebra.

Conversely, let A/F be an n-fold positive implicative triangle algebra and let $x \in A$ be such that $(\nu x^n \to 0) \to \nu x \in F$. Then $[\nu x] = ([\nu x]^n \to [0]) \to [\nu x] = [(\nu x^n \to 0) \to \nu x] = [1]$. Hence $[\nu x] = [1]$, that is, $\nu x \in F$. It follows from Theorem 3.3 that F is an n-fold IVRL-extended positive implicative filter. □

Proposition 3.11. *Let F_1, F_2 be IVRL-filters of A such that $F_1 \subseteq F_2$. If F_1 is an n-fold IVRL-extended positive implicative filter of A, then so is F_2.*

P r o o f. Let $x \in A$ be such that $(\nu x^n \to 0) \to \nu x \in F_2$. Since F_1 is an n-fold IVRL-extended positive implicative filter, Theorem 3.10 gives that A/F_1 is an n-fold positive implicative triangle algebra. Thus $[(\nu x^n \to 0) \to \nu x] = ([\nu x]^n \to [0]) \to [\nu x] = [\nu x]$. Hence $((\nu x^n \to 0) \to \nu x) \to \nu x \in F_1 \subseteq F_2$. Since F_2 is an IVRL-filter and $(\nu x^n \to 0) \to \nu x \in F_2$, we infer that $\nu x \in F_2$. Hence, by Theorem 3.3, F_2 is an n-fold IVRL-extended positive implicative filter. □

By Proposition 3.9 and Proposition 3.11, we have:

Corollary 3.12. $\{1\}$ *is an n-fold IVRL-extended positive implicative filter of A if and only if every IVRL-filter of A is an n-fold IVRL-extended positive implicative filter of A.*

Proposition 3.13. *Let $\nu x \vee \neg(\nu x^n) \in F$ for all x in A. Then F is an n-fold IVRL-extended positive implicative filter of A.*

P r o o f. Let $\nu x \vee \neg(\nu x^n) \in F$ for all x in A. Then by Lemma 2.2 (2),

$$\nu x \vee \neg(\nu x^n) \leqslant [\neg(\nu x^n) \to \nu x] \to \nu x.$$

So we have $[\neg(\nu x^n) \to \nu x] \to \nu x \in F$. If $\neg(\nu x^n) \to \nu x \in F$, then $\nu x \in F$. Thus F is an n-fold IVRL-extended positive implicative filter of A. □

Lemma 3.14. *Let F be an IVRL-filter of A. Then the following assertions are equivalent:*

(i) $\nu x \vee \neg(\nu x^n) \in F$ *for all x in A,*

(ii) $\nu x \vee (\nu x^n \to \nu y) \in F$.

P r o o f. (i) $\Rightarrow$ (ii): Let $\nu x \vee \neg(\nu x^n) \in F$. Lemma 2.2 gives $\nu x \vee \neg(\nu x^n) \leqslant \nu x \vee (\nu x^n \to y)$. Hence $\nu x \vee (\nu x^n \to \nu y) \in F$.

(ii) $\Rightarrow$ (i): It follows immediately by taking $y = 0$. □

Theorem 3.15. *Let F be an IVRL-filter of A. Then the following conditions are equivalent:*

(i) *F is an n-fold IVRL extended positive implicative filter,*

(ii) $(\neg(\nu x^n) \to \nu x) \to \nu x \in F$,

(iii) $(\neg(\nu x^n) \to \nu y) \to [(\nu y \to \nu x) \to \nu x] \in F$,

(iv) $(\neg((\nu x \vee \nu y)^n) \to \nu y) \to (\nu y \to \nu x) \in F$.

Proof. (i) $\Leftrightarrow$ (ii): Let F be an n-fold IVRL extended positive implicative filter. It follows from $\nu x \leqslant (\neg(\nu x^n) \to \nu x) \to \nu x$ and Lemma 2.2 (4) with $y = 0$ that $\neg(((\neg(\nu x^n) \to \nu x) \to \nu x)^n) \leqslant \neg(\nu x^n)$. Using Lemma 2.2 (4) with $y = (\neg(\nu x^n) \to \nu x) \to \nu x$ and Lemma 2.2 (5), we get $1 = (\neg(\nu x^n) \to \nu x) \to (\neg(\nu x^n) \to \nu x) = \neg(\nu x^n) \to ((\neg(\nu x^n) \to \nu x) \to \nu x) \leqslant \neg(((\neg(\nu x^n) \to \nu x) \to \nu x)^n) \to ((\neg(\nu x^n) \to \nu x) \to \nu x)$. Hence $\neg(((\neg(\nu x^n) \to \nu x) \to \nu x)^n) \to ((\neg(\nu x^n) \to \nu x) \to \nu x) \in F$. Theorem 3.3 (ii) gives $(\neg(\nu x^n) \to \nu x) \to \nu x \in F$.

The converse is obvious.

(ii) $\Leftrightarrow$ (iii): According to Lemma 2.2 we have $(\neg(\nu x^n) \to \nu y) * (\nu y \to \nu x) \leqslant \neg(\nu x^n) \to \nu x$ and $(\neg(\nu x^n) \to \nu x) \to \nu x \leqslant [(\neg(\nu x^n) \to \nu y) * (\nu y \to \nu x)] \to \nu x = (\neg(\nu x^n) \to \nu y) \to [(\nu y \to \nu x) \to \nu x]$, so $(\neg(\nu x^n) \to \nu y) \to [(\nu y \to \nu x) \to \nu x] \in F$.

Conversely, by taking $\nu y = \neg(\nu x^n)$, the proof is complete.

(iii) $\Leftrightarrow$ (iv): We substitute νx by $\nu x \vee \nu y$ in (iii), so we get $[\neg((\nu x \vee \nu y)^n) \to \nu y] \to (\nu x \vee \nu y) \in F$.

Conversely, by Lemma 2.2, $[\neg((\nu x \vee \nu y)^n) \to \nu y] \to (\nu x \vee \nu y) \leqslant (\neg(\nu x^n) \to \nu y) \to [(\nu y \to \nu x) \to \nu x]$. Hence $(\neg(\nu x^n) \to \nu y) \to [(\nu y \to \nu x) \to \nu x] \in F$. $\square$

Theorem 3.16. *Let F be an IVRL-filter of A. Then the following conditions are equivalent:*

(i) *F is an n-fold IVRL-extended positive implicative filter,*

(ii) $[(\nu x^n \to \nu y) \to \nu x] \to \nu x \in F$,

(iii) $[(\nu x^n \to \nu y) \to \nu z] \to [(\nu z \to \nu x) \to \nu x] \in F$,

(iv) $([(\nu x \vee \nu z)^n \to \nu y] \to \nu z) \to (\nu x \vee \nu z) \in F$.

Proof. The proof is similar to the proof of Theorem 3.15. $\square$

In the following theorem we prove that the converse of Proposition 3.13 holds.

Theorem 3.17. *Let F be an IVRL-filter of A. If F is an n-fold IVRL-extended positive implicative filter, then $\nu x \vee \neg(\nu x^n) \in F$.*

Proof. Let F be an n-fold IVRL-extended positive implicative filter. According to Theorem 3.15 we have $[\neg((\nu x \vee \nu y)^n) \to \nu y] \to (\nu x \vee \nu y) \in F$. Now, let $\nu y = \neg(\nu x^n)$. Then $[\neg((\nu x \vee \neg(\nu x^n))^n) \to \neg(\nu x^n)] \to (\nu x \vee \neg(\nu x^n)) = \nu x \vee \neg(\nu x^n) \in F$. $\square$

4. n-fold implicative filters in triangle algebras

In this section, we introduce the notion of n-fold IVRL-extended implicative filters in triangle algebras and consider them in details. We give some examples of them. The relationship between these IVRL-filters and the n-fold positive implicative filters will be determined. Also, the n-fold implicative triangle algebras will be defined and some of their properties will be given.

Definition 4.1. An IVRL-filter F of A is called n-fold IVRL-extended implicative filter of A if it satisfies: $1 \in F$ and $(\nu x^n \to (\nu y \to \nu z))$, $\nu x^n \to \nu y \in F$, implies $\nu x^n \to \nu z \in F$ for all $x, y, z \in A$.

Theorem 4.2. *For all $x, y, z \in A$, the following conditions are equivalent:*
(i) *F is an n-fold IVRL-extended implicative filter of A,*
(ii) $\nu x^n \to \nu x^{2n} \in F$,
(iii) $\nu x^{n+1} \to \nu y \in F$ *implies* $\nu x^n \to \nu y \in F$,
(iv) $\nu x^n \to (\nu y \to \nu z) \in F$ *implies* $(\nu x^n \to \nu y) \to (\nu x^n \to \nu z) \in F$.

Proof. (i) $\Rightarrow$ (iii): Let F be an n-fold IVRL-extended implicative filter and $\nu x^{n+1} \to \nu y \in F$. Then $\nu x^n * \nu x \to \nu y \in F$, by Lemma 2.2, $\nu x^n \to (\nu x \to \nu y) \in F$. Since $\nu x^n \leqslant \nu x$, we obtain $\nu x^n \to \nu x = 1 \in F$. By assumption, $\nu x^n \to \nu y \in F$.

(iii) $\Rightarrow$ (ii): We have $\nu x^{n+1} \to (\nu x^{n-1} \to \nu x^{2n}) = \nu x^{2n} \to \nu x^{2n} = 1 \in F$. According to (iii) one has $\nu x^n \to (\nu x^{n-1} \to \nu x^{2n}) \in F$. But $\nu x^{n+1} \to (\nu x^{n-2} \to \nu x^{2n}) = \nu x^{2n-1} \to \nu x^{2n} = \nu x^n \to (\nu x^{n-1} \to \nu x^{2n}) \in F$. That is, $\nu x^{n+1} \to (\nu x^{n-2} \to \nu x^{2n}) \in F$. Therefore $\nu x^n \to (\nu x^{n-2} \to \nu x^{2n}) \in F$. By repeating the process n times we have $\nu x^n \to (\nu x^0 \to \nu x^{2n}) = \nu x^n \to (1 \to \nu x^{2n}) = \nu x^n \to \nu x^{2n} \in F$.

(ii) $\Rightarrow$ (iv): Let $\nu x^n \to (\nu y \to \nu z) \in F$. According to Lemma 2.2,

$$\begin{aligned}
\nu x^n \to (\nu y \to \nu z) &\leqslant \nu x^n \to ((\nu x^n \to \nu y) \to (\nu x^n \to \nu z)) \\
&= \nu x^n \to (\nu x^n \to ((\nu x^n \to \nu y) \to \nu z)) \\
&= \nu x^{2n} \to ((\nu x^n \to \nu y) \to \nu z).
\end{aligned}$$

Hence $\nu x^{2n} \to ((\nu x^n \to \nu y) \to \nu z) \in F$. According to (ii) we have $\nu x^n \to \nu x^{2n} \in F$. Lemma 2.2 gives $\nu x^{2n} \to ((\nu x^n \to \nu y) \to \nu z) \leqslant (\nu x^n \to \nu x^{2n}) \to (\nu x^n \to ((\nu x^n \to \nu y) \to \nu z))$. Then we get $(\nu x^n \to \nu y) \to (\nu x^n \to \nu z) = (\nu x^n \to ((\nu x^n \to \nu y) \to \nu z)) \in F$.

(iv) $\Rightarrow$ (i): Let (iv) and $\nu x^n \to \nu y \in F$. Since F is an IVRL-filter, $\nu x^n \to \nu z \in F$. So F is an n-fold IVRL-extended implicative filter of A. $\square$

Proposition 4.3. *If F is an n-fold IVRL-extended implicative filter of A, then F is an $(n+1)$-fold IVRL-extended implicative filter of A.*

Proof. Let $x, y \in A$ be such that $\nu x^{n+2} \to \nu y \in F$. Lemma 2.2 forces $\nu x^{n+1} \to (\nu x \to \nu y) = \nu x^{n+2} \to \nu y$. Since F is an n-fold IVRL-extended implicative filter, Theorem 4.2 gives $\nu x^n \to (\nu x \to \nu y) \in F$. Hence $\nu x^{n+1} \to \nu y \in F$, that is, F is an $(n+1)$-fold implicative filter. $\square$

In the following example we show that the converse of the above proposition is not true.

Example 4.4. In Example 3.5, clearly $F = \{[1,1]\}$ is a 3-fold IVRL-extended implicative filter of A. Since $\nu[b,1]^3 \to \nu[0,1] = [1,1] \in F$, but $\nu[b,1]^2 \to \nu[0,1] = [b,b] \notin F$, F is not a 2-fold IVRL-extended implicative filter.

Proposition 4.5. *Let F_1 and F_2 be IVRL-filters of A such that $F_1 \subseteq F_2$. If F_1 is an n-fold IVRL-extended implicative filter of A, then so is F_2.*

Proof. Let F_1 be an n-fold IVRL-extended implicative filter of A. Then by Theorem 4.2, $\nu x^n \to \nu x^{2n} \in F_1 \subseteq F_2$ for all $x \in A$. Thus $\nu x^n \to \nu x^{2n} \in F_2$ for all $x \in A$. So F_2 is an n-fold IVRL-extended implicative filter. $\square$

Corollary 4.6. $\{1\}$ *is an n-fold IVRL-extended implicative filter of A if and only if every IVRL-filter F of A is an n-fold IVRL-extended implicative filter of A.*

Lemma 4.7. *Let A be a triangle algebra. Then*

$$(\nu x^{n+1} \to \nu y)^n \to (\nu x^n \to \nu y) \geqslant (\nu x^n \to \nu y)^n \to \nu y.$$

Proof. (1) According to Lemma 2.2 we have the following:

$$\begin{aligned}
&(\nu x^{n+1} \to \nu y)^n \to (\nu x^n \to \nu y)\\
&\quad = (\nu x^{n+1} \to \nu y)^{n-1} * (\nu x^{n+1} \to \nu y) \to (\nu x^n \to \nu y)\\
&\quad = (\nu x^{n+1} \to \nu y)^{n-1} \to ((\nu x^{n+1} \to \nu y) \to (\nu x^n \to \nu y))\\
&\quad = (\nu x^{n+1} \to \nu y)^{n-1} \to ((\nu x^{n+1} \to \nu y) \to (\nu x^{n-1} \to (\nu x \to \nu y)))\\
&\quad = (\nu x^{n+1} \to \nu y)^{n-1} \to (\nu x^{n-1} \to ((\nu x^{n+1} \to \nu y) \to (\nu x \to \nu y)))\\
&\quad = (\nu x^{n+1} \to \nu y)^{n-1} \to (\nu x^{n-1} \to ((\nu x \to (\nu x^n \to \nu y)) \to (\nu x \to \nu y))).
\end{aligned}$$

(2) So, (1) forces $(\nu x^{n+1} \to \nu y)^n \to (x^n \to \nu y) = [(\nu x^{n+1} \to \nu y)^{n-1}] \to [\nu x^{n-1} \to [(\nu x \to (\nu x^n \to \nu y)) \to (\nu x \to \nu y)]]$.

(3) We have $(\nu x^n \to \nu y) \to \nu y \leqslant [(\nu x \to (\nu x^n \to \nu y)) \to (\nu x \to \nu y)]$.

(4) According to (3) we have $[(\nu x^{n+1} \to \nu y)^{n-1}] \to [\nu x^{n-1} \to [(\nu x^n \to \nu y) \to \nu y]] \leqslant [(\nu x^{n+1} \to \nu y)^{n-1}] \to [\nu x^{n-1} \to [[(\nu x \to (\nu x^n \to \nu y)) \to (\nu x \to \nu y)]]]$.

(5) We have $((\nu x^{n+1} \to \nu y)^{n-1}) \to (\nu x^{n-1} \to ((\nu x^n \to \nu y) \to \nu y)) \leqslant (\nu x^{n+1} \to y)^n \to (\nu x^n \to \nu y)$ by (4) and (2).

(6) We have $[(\nu x^{n+1} \to \nu y)^{n-1}] \to [\nu x^{n-1} \to [(\nu x^n \to \nu y) \to \nu y]] = [(\nu x^{n+1} \to \nu y)^{n-1}] \to [(\nu x^n \to \nu y) \to [\nu x^{n-1} \to \nu y]] = [(\nu x^n \to \nu y)] \to [(\nu x^{n+1} \to \nu y)^{n-1} \to [\nu x^{n-1} \to \nu y]]$.

(7) We get $(\nu x^n \to \nu y) \to [(\nu x^{n+1} \to \nu y)^{n-1} \to (\nu x^{n-1} \to \nu y)] \leqslant (\nu x^{n+1} \to \nu y)^n \to (\nu x^n \to \nu y)$ by (6) and (5).

(8) We have $(\nu x^n \to \nu y) * (\nu x^{n+1} \to \nu y)^{n-1} * \nu x^{n-1} = (\nu x^n \to \nu y) * (\nu x^{n+1} \to \nu y)^{n-2} * (\nu x^{n+1} \to \nu y) * \nu x^{n-2} * \nu x = (\nu x^n \to \nu y) * (\nu x^{n+1} \to \nu y)^{n-2} * \nu x^{n-2} * \nu x * (\nu x^{n+1} \to \nu y)$.

(9) We also have $\nu x * (\nu x^{n+1} \to \nu y) = \nu x * [\nu x \to (\nu x^n \to \nu y)] \leqslant \nu x^n \to \nu y$.

(10) Therefore $(\nu x^n \to \nu y) * (\nu x^{n+1} \to \nu y)^{n-2} * \nu x^{n-2} * \nu x * (\nu x^{n+1} \to \nu y) \leqslant (\nu x^n \to \nu y) * (\nu x^{n+1} \to \nu y)^{n2} * \nu x^{n-2} * (\nu x^n \to \nu y)$.

(11) Thus $(\nu x^n \to \nu y) * (\nu x^{n+1} \to \nu y)^{n-1} * \nu x^{n-1} \leqslant (\nu x^n \to \nu y)^2 * (\nu x^{n+1} \to \nu y)^{n-2} * \nu x^{n-2}$.

(12) We get $[(\nu x^n \to \nu y)^2 * (\nu x^{n+1} \to \nu y)^{n-2} * \nu x^{n-2}] \to \nu y \leqslant [(\nu x^n \to \nu y) * (\nu x^{n+1} \to \nu y)^{n-1} * \nu x^{n-1}] \to y$ by (11).

(13) $((\nu x^n \to \nu y)^2 * (\nu x^{n+1} \to \nu y)^{n-2}) \to (\nu x^{n-2} \to \nu y) \leqslant ((\nu x^n \to \nu y) * (\nu x^{n+1} \to \nu y)^{n-1}) \to (\nu x^{n-1} \to \nu y)$ by (12).

(14) So $(\nu x^n \to \nu y)^2 \to ((\nu x^{n+1} \to \nu y)^{n-2} \to (\nu x^{n-2} \to \nu y)) \leqslant (\nu x^n \to \nu y) \to ((\nu x^{n+1} \to \nu y)^{n-1} \to (\nu x^{n-1} \to \nu y))$.

(15) According to (14) and (7) we have $(\nu x^n \to \nu y)^2 \to ((\nu x^{n+1} \to \nu y)^{n-2} \to (\nu x^{n-2} \to \nu y)) \leqslant (\nu x^{n+1} \to \nu y)^n \to (\nu x^n \to \nu y)$.

So by repeating the process n times,

$$
\begin{aligned}
&(\nu x^{n+1} \to \nu y)^n \to (\nu x^n \to \nu y) \\
&\qquad \geqslant (\nu x^n \to \nu y)^2 \to ((\nu x^{n+1} \to \nu y)^{n-2} \to (\nu x^{n-2} \to \nu y)) \\
&\qquad \geqslant \ \dots \\
&\qquad \geqslant (\nu x^n \to \nu y)^n \to ((\nu x^{n+1} \to \nu y)^0 \to (\nu x^0 \to \nu y)) \\
&\qquad = (\nu x^n \to \nu y)^n \to (1 \to (1 \to \nu y)) = (\nu x^n \to \nu y)^n \to \nu y.
\end{aligned}
$$

□

Theorem 4.8. *Every n-fold IVRL-extended positive implicative filter is an n-fold IVRL-extended implicative filter.*

P r o o f. Let F be an n-fold IVRL-extended positive implicative filter of A and

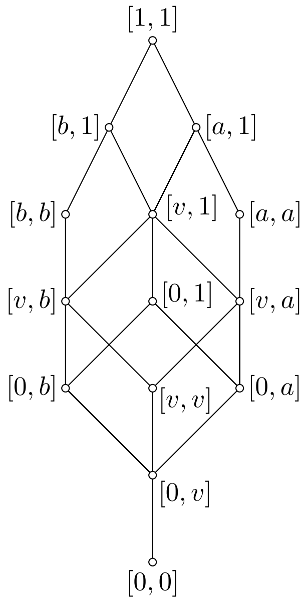
$x, y \in A$ be such that $\nu x^{n+1} \to \nu y \in F$. Lemma 4.7 gives $((\nu x^n \to \nu y)^n \to \nu y) \to ((\nu x^{n+1} \to \nu y)^n \to (\nu x^n \to \nu y)) = 1$. So $(\nu x^{n+1} \to \nu y)^n \to (((\nu x^n \to \nu y)^n \to \nu y) \to (\nu x^n \to \nu y)) = 1$. Since F is an IVRL-filter and $\nu x^{n+1} \to \nu y \in F$, $(\nu x^{n+1} \to \nu y)^n \in F$ and so $((\nu x^n \to \nu y)^n \to \nu y) \to (\nu x^n \to \nu y) \in F$. Since F is an n-fold IVRL-extended positive implicative filter, Theorem 3.3 gets $\nu x^n \to \nu y \in F$. Thus, F is an n-fold IVRL-extended implicative filter by Theorem 4.2. $\square$

In the following example we show that not every n-fold IVRL-extended implicative filter is an n-fold IVRL-extended positive implicative filter.

Example 4.9. Let $A = \{[0,0], [0,v], [0,a], [0,b], [0,1], [v,v], [v,a], [v,b], [v,1], [a,a], [a,1], [b,b], [b,1], [1,1]\}$. Define $\odot$ and $\Rightarrow$ as:

$\odot$	0	v	a	b	1
0	0	0	0	0	0
v	0	v	v	v	v
a	0	v	a	v	a
b	0	v	v	b	b
1	0	v	a	b	1

$\Rightarrow$	0	v	a	b	1
0	1	1	1	1	1
v	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	v	a	b	1



Define ν, μ, $*$ and $\to$ one as:

$$\nu[x_1, x_2] = [x_1, x_1], \quad \mu[x_1, x_2] = [x_2, x_2],$$
$$[x_1, x_2] * [y_1, y_2] = [x_1 \odot y_1, x_2 \odot y_2],$$
$$[x_1, x_2] \to [y_1, y_2] = [(x_1 \Rightarrow y_1) \wedge (x_2 \Rightarrow y_2), x_2 \Rightarrow y_2].$$

Then $(A, \vee, \wedge, *, \to, \nu, \mu, [0,0], [0,1], [1,1])$ is a triangle algebra with $[0,0]$ as the smallest and $[1,1]$ as the greatest element. It is clear that $F = \{[b,b], [b,1], [1,1]\}$ is a 2-fold IVRL-extended implicative filter but it is not a 2-fold IVRL-extended positive implicative filter because $(\nu[v,a]^2 \to [0,0]) \to \nu[v,a] = [1,1] \in F$ and $[v,v] \notin F$.

Definition 4.10. A is called an n-fold implicative triangle algebra if $\nu x^{n+1} = \nu x^n$ for all $x \in A$.

Example 4.11. In Example 4.9, A is an n-fold implicative triangle algebra for every natural number n.

Clearly, every Gödel triangle algebra is an n-fold implicative triangle algebra for every natural number n.

Proposition 4.12. *A is an n-fold implicative triangle algebra if and only if $\{1\}$ is an n-fold IVRL-extended implicative IVRL-filter of A.*

Proof. Let A be an n-fold implicative triangle algebra. Then $\nu x^{n+2} = \nu(x^{n+1} * x) = \nu x^n * \nu x = \nu x^{n+1} = \nu x^n$ for all $x \in A$. Similarly, $\nu x^{2n} = \nu x^n$, so $\nu x^n \to \nu x^{2n} = 1 \in \{1\}$ for all $x \in A$. By Theorem 4.2, $\{1\}$ is an n-fold implicative filter of A.

Conversely, $\{1\}$ is an n-fold IVRL-extended implicative filter of A by Theorem 4.2. Since $\nu x^n \to (\nu x^n \to \nu x^{n+1}) = \nu x^{2n} \to \nu x^{n+1} = 1 \in \{1\}$ and $\nu x^n \to \nu x^n = 1 \in \{1\}$, one has $\nu x^n \to \nu x^{n+1} \in \{1\}$, so $\nu x^n = \nu x^{n+1}$. Hence A is an n-fold implicative triangle algebra. $\square$

Corollary 4.13. *If A is a Gödel triangle algebra, then $\{1\}$ is an n-fold IVRL-extended implicative filter of A for every natural number n.*

In the following example we show that the converse of the above corollary is not true.

Example 4.14. In Example 3.7, $F = \{1\}$ is an n-fold IVRL-extended implicative filter of A for every natural number n but A is not a Gödel triangle algebra.

According to Theorem 4.2 (iv) and Lemma 2.2 (5) we have:

Lemma 4.15. *F is an n-fold IVRL-extended implicative filter of divisible triangle algebra A if and only if the following condition holds:*

$$\nu x^n * \nu y \to \nu z \in F, \text{ then } \nu x^n \wedge \nu y \to \nu z \in F \quad \forall x, y, z \in A.$$

Proposition 4.16. *Let $F = \{1\}$ be an 1-fold IVRL-extended implicative filter of triangle algebra A and $\mu(x^2) = \mu x$. Then A is a Gödel triangle algebra.*

P r o o f. If $\{1\}$ is 1-fold implicative filter, then A is 1-fold triangle implicative algebra by Proposition 4.12, i.e. $\nu x = \nu x^2$. Also we have $\mu(x^2) = \mu x$, hence $x^2 = x$. $\square$

E x a m p l e 4.17. In Example 3.5, $F = \{[1,1]\}$ is a 3-fold IVRL-extended implicative filter of triangle algebra A. But $\mu([a,a]^2) \neq \mu([a,a])$, $\nu([a,a]^2) \neq \nu([a,a])$ and so A is not a Gödel triangle algebra.

Theorem 4.18. *Let F be an IVRL-filter of A. Then F is an n-fold IVRL-extended implicative filter if and only if A/F is an n-fold implicative triangle algebra.*

P r o o f. Let F be an n-fold IVRL-extended implicative filter. Then $\nu x^n \to \nu x^{2n} \in F$ for all $x \in A$ by Theorem 4.2. Thus $[\nu x]^n \to [\nu x]^{2n} = [\nu x^n \to \nu x^{2n}] = [1]$. So $[\nu x]^n \leqslant [\nu x]^{2n}$. Hence $[\nu x]^n = [\nu x]^{2n}$ and A/F is an n-fold implicative triangle algebra.

Conversely, let A/F be an n-fold implicative triangle algebra. Then $[\nu x]^n = [\nu x]^{2n}$ for all $x \in A$. Thus $[\nu x^n \to \nu x^{2n}] = [\nu x]^n \to [\nu x]^{2n} = [1]$. Therefore, $\nu x^n \to \nu x^{2n} \in F$. According to Theorem 4.2, F is an n-fold IVRL-extended implicative filter of A. $\square$

Corollary 4.19. *F is a 1-fold IVRL-extended implicative filter of A and $\mu(x^2) = \mu x$ if and only if A/F is a Gödel triangle algebra.*

Conclusion

The concept of triangle algebra was introduced by Van Gasse et al., who defined IVRL-filters in triangle algebras as well as some of their properties. Following that, Zahiri et al. introduced a few specific sets, such as the radical of an IVRL-filter in triangle algebras, which were investigated in detail. In addition, they defined some types of IVRL-filters and discussed some of their properties.

In the present study, we introduced the n-fold filter theory in triangle algebras.

The concept of n-fold IVRL-extended (positive) implicative filters has been defined and studied in detail. Our finding confirmed that n-fold IVRL-extended filters possess extension properties, and every n-fold IVRL-extended filter is an $(n+1)$-fold IVRL-extended filter in triangle algebras. Furthermore, we defined the notion of n-fold (positive) implicative triangle algebras. The two mentioned algebras are special types of triangle algebras. Finally, the connection between this algebraic structure and n-fold IVRL-extended (positive) implicative filters was assessed.

In our future work we are going to continue our study on other types of n-fold filters in triangle algebra. We whould like to use these results to find some classification for this structure.

A c k n o w l e d g e m e n t s. The authors are very indebted to the referees for valuable suggestions that improved the readability of the paper.

References

[1] *P. Hájek*: Metamathematics of Fuzzy Logic. Trends in Logic—Studia Logica Library 4. Kluwer Academic Publishers, Dordrecht, 1998. zbl MR doi
[2] *D. Piciu*: Algebras of Fuzzy Logic. Ed. Universtaria, Craiova, 2007.
[3] *B. Van Gasse, C. Cornelis, G. Deschrijver, E. E. Kerre*: A characterization of interval-valued residuated lattices. Int. J. Approx. Reasoning *49* (2008), 478–487. zbl MR doi
[4] *B. Van Gasse, C. Cornelis, G. Deschrijver, E. E. Kerre*: Triangle algebras: A formal logic approach to interval-valued residuated lattices. Fuzzy Sets Syst. *159* (2008), 1042–1060. zbl MR doi
[5] *B. Van Gasse, G. Deschrijver, C. Cornelis, E. E. Kerre*: Filters of residuated lattices and triangle algebras. Inf. Sci. *180* (2010), 3006–3020. zbl MR doi
[6] *M. Ward, R. P. Dilworth*: Residuated lattices. Trans. Am. Math. Soc. *45* (1939), 335–354. zbl MR doi
[7] *S. Zahiri, A. Borumand Saeid, E. Eslami*: A new approach to filters in triangle algebras. Publ. Inst. Math., Nouv. Sér. *101 (115)* (2017), 267–283. MR doi

Authors' addresses: *Saeide Zahiri*, Department of Mathematics, Faculty of Science, Higher Education Center of Eghlid, Mostafa Khomeini St., Eghlid, Iran, e-mail: `saeede.zahiri@yahoo.com`, `s.zahiri@eghlid.ac.ir`; *Arsham Borumand Saeid*, Department of Pure Mathematics, Faculty of Mathematics and Computer, Bahman Blvd. 22, Shahid Bahonar University of Kerman, 76169-14111, Kerman, Iran, e-mail: `arsham@uk.ac.ir`.