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THE EQUIDISTRIBUTION OF FOURIER COEFFICIENTS OF HALF INTEGRAL WEIGHT MODULAR FORMS ON THE PLANE

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Abstract. Let $f = \sum\limits_{n=1}^{\infty} a(n)q^n \in S_{k+1/2}(N,\chi_0)$ be a nonzero cuspidal Hecke eigenform of weight $k+\frac{1}{2}$ and the trivial nebentypus χ_0, where the Fourier coefficients $a(n)$ are real. Bruinier and Kohnen conjectured that the signs of $a(n)$ are equidistributed. This conjecture was proved to be true by Inam, Wiese and Arias-de-Reyna for the subfamilies $\{a(tn^2)\}_n$, where t is a squarefree integer such that $a(t) \neq 0$. Let q and d be natural numbers such that $(d,q)=1$. In this work, we show that $\{a(tn^2)\}_n$ is equidistributed over any arithmetic progression $n \equiv d \bmod q$.

Keywords: Shimura lift; Fourier coefficient; half-integral weight; Sato-Tate equidistribution

MSC 2010: 11F30, 11F37

1. Introduction

Let $k \geqslant 2$, $4 \mid N$ be integers, $\chi \pmod N$ a Dirichlet character, and let $f = \sum\limits_{n=1}^{\infty} a(n)q^n \in S_{k+1/2}(N,\chi)$ be a nonzero cuspidal Hecke eigenform of weight $k+\frac{1}{2}$. Applying the Shimura lift to f for a fixed squarefree t such that $a(t) \neq 0$, we get $F_t = \sum\limits_{n=1}^{\infty} A_t(n)q^n \in S_{2k}(N/2,\chi^2)$ the Hecke eigenform of weight $2k$.

When $\chi = 1$, Bruinier and Kohnen suggested in [4] that half of the coefficients $a(n)$ are positive among all nonzero Fourier coefficients. This suggestion was formulated later explicitly as a conjecture in [8]. Assuming some error term for the convergence of the Sato-Tate distribution for integral weight modular forms in [6], Inam and Wise showed when F_t has no CM that half of the coefficients $a(tn^2)$ are positive. They formulated this result in terms of Dedekind-Dirichlet density. They also showed with Arias-de-Reyna in [2], that $(a(tn^2))_{n\in\mathbb{N}}$ are equidistributed when F_t has CM and the

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equidistribution was reformulated in both CM and not CM cases using Dedekind-Dirichlet and natural densities. Later, those results were obtained in [7] by removing the error term assumption.

The present work gives an improvement of the Bruinier-Kohnen conjecture. Indeed, under the error term hypothesis that we will explain below, our main result is the following theorem.

Theorem 1. *Assume the setting of the introduction and suppose that F_t does not have complex multiplication. Let q be a natural number. Suppose that for all Dirichlet characters $\varepsilon \pmod q$ and all roots of unity ξ such that $\xi \in \operatorname{Im}\varepsilon$, there are $C_{\varepsilon,\xi} > 0$ and $\alpha_{\varepsilon,\xi} > 0$ such that*

$$\left| \frac{\#\{p \leqslant x \text{ prime: } p \nmid N, \varepsilon(p) = \xi, \frac{A_t(p)}{2a(t)p^{(k-1)/2}\chi(p)} \in [a,b]\}}{\pi(x)} - \frac{\mu([a,b])}{\#\operatorname{Im}\varepsilon} \right| \leqslant \frac{C_{\varepsilon,\xi}}{x^{\alpha_{\varepsilon,\xi}}}. \tag{1}$$

Then for all integers d, $(d,q) = 1$, the sets

$$\Big\{n \in \mathbb{N}\colon\ (n,N) = 1,\ n \equiv d \bmod q,\ \frac{a(tn^2)}{\chi(n)} > 0\Big\} \quad \textit{and} \tag{2}$$
$$\Big\{n \in \mathbb{N}\colon\ (n,N) = 1,\ n \equiv d \bmod q,\ \frac{a(tn^2)}{\chi(n)} < 0\Big\}$$

have equal positive natural densities and both are half of the natural density of

$$\Big\{n \in \mathbb{N}\colon\ (n,N) = 1,\ n \equiv d \bmod q,\ \frac{a(tn^2)}{\chi(n)} \neq 0\Big\}. \tag{3}$$

We discuss here two aspects of this theorem. Consider first the case when $\chi = 1$ and the coefficients $a(n)$ are real. Then for all natural numbers q and d such that $(d,q) = 1$ we have

$$\lim_{x\to\infty} \frac{\#\{n \leqslant x\colon\ n \equiv d \bmod q,\ a(tn^2) \gtrless 0\}}{\#\{n \leqslant x\colon\ n \equiv d \bmod q,\ a(tn^2) \neq 0\}} = \frac{1}{2}.$$

This extends the results obtained in [7], and therefore one can ask if the Bruinier-Kohnen conjecture remains true over arithmetic progressions. We have no numerical experiments yet to support this hypothesis.

Consider now the general case $f \in S_{k+1/2}(N,\chi)$. Let q be a natural number, $\varepsilon \bmod q$ a Dirichlet character and $\xi \in \operatorname{Im}\varepsilon$. From the main theorem above and since the density of the set (3) is independent of d by Proposition 4 and Remark 2, the sets

$$\text{(4)} \qquad \Big\{n \in \mathbb{N}\colon\ (n,N)=1,\ \varepsilon(n)=\xi,\ \frac{a(tn^2)}{\chi(n)}>0\Big\} \quad \text{and}$$
$$\Big\{n \in \mathbb{N}\colon\ (n,N)=1,\ \varepsilon(n)=\xi,\ \frac{a(tn^2)}{\chi(n)}<0\Big\}$$

have equal positive natural densities and both are half of the natural density of

$$\Big\{n \in \mathbb{N}\colon\ (n,N)=1,\ \varepsilon(n)=\xi,\ \frac{a(tn^2)}{\chi(n)}\neq 0\Big\}.$$

In the particular case $q=N$ and $\varepsilon=\chi$, we deduce that when $\xi \neq \pm\mathrm{i}$, the sets

$$\text{(5)} \qquad \{n \in \mathbb{N}\colon\ \chi(n)=\xi,\ \mathrm{Re}(a(tn^2))>0\} \quad \text{and}$$
$$\{n \in \mathbb{N}\colon\ \chi(n)=\xi,\ \mathrm{Re}(a(tn^2))<0\}$$

have equal positive natural densities and both are half of the natural density of

$$\{n \in \mathbb{N}\colon\ \chi(n)=\xi,\ a(tn^2)\neq 0\}.$$

Geometrically, the coefficients $a(tn^2)$ with $\chi(n)=\xi$ belong to the same line and they are equidistributed over it. When $\xi=\pm\mathrm{i}$, we obtain a similar result and the coefficients $a(tn^2)$ with $\chi(n)=\mathrm{i}$ or $-\mathrm{i}$ are equidistributed over the vertical line that passes through i and $-$i. Once again, one can ask more generally if the Fourier coefficients $a(n)$ with $(n,N)=1$, that belong to the same line, are equidistributed geometrically as above.

2. Notions of density

Recall that the set of primes (or the set of natural numbers) $S \subseteq \mathbb{P}$ (or $A \subseteq \mathbb{N}$) has a natural density $d(S)$ (or $d(A)$) if the limit

$$d(S)=\lim_{x\to\infty}\frac{\pi_S(x)}{\pi(x)} \quad \Big(\text{or } d(A)=\lim_{x\to\infty}\frac{\#\{n\leqslant x\colon\ n\in A\}}{x}\Big)$$

exists, where $\pi_S(x)$ and $\pi(x)$ are defined by

$$\pi(x)=\#\{p\leqslant x\colon\ p\in\mathbb{P}\} \quad \text{and} \quad \pi_S(x)=\#\{p\leqslant x\colon\ p\in S\}.$$

The set of primes (or of natural numbers) S (or A) is said to have Dirichlet density $\delta(S)$ (or Dedekind-Dirichlet density $\delta(A)$) if the limit

$$\delta(S)=\lim_{z\to 1^+}\frac{\sum_{p\in S}p^{-z}}{\log(z-1)^{-1}}\quad\Bigl(\text{or }\delta(A)=\lim_{z\to 1^+}(z-1)\sum_{n\in A}\frac{1}{n^z}\Bigr)$$

exists. Recall that if the set A of natural numbers has natural density $d(A)$, then it also has Dedekind-Dirichlet density $\delta(A)$ with $d(A)=\delta(A)$. Further, the set of primes S is said to be regular if there is a holomorphic function $g(z)$ on $\mathrm{Re}(z)\geqslant 1$ such that

$$\sum_{p\in S}\frac{1}{p^z}=\delta(S)\log\frac{1}{z-1}+g(z).$$

We need the following technical lemma (see [6], Lemma 2.1).

Lemma 1. *Let S_1 and S_2 be two regular sets of primes such that $\delta(S_1)=\delta(S_2)$. Then the function $\sum\limits_{p\in S_1}p^{-z}-\sum\limits_{q\in S_1}q^{-z}$ is analytic on* $\mathrm{Re}(z)\geqslant 1$.

The following proposition shows that the set of primes S is regular if it has a natural density that satisfies certain error term (see [6], Proposition 2.2).

Proposition 1. *Let $S\subseteq\mathbb{P}$ be a set of primes that have natural density $d(S)$. Define $E(x)=\pi_S(x)/\pi(x)-d(S)$ to be the error function. Suppose that there are $\alpha>0$, $C>0$ and $M>0$ such that for all $x>M$ we have $|E(x)|\leqslant Cx^{-\alpha}$. Then S is a regular set of primes.*

3. The Chebotarev-Sato-Tate equidistribution

We recall now some properties of the Shimura lift (see [12]). The Fourier coefficients of f and F_t are related by the formula

$$A_t(n)=\sum_{d\mid n}\chi_{t,N}(d)d^{k-1}a\Bigl(\frac{n^2}{d^2}t\Bigr), \tag{6}$$

where $\chi_{t,N}$ denotes the character $\chi_{t,N}(d):=\chi(d)(-1)^kN^2t/d$. Since f is the Hecke eigenform for the Hecke operator T_{p^2}, F_t is an eigenform for the Hecke operator T_p, for all primes $p\nmid N$. Further, we have $F_t=a(t)F$, where F is a normalised Hecke eigenform independent of t.

Applying the Ramanujan-Petersson bound to the Fourier coefficients of F_t, $|A_t(p)/a(t)| \leqslant 2p^{(k-1)/2}$. Since $F_t \in S_{2k}(\frac{1}{2}N, \chi^2)$, $A_t(p) = \chi^2(p)\overline{A_t(p)}$. Therefore $A_t(p)/\chi(p) \in \mathbb{R}$ and define

$$B_t(p) := \frac{A_t(p)}{2a(t)p^{(k-1)/2}\chi(p)} \in [-1, 1].$$

Notice that $a(t) \in \mathbb{R}$ since $a(t) = A_t(1)/\chi(1)$.

Recall that the Sato-Tate measure μ is the measure on the interval $[-1, 1]$ given by $(2/\pi)\sqrt{1-t^2}\,\mathrm{d}t$. We state the important Sato-Tate equidistribution theorem for $\Gamma_0(N)$ (see Theorem B of [3]).

Theorem 2 (Barnet-Lamb, Geraghty, Harris, Taylor). *Let $k \geqslant 1$ and let $F_t = \sum_{n\geqslant 1} A(n)q^n \in S_{2k}(\frac{1}{2}N, \chi^2)$ be a cuspidal Hecke eigenform of weight $2k$ for $\Gamma_0(N)$. Suppose that F_t is without multiplication. Denote by $\operatorname{Im}\chi$ the image of χ and let $\xi \in \operatorname{Im}\chi$. Then, when p runs through the primes $p \nmid N$ such that $\chi(p) = \xi$, the numbers*

$$B(p) = \frac{A_t(p)}{2a(t)p^{(k-1)/2}\chi(p)} \in [-1, 1]$$

are μ-equidistributed in $[-1, 1]$.

Inam et al. in [2], [6], [7] obtained the equidistribution of the coefficients $a(tn^2)$ by using Theorem 2. In order to prove the geometric equidistribution on the plane as was explained in the introduction, we need the following hybrid Chebotarev-Sato-Tate equidistribution proved for elliptic curves in [11] for the first time, and it has been generalized recently by Wong (see [13]) particularly to non-CM Hecke eigenforms.

Proposition 2 (Wong). *Let q be a natural number and d an integer with $(d, q) = 1$. Let $[a, b] \subset [-1, 1]$ and put $S_{[a,b]} := \{p \text{ prime}\colon\ p \equiv d \pmod q, B_t(p) \in [a, b]\}$. The set $S_{[a,b]}$ has natural density equal to $(2/\pi\varphi(q)) \int_a^b \sqrt{1-t^2}\,\mathrm{d}t$.*

Using Dirichlet's theorem on arithmetic progressions, this proposition could be rewritten as follows.

Proposition 3. *Let q be a natural number, $\varepsilon \pmod q$ a Dirichlet character and ξ a root of unity such that $\xi \in \operatorname{Im}\varepsilon$. Let $[a, b] \subset [-1, 1]$ and put $S_{[a,b]} := \{p \text{ prime}\colon \varepsilon(p) = \xi,\ B_t(p) \in [a, b]\}$. The set $S_{[a,b]}$ has natural density equal to*

$$\frac{1}{\#\operatorname{Im}\varepsilon}\frac{2}{\pi}\int_a^b \sqrt{1-t^2}\,\mathrm{d}t,$$

where $\#\operatorname{Im}\varepsilon$ is the cardinality of the image of ε.

We will use frequently throughout the paper the following lemma (see [10]).

Lemma 2. *Under the hypothesis fixed in the introduction, let n be an integer such that $(n, N) = 1$. Then $a(tn^2)/\chi(n) \in \mathbb{R}$.*

4. Preliminaries results

We next show that the Chebotarev-Sato-Tate theorem (see [13], Proposition 2.2) gives the equidistribution of the coefficients $a(tp^2)$ when primes p run over arithmetic progressions.

Theorem 3. *We use the assumptions fixed in the introduction and suppose that F_t has no CM. Let q be a natural number, $\varepsilon \pmod q$ a Dirichlet character and ξ a root of unity such that $\xi \in \operatorname{Im}\varepsilon$. Define the set of primes*

$$\mathbb{P}_{\varepsilon,\xi,>} := \Big\{p \in \mathbb{P}\colon\ \varepsilon(p) = \xi,\ \frac{a(tp^2)}{\chi(p)} > 0\Big\},$$

and similarly $\mathbb{P}_{\varepsilon,\xi}$, $\mathbb{P}_{\varepsilon,\xi,<}$, $\mathbb{P}_{\varepsilon,\xi,\geqslant}$, $\mathbb{P}_{\varepsilon,\xi,\leqslant}$, and $\mathbb{P}_{\varepsilon,\xi,=0}$. Let d be an integer such that $(d, q) = 1$. Define also

$$\mathbb{P}_{d,q,>} := \Big\{p \in \mathbb{P}\colon\ p \equiv d \bmod q,\ \frac{a(tp^2)}{\chi(p)} > 0\Big\},$$

and similarly $\mathbb{P}_{d,q}$, $\mathbb{P}_{d,q,<}$, $\mathbb{P}_{d,q,\geqslant}$, $\mathbb{P}_{d,q,\leqslant}$, $\mathbb{P}_{d,q,=0}$.

The sets $\mathbb{P}_{d,q,>}$, $\mathbb{P}_{d,q,<}$, $\mathbb{P}_{d,q,\geqslant}$, $\mathbb{P}_{d,q,\leqslant}$ have natural density $1/(2\varphi(q))$ and $\mathbb{P}_{d,q,=0}$ has natural density 0. Further, the sets $\mathbb{P}_{\varepsilon,\xi,>}$, $\mathbb{P}_{\varepsilon,\xi,<}$, $\mathbb{P}_{\varepsilon,\xi,\geqslant}$, $\mathbb{P}_{\varepsilon,\xi,\leqslant}$ have natural density $1/(2\#\operatorname{Im}\varepsilon)$ and $\mathbb{P}_{\varepsilon,\xi,=0}$ has natural density 0, where $\#\operatorname{Im}\varepsilon$ is the cardinality of the image of ε.

P r o o f of Theorem 3. Define the sets

$$\pi_{d,q,>}(x) := \#\Big\{p \leqslant x\colon\ p \equiv d \bmod q,\ \frac{a(tp^2)}{\chi(p)} > 0\Big\},$$

and similarly, $\pi_{d,q}(x)$, $\pi_{d,q,<}(x)$, $\pi_{d,q,\geqslant}(x)$, $\pi_{d,q,\leqslant}(x)$, and $\pi_{d,q,=0}(x)$. Without loss of generality, we can assume that F_t is normalised and thus $a(t) = 1$. Denote the character $(-1)^k N^2 t/\cdot$ by $\chi_1(\cdot) = (-1)^k N^2 t/\cdot$. The formula (6) yields

$$\frac{a(tp^2)}{\chi(p)} > 0 \iff B_t(p) > \frac{\chi_1(p)}{2\sqrt{p}}.$$

Let $\varepsilon > 0$. Since for all $p > 1/(4\varepsilon^2)$ we have $\chi_1(p)/(2\sqrt{p}) = 1/(2\sqrt{p}) < \varepsilon$, then

$$\begin{aligned}(7)\qquad \pi_{d,q,>}(x) + \#\Big\{p \leqslant x \text{ prime}\colon\ p \equiv d \bmod q,\ p \leqslant \frac{1}{4\varepsilon^2}\Big\}& \\ \geqslant \#\{p \leqslant x \text{ prime}\colon\ p \equiv d \bmod q,\ B_t(p) > \varepsilon\}&.\end{aligned}$$

Applying Proposition 2 we get

$$\lim_{x\to\infty} \frac{\#\{p \leqslant x \text{ prime}\colon\ p \equiv d \bmod q,\ B_t(p) > \varepsilon\}}{\pi(x)} = \frac{\mu([\varepsilon,1])}{\varphi(q)}$$

and then

$$\lim_{x\to\infty} \frac{\#\{p \leqslant x \text{ prime}\colon\ p \equiv d \bmod q,\ B_t(p) > \varepsilon\}}{\pi_{d,q}(x)} = \mu([\varepsilon,1]).$$

It follows that $\liminf\limits_{x\to\infty} \pi_{d,q,>}(x)/\pi_{d,q}(x) \geqslant \mu([\varepsilon,1])$ for all $\varepsilon > 0$, hence

$$\liminf_{x\to\infty} \frac{\pi_{d,q,>}(x)}{\pi_{d,q}(x)} \geqslant \mu([0,1]) = \frac{1}{2}.$$

Similarly, we have

$$\liminf_{x\to\infty} \frac{\pi_{d,q,\leqslant}(x)}{\pi_{d,q}(x)} \geqslant \mu([0,1]) = \frac{1}{2}.$$

Since $\pi_{d,q,\leqslant}(x) = \pi_{d,q}(x) - \pi_{d,q,>}(x)$, then $\limsup\limits_{x\to\infty} \pi_{d,q,>}(x)/\pi_{d,q}(x) = \frac{1}{2}$. Using the same method, we obtain the densities of $\mathbb{P}_{d,q,<}$, $\mathbb{P}_{d,q,\geqslant}$ and $\mathbb{P}_{d,q,\leqslant}$. Finally, since $\pi_{d,q,=0}(x) = \pi_{d,q,\geqslant}(x) - \pi_{d,q,>}(x)$, then the density of $\mathbb{P}_{d,q,=0}(x)$ is zero.

The densities of the sets $\mathbb{P}_{\varepsilon,\xi,>}$, $\mathbb{P}_{\varepsilon,\xi,<}$, $\mathbb{P}_{\varepsilon,\xi,\geqslant}$, $\mathbb{P}_{\varepsilon,\xi,\leqslant}$ and $\mathbb{P}_{\varepsilon,\xi,=0}$ are obtained similarly by using Proposition 3. □

The following theorem shows that the set of primes of Theorem 3 is regular if the Chebotarev-Sato-Tate theorem satisfies certain error term. The proof is closely similar to that of [6], Theorem 4.2.

Theorem 4. *Under the assumptions of Theorem 3, suppose there are $C > 0$ and $\alpha > 0$ such that*

$$\left| \frac{\#\{p \leqslant x \text{ prime}\colon\ \varepsilon(p) = \xi,\ \frac{A_t(p)}{2a(t)p^{(k-1)/2}\chi(p)} \in [a,b]\}}{\pi(x)} - \frac{\mu([a,b])}{\#\operatorname{Im}\varepsilon} \right| \leqslant \frac{C}{x^\alpha}.$$

Then the sets $\mathbb{P}_{\varepsilon,\xi,\geqslant}$, $\mathbb{P}_{\varepsilon,\xi,\leqslant}$, $\mathbb{P}_{\varepsilon,\xi,>}$, $\mathbb{P}_{\varepsilon,\xi,<}$ and $\mathbb{P}_{\varepsilon,\xi,=0}$ are regular sets of primes.

Remark 1. Let ξ_q be a qth root of unity. The previous error term is weaker than the one conjectured by Akiyama and Tanigawa (see [1]) and it can be obtained by [13], Theorem 1.3 if GRH is assumed and also if $L(z, \mathrm{Sym}^m(F_t/a(t)) \otimes \eta)$ is automorphic over $\mathbb{Q}$ for every m and for all irreducible characters η of $G(\mathbb{Q}(\xi_q)/\mathbb{Q})$.

To proceed with the proof of Theorem 1, we establish the following two lemmas.

Lemma 3. *Assume the assumptions fixed in the introduction and suppose that F_t has no CM. Let q be a natural number. Suppose that for all $\varepsilon \pmod q$ Dirichlet characters and all roots of unity ξ such that $\xi \in \mathrm{Im}\,\varepsilon$ there are $C_{\varepsilon,\xi} > 0$ and $\alpha_{\varepsilon,\xi} > 0$ such that*

$$\left| \frac{\#\{p \leqslant x \text{ prime: } p \nmid N, \varepsilon(p) = \xi, \frac{A_t(p)}{2a(t)p^{(k-1)/2}\chi(p)} \in [a,b]\}}{\pi(x)} - \frac{\mu([a,b])}{\#\,\mathrm{Im}\,\varepsilon} \right| \leqslant \frac{C_{\varepsilon,\xi}}{x^{\alpha_{\varepsilon,\xi}}}. \tag{8}$$

Suppose further that $a(t) > 0$. Define the multiplicative function for all $n \in \mathbb{N}$,

$$f(n) = \begin{cases} 1 & \text{if } \dfrac{a(tn^2)}{\chi(n)} > 0 \text{ and } (n,N) = 1, \\ -1 & \text{if } \dfrac{a(tn^2)}{\chi(n)} < 0 \text{ and } (n,N) = 1, \\ 0 & \text{if } a(tn^2) = 0 \text{ and } (n,N) = 1, \\ 0 & \text{if } (n,N) \neq 1. \end{cases}$$

Let d be an integer with $(d,q) = 1$. Then the Dirichlet series

$$F(z) = \sum_{\substack{n \geqslant 1 \\ n \equiv d \bmod q}} \frac{f(n)}{n^z}$$

is holomorphic on $\mathrm{Re}(z) \geqslant 1$.

P r o o f of Lemma 3. We have

$$\begin{aligned} \sum_{\substack{n \geqslant 1 \\ n \equiv d \bmod q}} \frac{f(n)}{n^z} &= \frac{1}{\varphi(q)} \sum_{n=1}^{\infty} \frac{f(n)}{n^z} \times \left(\sum_{\varepsilon \bmod q} \varepsilon(n)\overline{\varepsilon(d)} \right) \\ &= \frac{1}{\varphi(q)} \sum_{\varepsilon \bmod q} \left(\sum_{n=1}^{\infty} \frac{f(n)\varepsilon(n)}{n^z} \right) \times \overline{\varepsilon(d)}. \end{aligned}$$

Since the first sum is finite, it suffices to show that $G_\varepsilon(z) = \sum\limits_{n=1}^{\infty} f(n)\varepsilon(n)/n^z$ is holomorphic on $\mathrm{Re}(z) \geqslant 1$.

Since $a(t) > 0$ and for all $m, n \in \mathbb{N}$, $(m, N) = 1$, $(n, N) = 1$,

$$\frac{a(tm^2)}{\chi(m)} \frac{a(tn^2)}{\chi(n)} = a(t) \frac{a(tm^2 n^2)}{\chi(mn)},$$

then $f(n)$ is multiplicative.

Applying [2], Lemma 2.1.2, we obtain

$$\log G_\varepsilon(z) = \sum_{p \in \mathbb{P}} \frac{f(p)\varepsilon(p)}{p^z} + g(z),$$

where $g(z)$ is a function that is holomorphic on $\mathrm{Re}(z) > \frac{1}{2}$. Hence

$$\begin{aligned}\log G_\varepsilon(z) &= \sum_{p \in \mathbb{P}} \frac{f(p)\varepsilon(p)}{p^z} + g(z) = \sum_{\xi \in \mathrm{Im}(\varepsilon)} \xi \sum_{p \in \mathbb{P}_{\varepsilon,\xi}} \frac{f(p)}{p^z} + g(z) \\ &= \sum_{\xi \in \mathrm{Im}(\varepsilon)} \xi \Bigg(\sum_{p \in \mathbb{P}_{\varepsilon,\xi,>}} \frac{1}{p^z} - \sum_{p \in \mathbb{P}_{\varepsilon,\xi,<}} \frac{1}{p^z} \Bigg) + g(z).\end{aligned}$$

The sets $\mathbb{P}_{\varepsilon,\xi,>}$ and $\mathbb{P}_{\varepsilon,\xi,<}$ are regular sets of primes, and they have the same density $1/(2\#\,\mathrm{Im}\,\varepsilon)$ by Theorem 3. Therefore by Lemma 1, $\log G_\varepsilon(z)$ is holomorphic on $R(z) \geqslant 1$, and consequently $G_\varepsilon(z)$ is also holomorphic. $\square$

Lemma 4. *We use the assumptions fixed in the introduction and suppose that F_t has no CM. Let q be a natural number. Suppose that for all Dirichlet characters ε $(\mathrm{mod}\ q)$ and all roots of unity ξ such that $\xi \in \mathrm{Im}\,\varepsilon$ there are $C_{\varepsilon,\xi} > 0$ and $\alpha_{\varepsilon,\xi} > 0$ such that*

$$\left| \frac{\#\{p \leqslant x \text{ prime}\colon\ p \nmid N, \varepsilon(p) = \xi, \frac{A_t(p)}{2a(t)p^{(k-1)/2}\chi(p)} \in [a,b]\}}{\pi(x)} - \frac{\mu([a,b])}{\#\,\mathrm{Im}\,\varepsilon} \right| \leqslant \frac{C_{\varepsilon,\xi}}{x^{\alpha_{\varepsilon,\xi}}}. \tag{9}$$

Then for all integers d, $(d,q) = 1$, the set

$$\{n \in \mathbb{N}\colon\ (n,N) = 1,\ n \equiv d \bmod q,\ a(tn^2) \neq 0\}$$

has natural density.

P r o o f of Lemma 4. We have

$$\sum_{\substack{n \geqslant 1 \\ n \equiv d \bmod q}} \frac{f(n)^2}{n^z} = \frac{1}{\varphi(q)} \sum_{\varepsilon \bmod q} \Bigg(\sum_{n=1}^{\infty} \frac{f(n)^2 \varepsilon(n)}{n^z} \Bigg) \times \overline{\varepsilon(d)}.$$

We shall define

$$H_\varepsilon(z) = \sum_{n=1}^{\infty} \frac{f(n)^2\varepsilon(n)}{n^z}.$$

Apply [2], Lemma 2.1.2 to get

$$\log H_\varepsilon(z) := \sum_{p\in\mathbb{P}} \frac{f(p)^2\varepsilon(p)}{p^z} + g_\varepsilon(z) = \sum_{\xi\in\operatorname{Im}\varepsilon} \xi \sum_{p\in\mathbb{P}_{\varepsilon,\xi,>}\cup\mathbb{P}_{\varepsilon,\xi,<}} \frac{1}{p^z} + g_\varepsilon(z),$$

where $g_\varepsilon(z)$ is a function that is holomorphic on $\operatorname{Re}(z) > \frac{1}{2}$. Applying Theorem 4, the sets $\mathbb{P}_{\varepsilon,\xi,>}$ and $\mathbb{P}_{\varepsilon,\xi,<}$ are regular sets of primes of natural density $1/(2\#\operatorname{Im}\varepsilon)$. Then

$$\sum_{p\in\mathbb{P}_{\varepsilon,\xi,>}\cup\mathbb{P}_{\varepsilon,\xi,<}} \frac{1}{p^z} = \frac{1}{\#\operatorname{Im}\varepsilon}\log\frac{1}{z-1} + h_\xi(z),$$

where h_ξ is a holomorphic function on $\operatorname{Re}(z) \geqslant 1$. It follows that

$$\begin{aligned}\log H_\varepsilon(z) &:= \sum_{\xi\in\operatorname{Im}\varepsilon} \xi \sum_{p\in\mathbb{P}_{\varepsilon,\xi,>}\cup\mathbb{P}_{\varepsilon,\xi,<}} \frac{1}{p^z} + g_\varepsilon(z)\\ &= \frac{\sum_{\xi\in\operatorname{Im}\varepsilon}\xi}{\#\operatorname{Im}\varepsilon}\log\frac{1}{z-1} + \sum_{\xi\in\operatorname{Im}\varepsilon}\xi h_\xi(z) + g_\varepsilon(z).\end{aligned}$$

Thus, $\log H_{\varepsilon_0}(z) = \log(z-1)^{-1} + h_1(z) + g_{\varepsilon_0(z)}$, where ε_0 is the principal Dirichlet character modulo q, and $\log H_\varepsilon(z) = \sum\limits_{\xi\in\operatorname{Im}\varepsilon} \xi\, h_\xi(z) + g_\varepsilon(z)$ when $\varepsilon \neq \varepsilon_0$. From this we see that in all cases, there is $b_\varepsilon \in \mathbb{C}$ satisfying

$$H_\varepsilon(z) = \frac{b_\varepsilon}{z-1} + k_\varepsilon(z),$$

where k_ε is holomorphic on $\operatorname{Re}(z) \geqslant 1$. Therefore

$$\sum_{\substack{n\geqslant 1\\ n\equiv d \bmod q}} \frac{f(n)^2}{n^z} = \frac{b}{z-1} + k(z),$$

where $b \in \mathbb{C}$ and k is holomorphic on $\operatorname{Re}(z) \geqslant 1$. We can now apply Wiener-Ikehara's theorem (see [9]) to deduce the result. $\square$

Remark 2. Notice that the natural density of the set

$$\{n \in \mathbb{N}\colon\ (n, N) = 1,\ n \equiv d \bmod q,\ a(tn^2) \neq 0\}$$

is independent of the choice of d. Indeed, from Wiener-Ikehara's theorem we know that this density is equal to $(h_1(1) + g_{\varepsilon_0}(1))/\varphi(q)$.

5. Proof of Theorem 1

Before starting the proof, recall the theorem of Delange (see [5]).

Theorem 5. *Let $g\colon \mathbb{N}\to\mathbb{C}$ be a multiplicative arithmetic function for which:*

(1) *for all $n\in\mathbb{N}$, $|g(n)|\leqslant 1$;*

(2) *there exists $a\in\mathbb{C}$ such that $a\neq 1$ and satisfying $\lim\limits_{x\to\infty}\sum\limits_{\substack{p\text{ prime}\\ p\leqslant x}} g(p)/\pi(x)=a$.*

Then we have

$$\lim_{x\to\infty}\sum_{n\leqslant x} g(n)/x=0.$$

We can now piece together the previous lemmas to prove Theorem 1.

Proof of Theorem 1. We have

$$\sum_{\substack{1\leqslant n\leqslant x\\ n\equiv d \bmod q}} f(n)=\frac{1}{\varphi(q)}\sum_{\varepsilon \bmod q}\Bigg(\sum_{1\leqslant n\leqslant x} f(n)\varepsilon(n)\Bigg)\times\overline{\varepsilon(d)}. \tag{10}$$

For a Dirichlet character ε modulo q we have

$$\begin{aligned}&\lim_{x\to\infty}\sum_{1\leqslant p\leqslant x} f(p)\varepsilon(p)/\pi(x)\\ &\qquad=\lim_{x\to\infty}\sum_{\xi\in\operatorname{Im}\varepsilon}\xi\frac{\#\{p\leqslant x\colon\ p\in\mathbb{P}_{\varepsilon,\xi,>}\}}{\pi(x)}-\xi\frac{\#\{p\leqslant x\colon\ p\in\mathbb{P}_{\varepsilon,\xi,<}\}}{\pi(x)}=0,\end{aligned}$$

since $\mathbb{P}_{\varepsilon,\xi,>}$ and $\mathbb{P}_{\varepsilon,\xi,<}$ have the same natural density $1/(2\#\operatorname{Im}\varepsilon)$. Applying Delange's theorem, we get $\lim\limits_{x\to\infty}\sum\limits_{1\leqslant n\leqslant x} f(n)\varepsilon(n)/x=0$, and consequently,

$$\lim_{x\to\infty}\sum_{\substack{1\leqslant n\leqslant x\\ n\equiv d \bmod q}} f(n)/x=0.$$

From this we have

$$\begin{aligned}\lim_{x\to\infty}&\frac{\#\{n\leqslant x\colon\ (n,N)=1,\ n\equiv d \bmod q,\ a(tn^2)/\chi(n)>0\}}{x}\\ &-\frac{\#\{n\leqslant x\colon\ (n,N)=1,\ n\equiv d \bmod q,\ a(tn^2)/\chi(n)<0\}}{x}=0.\end{aligned} \tag{11}$$

By Lemma 4, there is $b>0$ such that

$$\begin{aligned}\lim_{x\to\infty}&\frac{\#\{n\leqslant x\colon\ (n,N)=1,\ n\equiv d \bmod q,\ a(tn^2)/\chi(n)>0\}}{x}\\ &+\frac{\#\{n\leqslant x\colon\ (n,N)=1,\ n\equiv d \bmod q,\ a(tn^2)/\chi(n)<0\}}{x}=b.\end{aligned} \tag{12}$$

The result follows from (11) and (12). □

We show finally by another method how the natural density of the set defined in Lemma 4 is independent of d.

Proposition 4. *Under the assumptions of Theorem* 1, *the natural density of the set*

$$\{n \in \mathbb{N} \colon\ (n, N) = 1,\ n \equiv d \bmod q,\ a(tn^2) \neq 0\}$$

is equal to

$$\frac{1}{\varphi(q)} \lim_{z \to 1^+} (z-1) \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{f(n)^2}{n^z}.$$

P r o o f of Proposition 4. Since $\{n \in \mathbb{N} \colon\ (n, N) = 1,\ n \equiv d \bmod q,\ a(tn^2) \neq 0\}$ has natural density by Lemma 4, it suffices to prove that the Dedekind-Dirichlet density of this set is equal to

$$\frac{1}{\varphi(q)} \lim_{z \to 1^+} (z-1) \sum_{\substack{n=1 \\ (n,q)=1^{\infty}}} \frac{f(n)^2}{n^z}.$$

We shall define

$$B(z) = \sum_{\substack{n=1 \\ n \equiv d \bmod q^{\infty}}} \frac{f(n)^2}{n^z}$$

and

$$C_\varepsilon(z) = \sum_{n=1}^{\infty} \frac{f(n)^2 \varepsilon(n)}{n^z},$$

where ε runs over Dirichlet characters modulo q.

We must now compute $\lim\limits_{z \to 1^+} (z-1) B(z)$. By the same computations as in the previous theorem, it suffices to compute $\lim\limits_{z \to 1^+} (z-1) C_\varepsilon(z)$. We have

$$\begin{aligned}
\frac{C_\varepsilon(z)}{L(z,\varepsilon)} &= \prod_{p \in \mathbb{P}} \sum_{k=0}^{\infty} f(p^k)^2 \varepsilon(p^k) p^{-kz} \times \prod_{p \in \mathbb{P}} \Big(1 - \frac{\varepsilon(p)}{p^z}\Big) \\
&= \prod_{p \in \mathbb{P}} \Big(1 - \frac{\varepsilon(p)}{p^z}\Big) \times \prod_{p \in \mathbb{P}} \Bigg(1 + \sum_{\substack{k=1 \\ a(tp^{2k}) \neq 0}}^{\infty} \frac{\varepsilon(p^k)}{p^{kz}}\Bigg) \\
&= \prod_{\substack{p \in \mathbb{P} \\ a(tp^2) \neq 0}} \Bigg[\Big(1 - \frac{\varepsilon(p)}{p^z}\Big)\Bigg(1 + \frac{\varepsilon(p)}{p^z} + \sum_{\substack{k=2 \\ a(tp^{2k}) \neq 0}}^{\infty} \frac{\varepsilon(p^k)}{p^{kz}}\Bigg)\Bigg] \\
&\quad \times \prod_{\substack{p \in \mathbb{P} \\ a(tp^2) = 0}} \Bigg[\Big(1 - \frac{\varepsilon(p)}{p^z}\Big)\Bigg(1 + \sum_{\substack{k=2 \\ a(tp^{2k}) \neq 0}}^{\infty} \frac{\varepsilon(p^k)}{p^{kz}}\Bigg)\Bigg]
\end{aligned}$$

$$= \prod_{\substack{p\in\mathbb{P}\\ a(tp^2)\neq 0}} \Big(1 - \frac{\varepsilon(p^2)}{p^{2z}} + h_1(z,p)\Big) \times \prod_{\substack{p\in\mathbb{P}\\ a(tp^2)=0}} \Big(1 - \frac{\varepsilon(p)}{p^{z}} + h_2(z,p)\Big),$$

where $h_1(z,p)$ and $h_2(z,p)$ are the remaining terms. Apply logarithm to the ratio $C_\varepsilon(z)/L(z,\varepsilon)$ and notice that

$$\sum_{\substack{p\in\mathbb{P}\\ a(tp^2)\neq 0}} \log \Big(1 - \frac{\varepsilon(p^2)}{p^{2z}} + h_1(z,p)\Big)$$

is holomorphic on $\mathrm{Re}(z) \geqslant 1$. On the other hand, we have

$$\sum_{\substack{p\in\mathbb{P}\\ a(tp^2)=0}} \log \Big(1 - \frac{\varepsilon(p)}{p^{z}} + h_2(z,p)\Big) = \sum_{\substack{p\in\mathbb{P}\\ a(tp^2)=0}} \frac{\varepsilon(p)}{p^{z}} + h_3(z,p),$$

where $h_3(z,p)$ is holomorphic on $\mathrm{Re}(z) \geqslant 1$. Further, since for all roots of unity ξ such that $\xi \in \mathrm{Im}\,\varepsilon$, the set $\mathbb{P}_{\varepsilon,\xi,=0}$ is a regular set of primes of density 0 by Theorem 3, then

$$\sum_{\substack{p\in\mathbb{P}\\ a(tp^2)=0}} \frac{\varepsilon(p)}{p^{z}} = \sum_{\xi\in\mathrm{Im}\,\varepsilon} \xi \sum_{p\in\mathbb{P}_{\varepsilon,\xi,=0}} \frac{1}{p^{z}}$$

is also holomorphic on $\mathrm{Re}(z) \geqslant 1$. Thus $\log C_\varepsilon(z)/L(z,\varepsilon)$ is holomorphic on $\mathrm{Re}(z) \geqslant 1$ and by taking exponential we see that $C_\varepsilon(z)/L(z,\varepsilon)$ is also holomorphic on $\mathrm{Re}(z) \geqslant 1$. Then the limit $\lim_{z\to 1^+} (z-1)C_{\varepsilon_0}(z)$ exists, where ε_0 is the principal character modulo q, and $\lim_{z\to 1^+} (z-1)C_\varepsilon(z) = 0$ when $\varepsilon \neq \varepsilon_0$,

$$\lim_{z\to 1^+} (z-1)B(z) = \frac{1}{\varphi(q)} \lim_{z\to 1^+} (z-1)C_{\varepsilon_0}(z) = \frac{1}{\varphi(q)} \lim_{z\to 1^+} (z-1) \sum_{\substack{n=1\\ (n,q)=1}}^{\infty} \frac{f(n)^2}{n^z}.$$

□

We conclude with some related remarks.

Remark 3. When $q = N$ or $(q,N) = 1$, the Dedekind-Dirichlet density of the set $\{n \in \mathbb{N}\colon\ (n,N) = 1,\ n \equiv d \bmod q,\ a(tn^2) = 0\}$ exists. Indeed, we have

$$\lim_{z\to 1^+} (z-1) \sum_{\substack{n\geqslant 1\\ n\equiv d \bmod q}} \frac{1}{n^z} = \frac{1}{q}.$$

By Lemma 3, it follows that

$$\lim_{z\to 1^+}(z-1)\left(2\sum_{\substack{(n,N)=1\\ a(tn^2)/\chi(n)>0\\ n\equiv d \bmod q}}\frac{1}{n^z}+\sum_{\substack{(n,N)=1\\ a(tn^2)=0\\ n\equiv d \bmod q}}\frac{1}{n^z}+\sum_{\substack{(n,N)\neq 1\\ n\equiv d \bmod q}}\frac{1}{n^z}\right)=\frac{1}{q}. \tag{13}$$

Let χ_0 be a principal character modulo N. We have

$$\begin{aligned}\sum_{\substack{(n,N)=1\\ n\equiv d \bmod q}}\frac{1}{n^z}&=\sum_{n\equiv d \bmod q}\frac{\chi_0(n)}{n^z}=\frac{1}{\varphi(q)}\sum_{n\geqslant 0}\frac{\chi_0(n)}{n^z}\sum_{\varepsilon \bmod q}\overline{\varepsilon(d)}\varepsilon(n)\\&=\frac{1}{\varphi(q)}\sum_{\varepsilon \bmod q}\overline{\varepsilon(d)}\sum_{n\geqslant 0}\frac{\chi_0(n)\varepsilon(n)}{n^z}.\end{aligned}$$

Following our hypothesis, if $q=N$, we consider $\chi_0\varepsilon$ as a character modulo N, if $(q,N)=1$, we consider it as a character modulo qN. Therefore

$$\lim_{z\to 1+}\sum_{\substack{(n,N)=1\\ n\equiv d \bmod q}}\frac{1}{n^z}$$

exists and thus

$$\lim_{z\to 1+}\sum_{\substack{(n,N)\neq 1\\ n\equiv d \bmod q}}\frac{1}{n^z}$$

also exists. Replace this in (13) and the result follows.

Remark 4. A weaker version of Theorem 1 could be obtained using Proposition 4. Indeed, in the proof of the previous proposition there is $b>0$ such that $\lim_{z\to 1^+}(z-1)B(z)=b$. Hence $\{n\in\mathbb{N}\colon (n,N)=1,\ n\equiv d \bmod q \text{ and } a(tn^2)\neq 0\}$ has a Dedekind-Dirichlet density equal to b. It follows from (13) that

$$\lim_{z\to 1^+}(z-1)\left(\sum_{\substack{(n,N)=1\\ n\equiv d \bmod q\\ a(tn^2)=0}}\frac{1}{n^z}+\sum_{\substack{(n,N)\neq 1\\ n\equiv d \bmod q}}\frac{1}{n^z}\right)=\frac{1}{q}-b.$$

Replace this in (13) to get

$$\lim_{z\to 1^+}(z-1)\sum_{\substack{(n,N)=1\\ n\equiv d \bmod q\\ a(tn^2)/\chi(n)>0}}\frac{1}{n^z}=\frac{b}{2}.$$

The equidistribution obtained here is in terms of the Dedekind-Dirichlet density only.

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