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FUNCTIONAL INEQUALITIES AND MANIFOLDS WITH NONNEGATIVE WEIGHTED RICCI CURVATURE

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Abstract. We show that n-dimensional ($n \geqslant 2$) complete and noncompact metric measure spaces with nonnegative weighted Ricci curvature in which some Caffarelli-Kohn-Nirenberg type inequality holds are isometric to the model metric measure n-space (i.e. the Euclidean metric n-space). We also show that the Euclidean metric spaces are the only complete and noncompact metric measure spaces of nonnegative weighted Ricci curvature satisfying some prescribed Sobolev type inequality.

Keywords: Caffarelli-Kohn-Nirenberg type inequality; weighted Ricci curvature; volume comparison

MSC 2010: 53C21, 31C12

1. Introduction

Functional inequalities, such as the Sobolev inequality, the Caffarelli-Kohn-Nirenberg inequality, the Gagliardo-Nirenberg inequality, and so on, in the Euclidean space or on general Riemannian manifolds have been studied intensively (see, e.g., [2], [3], [6], [7], [10], [13], [14], [17], [18], [31] and the references therein). In this subject, an interesting and difficult topic is trying to find the *best* constant for the functional inequality of a given type. Many works focus on this topic and some interesting results have been obtained (see, e.g., [1], [28], [30], [32]). Denote by $C_0^\infty(\mathbb{R}^n)$ the space of smooth functions with compact support on the n-dimensional Euclidean

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space $\mathbb{R}^n$. In [6], Caffarelli, Kohn and Nirenberg have proven the validity of a functional inequality for any function in the space $C_0^\infty(\mathbb{R}^n)$. However, they did not give the possible best constant therein. Following the convention, we call functional inequalities having the same type as the one obtained in [6] the *Caffarelli-Kohn-Nirenberg type* inequalities. By considering some special cases, Xia in [32] has given the exact value of the smallest admissible constant for the Caffarelli-Kohn-Nirenberg type equality given in [6]. In fact, he proved the following.

Theorem 1.1 ([32]). *Let $n \geqslant 2$, $r > p > 1$, and α, β be fixed real numbers satisfying*

$$\frac{1}{p} + \frac{\alpha}{n},\ \frac{p-1}{p(r-1)} + \frac{\beta}{n},\ \frac{1}{r} + \frac{\gamma}{n} > 0,$$

where

$$\gamma = \frac{1}{r}(\alpha - 1) + \frac{p-1}{pr}\beta.$$

Then for all $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned}\text{(1.1)}\quad &\int_{\mathbb{R}^n} |x|^{r\gamma} |f|^r \,\mathrm{d}x \\ &\leqslant \frac{r}{n + r\gamma} \left(\int_{\mathbb{R}^n} |x|^{\alpha p} |\nabla f|^p \,\mathrm{d}x \right)^{1/p} \left(\int_{\mathbb{R}^n} |x|^{\beta} |f|^{p(r-1)/(p-1)} \,\mathrm{d}x \right)^{(p-1)/p},\end{aligned}$$

where $|x|$ is the Euclidean length of $x \in \mathbb{R}^n$. Moreover, when

$$n + \beta < \left(1 - \alpha + \frac{\beta}{p}\right) \frac{(r-1)p}{r-p}, \tag{1.2}$$

the inequality is the best possible in the sense that

$$\inf_{f \in C_0^\infty(\mathbb{R}^n) \setminus \{0\}} \frac{\left(\int_{\mathbb{R}^n} |x|^{\alpha p} |\nabla f|^p \,\mathrm{d}x\right)^{1/p} \left(\int_{\mathbb{R}^n} |x|^{\beta} |f|^{p(r-1)/(p-1)}\right)^{(p-1)/p}}{\int_{\mathbb{R}^n} |x|^{r\gamma} |f|^r \,\mathrm{d}x} = \frac{r}{n + r\gamma} \tag{1.3}$$

and a family of minimizers of (1.3) *is given by*

$$f_{\min}(x) = (\lambda + |x|^{1-\alpha+\beta/p})^{(1-p)/(r-p)}, \quad \lambda > 0.$$

For a given complete Riemannian manifold M let $C_0^\infty(M)$ be the space of smooth functions on M with compact support, and $\mathrm{d}v_g$ be the volume element (i.e. Riemannian measure) related to the Riemannian metric g. By applying the Caffarelli-Kohn-Nirenberg type inequality (1.1) which has the best constant, the Bishop-Gromov's volume comparison theorem for manifolds with Ricci curvature bounded from below (see, e.g., [8], pages 71–73), and constructing functions based on the minimizer $f_{\min}$ above, Xia in [32] has proven the following rigidity theorem.

Theorem 1.2 ([32]). *Let n, r, p, α, β, γ be as in Theorem 1.1, and assume that (1.2) holds. Let M be an n-dimensional complete open Riemannian manifold with nonnegative Ricci curvature. Fix a point $x_0 \in M$ and denote by μ the distance function on M from x_0. If for any $f \in C_0^\infty(M)$ we have*

$$\int_M |\mu|^{r\gamma}|f|^r \,\mathrm{d}v_g \leqslant \frac{r}{n+r\gamma}\left(\int_M |\mu|^{\alpha p}|\nabla f|^p \,\mathrm{d}v_g\right)^{1/p}\left(\int_M |\mu|^{\beta}|f|^{p(r-1)/(p-1)} \,\mathrm{d}v_g\right)^{(p-1)/p},$$

then M is isometric to $\mathbb{R}^n$.

It is interesting to know *under what kind of conditions* a complete open n-manifold ($n \geqslant 2$) is isometric to $\mathbb{R}^n$ or has finite topological type, which in essence has relation with the splittingness of the prescribed manifold. This is a classical topic in the global geometry, which has been investigated intensively (see e.g. [9], [20], [25]).

One purpose of this paper is to generalize Theorem 1.2. For that we need to use the following notions of smooth metric measure spaces and the weighted Ricci curvature.

A smooth metric measure space (also known as the weighted measure space) is actually a Riemannian manifold equipped with some measure which is conformal to the usual Riemannian measure. More precisely, for a given complete n-dimensional Riemannian manifold (M,g) with the metric g, the triple $(M, g, \mathrm{e}^{-\varphi}\,\mathrm{d}v_g)$ is called a smooth metric measure space, where φ is a *smooth real-valued* function on M and, as before, $\mathrm{d}v_g$ is the Riemannian volume element related to g (sometimes, we also call $\mathrm{d}v_g$ the volume density). Correspondingly, for a geodesic ball $B(x_0, r)$ on M, with center $x_0 \in M$ and radius r, one can also define its *weighted* (or φ-)*volume* $\mathrm{vol}_\varphi[B(x_0,r)]$ as

$$\mathrm{vol}_\varphi[B(x_0,r)] := \int_{B(x_0,r)} \mathrm{e}^{-\varphi}\,\mathrm{d}v_g.$$

Now, for convenience, *we also make an agreement that in this paper* $\mathrm{vol}_\varphi(\cdot)$ *represents the weighted (or φ-)volume of the given geometric object on a metric measure space.*

For a given smooth metric measure space $(M, g, \mathrm{e}^{-\varphi}\,\mathrm{d}v_g)$, the following N-Bakry-Émery tensor

$$\mathrm{Ric}_\varphi^N := \mathrm{Ric} + \mathrm{Hess}\,\varphi - \frac{\mathrm{d}\varphi \otimes \mathrm{d}\varphi}{N},$$

with Ric and Hess being the Ricci and the Hessian operators on M, respectively, can be considered. Especially, when $N = \infty$, the N-Bakry-Émery tensor Ric_φ^N

degenerates into the so-called ∞-*Bakry-Émery Ricci tensor* Ric_φ which is given by

$$\mathrm{Ric}_\varphi = \mathrm{Ric} + \mathrm{Hess}\,\varphi.$$

The ∞-Bakry-Émery Ricci tensor is also called the *weighted Ricci tensor*. Bakry and Émery in [4], [5] introduced firstly and extensively investigated the generalized Ricci tensor above and its relationship with diffusion processes.

Similarly to the p-norm of smooth functions with compact support on the manifold (M, g) defined in Theorem 1.2, for the smooth metric measure space $(M, g, \mathrm{e}^{-\varphi}\,\mathrm{d}v_g)$ and any $u \in C_0^\infty(M)$, we can define the *weighted p-norm* $\|u\|_{p;MMS}$ of u as

$$\|u\|_{p;MMS} := \left(\int_M |u|^p \cdot \mathrm{e}^{-\varphi}\,\mathrm{d}v_g\right)^{1/p}.$$

Clearly, when $\varphi \equiv 0$, the weighted p-norm is just the p-norm.

One might have an illusion that smooth metric measure spaces are not necessary to be studied since they are simply obtained from corresponding Riemannian manifolds by adding a conformal measure to the Riemannian measure. However, the opposite is true; they do have many differences. For instance, when Ric_φ is bounded from below, the Myer's theorem, Bishop-Gromov's volume comparison, Cheeger-Gromoll's splitting theorem and Abresch-Gromoll's excess estimate cannot hold as in the Riemannian case. Here, for the purpose of comprehension, we would like to repeat an example given in [29], Example 2.1. That is, for the metric measure space $(\mathbb{R}^n, g_{\mathbb{R}^n}, \mathrm{e}^{-\varphi}\,\mathrm{d}v_{g_{\mathbb{R}^n}})$, where $g_{\mathbb{R}^n}$ and $\mathrm{d}v_{g_{\mathbb{R}^n}}$ are the usual Euclidean metric and the Euclidean volume density related to $g_{\mathbb{R}^n}$, respectively, if $\varphi(x) = \frac{1}{2}\lambda|x|^2$ for $x \in \mathbb{R}^n$, then we have $\mathrm{Hess} = \lambda g_{\mathbb{R}^n}$ and $\mathrm{Ric}_\varphi = \lambda g_{\mathbb{R}^n}$. Therefore, from this example we know that unlike in the case of Ricci curvature bounded from below uniformly by some positive constant, a metric measure space is not necessarily compact provided $\mathrm{Ric}_\varphi \geqslant \lambda$ and $\lambda > 0$. So, it is meaningful to study the geometry of smooth metric measure spaces. For the basic and necessary knowledge about the metric measure spaces, we refer the readers to the excellent work of Wei and Wylie (see [29]). The subject on the metric measure space and the related weighted Ricci tensor occurs naturally in many different subjects and has many important applications (see e.g. [19], [24], [29]).

By applying the volume comparison result [29], Theorem 1.2 for smooth metric measure spaces with weighted Ricci curvature bounded from below (see also Theorem 2.7) and considering the notion of the weighted p-norm briefly introduced above, we can obtain the following.

Theorem 1.3. *Let n, r, p, α, β, γ be as in Theorem* 1.1, *and assume that* (1.2) *holds. Assume that $(M, g, \mathrm{e}^{-\varphi}\,\mathrm{d}v_g)$ is an n-dimensional ($n \geqslant 2$) complete and non-compact smooth metric measure space with nonnegative weighted Ricci curvature.*

For a point $x_0 \in M$ at which $\varphi(x_0)$ is away from $-\infty$, assume that the radial derivative $\partial_t \varphi$ satisfies $\partial_t \varphi \geqslant 0$ along all minimal geodesic segments from x_0, with $t := d(x_0, \cdot)$ being the distance to x_0 (on M). If furthermore for any $f \in C_0^\infty(M)$ the Caffarelli-Kohn-Nirenberg type inequality

$$\begin{aligned}(1.4)\quad &\int_M |t|^{r\gamma}|f|^r \mathrm{e}^{-\varphi}\,\mathrm{d}v_g \\ &\leqslant \frac{r}{n+r\gamma}\Big(\int_M |t|^{\alpha p}|\nabla f|^p \mathrm{e}^{-\varphi}\,\mathrm{d}v_g\Big)^{1/p}\Big(\int_M |t|^{\beta}|f|^{p(r-1)/(p-1)}\mathrm{e}^{-\varphi}\,\mathrm{d}v_g\Big)^{(p-1)/p}\end{aligned}$$

holds, then (M,g) is isometric to $(\mathbb{R}^n, g_{\mathbb{R}^n})$, and moreover, in this case we have that $\varphi \equiv \varphi(x_0)$ is a constant function with respect to the variable t, and $\mathrm{e}^{-\varphi}\,\mathrm{d}v_g = \mathrm{e}^{-\varphi(x_0)}\,\mathrm{d}v_{g_{\mathbb{R}^n}}$.

Remark 1.4. Since φ is a smooth real-valued function on the complete noncompact manifold M, we have that if $\varphi(x)$ does not tend to $-\infty$ as x tends to the infinity, then x_0 can be chosen arbitrarily; if $\varphi(x) \to -\infty$ as $x \to \infty$, then x_0 can be chosen to be any point except those points near the infinity. If $\varphi \equiv 0$ on M, then the metric measure space $(M, g, \mathrm{e}^{-\varphi}\,\mathrm{d}v_g)$ can be seen as the Riemannian manifold (M,g) directly. Clearly, in this case, Theorem 1.3 is totally the same as Theorem 1.2 above. So, we can equivalently say that Theorem 1.3 in [32] is only a special case of Theorem 1.3. See Subsection 2.1 for the precise explanation about the radial derivative ∂_t and the radial direction w.r.t. the point x_0.

Now, we would like to review some existent results and a recent new conclusion of myself (see [23]) briefly to *reveal the affection of functional inequalities to the geometric structure of a given complete noncompact manifold*, which is actually the purpose of writing this paper.

Given $q \in [1, n)$, let $\widetilde{q} = nq/(n-q)$. Let $K(n,q)$ be the best constant for the Euclidean Sobolev inequality, which means that

$$(1.5)\qquad K(n,q)^{-1} = \inf_{f \in C_0^\infty(\mathbb{R}^n)\setminus\{0\}} \frac{\big(\int_{\mathbb{R}^n}|\nabla f|^q\big)^{1/q}}{\big(\int_{\mathbb{R}^n}|f|^{\widetilde{q}}\big)^{1/\widetilde{q}}}.$$

For this best constant, we know that (see [2], [3], [14], [28])

$$K(n,1) = n^{-1}D_n^{-1/n},$$

and for $q > 1$

$$K(n,q) = \frac{1}{n}\Big[\frac{n(q-1)}{n-q}\Big]^{(q-1)/q}\Big[\frac{\Gamma(n+1)}{nD_n\Gamma(n/q)\Gamma(n+1-n/q)}\Big]^{1/n},$$

where D_n is the volume of the unit ball in $\mathbb{R}^n$ and Γ is the Euler function. Besides, for $q > 1$ the infimum in (1.5) can be achieved by the function $(\lambda + |x|^{q/(q-1)})^{1-n/q}$, $\lambda > 0$ with $|x|$ being the Euclidean length of the vector x in $\mathbb{R}^n$. In [30], Xia showed that an n-dimensional ($n \geqslant 2$) complete open manifold M with nonnegative Ricci curvature, in which the following Sobolev inequality

$$\text{(1.6)} \qquad \Big(\int_M |u|^{\widetilde{q}} \,\mathrm{d}v_g\Big)^{1/\widetilde{q}} \leqslant K(n,q)\Big(\int_M |\nabla u|^q \,\mathrm{d}v_g\Big)^{1/q} \quad \forall\, u \in C_0^\infty(M),$$

holds, is *isometric* to $\mathbb{R}^n$. This fact generalizes Ledoux's corresponding result in [17].

Assume now that $1 < p < n$, $p < q \leqslant p(n-1)/(n-p)$ and denote by δ, r and θ the following

$$\text{(1.7)} \qquad \delta = np - (n-p)q, \quad r = p\frac{q-1}{p-1}, \quad \theta = \frac{(q-p)n}{(q-1)(np-(n-p)q)}.$$

Mao in [23] showed that for an n-dimensional ($n \geqslant 2$) complete and noncompact smooth metric measure space $(M, g, \mathrm{e}^{-\varphi}\,\mathrm{d}v_g)$ with nonnegative *weighted* Ricci curvature, if for a point $x_0 \in M$ at which $\varphi(x_0)$ is away from $-\infty$, the radial derivative $\partial_t\varphi$ satisfies $\partial_t\varphi \geqslant 0$ along all minimal geodesic segments from x_0, with $t := d(x_0, \cdot)$ being the distance to x_0 (on M), and a Gagliardo-Nirenberg type inequality

$$\text{(1.8)} \qquad \Big(\int_M |u|^r \cdot \mathrm{e}^{-\varphi}\,\mathrm{d}v_g\Big)^{1/r} \leqslant \Phi\Big(\int_M |\nabla u|^p \cdot \mathrm{e}^{-\varphi}\,\mathrm{d}v_g\Big)^{\theta/p}\Big(\int_M |u|^q \cdot \mathrm{e}^{-\varphi}\,\mathrm{d}v_g\Big)^{(1-\theta)/q} \quad \forall\, u \in C_0^\infty(M),$$

i.e.

$$\|u\|_{r;MMS} \leqslant \Phi \cdot \|\nabla u\|_{p;MMS}^{\theta} \cdot \|u\|_{q;MMS}^{1-\theta} \quad \forall\, u \in C_0^\infty(M),$$

with Φ given by

$$\Phi = \Big(\frac{q-p}{p\sqrt{\pi}}\Big)^{\theta}\Big(\frac{pq}{n(q-p)}\Big)^{\theta/p}\Big(\frac{\theta}{pq}\Big)^{1/r} \times \Big(\frac{\Gamma(q(p-1)/(q-p))\Gamma(\frac{1}{2}n+1)}{\Gamma(((p-1)/p)(\delta/(q-p)))\Gamma(n(p-1)/p+1)}\Big)^{\theta/n},$$

is satisfied on M, then (M, g) is isometric to $(\mathbb{R}^n, g_{\mathbb{R}^n})$. Moreover, in this case we have that $\varphi \equiv \varphi(x_0)$ is a constant function with respect to the variable t, and $\mathrm{e}^{-\varphi}\,\mathrm{d}v_g = \mathrm{e}^{-\varphi(x_0)}\,\mathrm{d}v_{g_{\mathbb{R}^n}}$. Clearly, when q in (1.7) is chosen to be $q = (n-1)p/(n-p)$, one has $\theta = 1$, $r = np/(n-p)$. If *furthermore* we require that $\varphi \equiv 0$, then the Gagliardo-Nirenberg inequality (1.8) degenerates into the Sobolev inequality (1.6), and Mao's conclusion in [23] is the same as Xia's result in [30] *for the case of* $q > 1$. So, when $q > 1$, Mao's conclusion in [23] generalizes *a lot* the corresponding results in [17], [30].

The above argument naturally leads us to consider the following problem.

Problem. Suppose that $(M, g, \mathrm{e}^{-\varphi}\,\mathrm{d}v_g)$ is an n-dimensional ($n \geqslant 2$) complete and noncompact smooth metric measure space with nonnegative weighted Ricci curvature. For a point $x_0 \in M$ at which $\varphi(x_0)$ is away from $-\infty$, assume that the radial derivative $\partial_t\varphi$ satisfies $\partial_t\varphi \geqslant 0$ along all minimal geodesic segments from x_0, with $t := d(x_0, \cdot)$ being the distance to x_0 (on M). Moreover, the Sobolev type inequality

$$\begin{aligned}(1.9)\qquad &\left(\int_M |u|^{n/(n-1)} \cdot \mathrm{e}^{-\varphi}\,\mathrm{d}v_g\right)^{(n-1)/n} \\ &\qquad \leqslant n^{-1} D_n^{-1/n} \int_M |\nabla u| \cdot \mathrm{e}^{-\varphi}\,\mathrm{d}v_g \quad \forall\, u \in C_0^\infty(M),\end{aligned}$$

holds. Could we get the assertion that "(M, g) *is isometric to* $(\mathbb{R}^n, g_{\mathbb{R}^n})$, *and moreover, in this case,* $\varphi \equiv \varphi(x_0)$ *is a constant function with respect to the variable* t, *and* $\mathrm{e}^{-\varphi}\,\mathrm{d}v_g = \mathrm{e}^{-\varphi(x_0)}\,\mathrm{d}v_{g_{\mathbb{R}^n}}$"?

Remark 1.5. Clearly, when $\varphi \equiv 0$, the assertion of the above *Problem* is the same with Xia's result (only *when* $q = 1$) in [30] mentioned above. Besides, by choosing $q = (n-1)p/(n-p)$ in (1.7) and applying Mao's result in [23] mentioned above directly, *for* $q > 1$ we get that for an n-dimensional ($n \geqslant 2$) complete and noncompact smooth metric measure space $(M, g, \mathrm{e}^{-\varphi}\,\mathrm{d}v_g)$, if for a point $x_0 \in M$ at which $\varphi(x_0)$ is away from $-\infty$, the radial derivative $\partial_t\varphi$ satisfies $\partial_t\varphi \geqslant 0$ along all minimal geodesic segments from x_0, with $t := d(x_0, \cdot)$ being the distance to x_0 (on M), and the Sobolev type inequality

$$\left(\int_M |u|^{\widetilde{q}} \cdot \mathrm{e}^{-\varphi}\,\mathrm{d}v_g\right)^{1/\widetilde{q}} \leqslant K(n,q)\left(\int_M |\nabla u|^q \cdot \mathrm{e}^{-\varphi}\,\mathrm{d}v_g\right)^{1/q} \quad \forall u \in C_0^\infty(M),$$

holds, where $\widetilde{q} = nq/(n-q)$ and $K(n,q)$ is defined by (1.5), then (M, g) is isometric to $(\mathbb{R}^n, g_{\mathbb{R}^n})$. Based on these facts, we think the assertion of *Problem* above might be *true*.

As we know, when $\varphi \equiv 0$, the Sobolev type inequality (1.9) is equivalent to the isoperimetric inequality (see [26])

$$[\mathrm{vol}(\Omega)]^{n(-1)/n} \leqslant n^{-1} D_n^{-1/n} \cdot \mathrm{vol}(\partial\Omega),$$

where $\partial\Omega$ is the boundary of an open bounded domain $\Omega \subset M$, and $\mathrm{vol}(\Omega)$ and $\mathrm{vol}(\partial\Omega)$ denote the volumes of Ω and $\partial\Omega$, respectively. Isoperimetric inequalities generally link with the geometric structure of manifolds, so, from this aspect, it is also interesting to consider the above *Problem*. In this paper, for convenience, *we make an agreement that* $\mathrm{vol}(\cdot)$ *represents the volume of the given geometric object.*

The paper is organized as follows. Some useful facts, including two volume comparison theorems for complete manifolds with radial curvature bounded and a volume comparison result for smooth metric measure spaces with weighted Ricci curvature bounded from below, will be reviewed in Section 2. The proof of Theorem 1.3 will be shown in Section 3. A partial answer to *Problem* above will be given in Section 4.

2. Useful facts

We would like to review [29], Theorem 1.2 and [11], Theorem 3.3, Corollary 3.4 and Theorem 4.2, which consist of the key point of the proof of Theorem 1.3 shown in the next section. However, some necessary preliminaries should be introduced first. In fact, one can find a similar version (see [23], Section 2) of this section, but we still give it here so that the readers can understand [29], Theorem 1.2 and [11], Theorem 3.3, Corollary 3.4 and Theorem 4.2 completely and clearly.

2.1. Preliminaries. Denote by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$. Given an n-dimensional ($n \geqslant 2$) complete Riemannian manifold (M, g) with the metric g, for a point $x \in M$ let S_x^{n-1} be the unit sphere with center x in the tangent space T_xM, and let $\operatorname{Cut}(x)$ be the cut-locus of x, which is a closed set of zero n-Hausdorff measure. Clearly,

$$\mathbb{D}_x = \{t\xi\colon\ 0 \leqslant t < d_\xi,\ \xi \in S_x^{n-1}\}$$

is a star-shaped open set of T_xM through which the exponential map $\exp_x\colon\ \mathbb{D}_x \to M \setminus \operatorname{Cut}(x)$ gives a diffeomorphism from $\mathbb{D}_x$ to the open set $M \setminus \operatorname{Cut}(x)$, where d_ξ is defined by

$$\begin{aligned} d_\xi = d_\xi(x) := \sup\{t > 0\colon\ &\gamma_\xi(s) := \exp_x(s\xi) \\ &\text{is the unique minimal geodesic joining } x \text{ and } \gamma_\xi(t)\}. \end{aligned}$$

As in [8], we can introduce two important maps used to construct the geodesic spherical coordinate chart at a prescribed point on a Riemannian manifold. For a fixed vector $\xi \in T_xM$, $|\xi| = 1$, let $\xi^\perp$ be the orthogonal complement of $\{\mathbb{R}\xi\}$ in T_xM, and let $\tau_t\colon\ T_xM \to T_{\exp_x(t\xi)}M$ be the parallel translation along $\gamma_\xi(t)$. The path of linear transformations $\mathbb{A}(t,\xi)\colon\ \xi^\perp \to \xi^\perp$ is defined by

$$\mathbb{A}(t,\xi)\eta = (\tau_t)^{-1}Y_\eta(t),$$

where $Y_\eta(t) = d(\exp_x)_{(t\xi)}(t\eta)$ is the Jacobi field along $\gamma_\xi(t)$ satisfying $Y_\eta(0) = 0$, and $(\nabla_t Y_\eta)(0) = \eta$. Moreover, for $\eta \in \xi^\perp$, set

$$\mathcal{R}(t)\eta = (\tau_t)^{-1}R(\gamma_\xi'(t), \tau_t\eta)\gamma_\xi'(t),$$

where the curvature tensor $R(X,Y)Z$ is defined by $R(X,Y)Z = -[\nabla_X, \nabla_Y]Z + \nabla_{[X,Y]}Z$. Then $\mathcal{R}(t)$ is a self-adjoint operator on $\xi^\perp$, whose trace is the radial Ricci tensor $\mathrm{Ric}_{\gamma_\xi(t)}(\gamma'_\xi(t), \gamma'_\xi(t))$. Clearly, the map $\mathbb{A}(t,\xi)$ satisfies the Jacobi equation $\mathbb{A}'' + \mathcal{R}\mathbb{A} = 0$ with initial conditions $\mathbb{A}(0,\xi) = 0$, $\mathbb{A}'(0,\xi) = I$. By Gauss's lemma, the Riemannian metric of $M \setminus \mathrm{Cut}(x)$ in the geodesic spherical coordinate chart can be expressed by

$$\mathrm{d}s^2(\exp_x(t\xi)) = \mathrm{d}t^2 + |\mathbb{A}(t,\xi)\,\mathrm{d}\xi|^2 \quad \forall\, t\xi \in \mathbb{D}_x. \tag{2.1}$$

We consider the metric components $g_{ij}(t,\xi)$, $i,j \geqslant 1$, in a coordinate system $\{t, \xi_a\}$ formed by fixing an orthonormal basis $\{\eta_a, a \geqslant 2\}$ of $\xi^\perp = T_\xi S_x^{n-1}$, and then extending it to a local frame $\{\xi_a, a \geqslant 2\}$ of S_x^{n-1}. Define a function $J > 0$ on $\mathbb{D}_x \setminus \{x\}$ by

$$J^{n-1} = \sqrt{|g|} := \sqrt{\det[g_{ij}]}. \tag{2.2}$$

Since $\tau_t \colon S_x^{n-1} \to S_{\gamma_\xi(t)}^{n-1}$ is an isometry, we have

$$\langle \mathrm{d}(\exp_x)_{t\xi}(t\eta_a), \mathrm{d}(\exp_x)_{t\xi}(t\eta_b)\rangle_g = \langle \mathbb{A}(t,\xi)(\eta_a), \mathbb{A}(t,\xi)(\eta_b)\rangle_g,$$

and then

$$\sqrt{|g|} = \det \mathbb{A}(t,\xi).$$

So, by applying (2.1) and (2.2), the volume $\mathrm{vol}[B(x,r)]$ of a geodesic ball $B(x,r)$, with radius r and center x, on M is given by

$$\begin{aligned} \mathrm{vol}[B(x,r)] &= \int_{S_x^{n-1}} \int_0^{\min\{r, d_\xi\}} \sqrt{|g|}\,\mathrm{d}t\,\mathrm{d}\sigma \\ &= \int_{S_x^{n-1}} \left(\int_0^{\min\{r, d_\xi\}} \det(\mathbb{A}(t,\xi))\,\mathrm{d}t \right) \mathrm{d}\sigma, \end{aligned} \tag{2.3}$$

where $\mathrm{d}\sigma$ denotes the $(n-1)$-dimensional volume element on $\mathbb{S}^{n-1} \equiv S_x^{n-1} \subseteq T_xM$. As in Section 1, let $r(z) = d(x,z)$ be the intrinsic distance to the point $x \in M$. Since for any $\xi \in S_x^{n-1}$ and $t_0 > 0$ we have $\nabla r(\gamma_\xi(t_0)) = \gamma'_\xi(t_0)$ when the point $\gamma_\xi(t_0) = \exp_x(t_0\xi)$ is away from the cut locus of x (see [12]), then, by the definition of a nonzero tangent vector "*radial*" to a prescribed point on a manifold given in the first page of [15], we know that for $z \in M \setminus (\mathrm{Cut}(x) \cup x)$ the unit vector field

$$v_z := \nabla r(z)$$

is the radial unit tangent vector at z. Set

$$l(x) := \max_{z \in M} r(z) = \max_{z \in M} d(x,z). \tag{2.4}$$

Then we have $l(x) = \max_\xi d_\xi$ (see [11], Section 2). We also need the following fact about $r(z)$ (see [25], Proposition 39, page 266):

$$\partial_r \Delta r + \frac{(\Delta r)^2}{n-1} \leqslant \partial_r \Delta r + |\mathrm{Hess}\, r|^2 = -\mathrm{Ric}(\partial_r, \partial_r) \text{ with } \Delta r = \partial_r \ln \sqrt{|g|},$$

when $\partial_r = \nabla r$ is a differentiable vector (see [25], Proposition 7, page 47 for the differentiation of ∂_r), and Δ is the Laplace operator on M and $\mathrm{Hess}\, r$ is the Hessian of $r(z)$. Then, using also (2.2), we have

$$J'' + \frac{1}{(n-1)}\mathrm{Ric}(\gamma'_\xi(t), \quad \gamma'_\xi(t))J \leqslant 0, \tag{2.5}$$

$$J(t,\xi) = t + O(t^2), \quad J'(t,\xi) = 1 + O(t). \tag{2.6}$$

As shown in [11] and also pointed out in [22], facts (2.5) and (2.6) make a fundamental role in the derivation of *the generalized Bishop's volume comparison theorem I* below (see Theorem 2.5 for the precise statement). One can also find that (2.6) is also necessary in the proof of Theorem 1.3 in Section 3.

Denote by $\mathrm{inj}(x)$ the injectivity radius of a point $x \in M$. Now, we would like to introduce a notion of spherically symmetric manifold which actually acts as the model space in this paper.

Definition 2.1. A domain $\Omega = \exp_x([0,l) \times S_x^{n-1}) \subset M \setminus \mathrm{Cut}(x)$ with $l < \mathrm{inj}(x)$ is said to be spherically symmetric with respect to a point $x \in \Omega$ if the matrix $\mathbb{A}(t,\xi)$ satisfies $\mathbb{A}(t,\xi) = h(t)I$ for a function $h \in C^2([0,l))$ with $h(0) = 0$, $h'(0) = 1$ and $h|(0,l) > 0$.

Naturally, Ω in Definition 2.1 is a spherically symmetric manifold and x is called its *base point*. Together with (2.1), on the set Ω given in Definition 2.1, the Riemannian metric of M can be expressed by

$$\mathrm{d}s^2(\exp_x(t\xi)) = \mathrm{d}t^2 + h^2(t)|\mathrm{d}\xi|^2, \quad \xi \in S_x^{n-1},\ 0 \leqslant t < l, \tag{2.7}$$

with $|\mathrm{d}\xi|^2$ being the round metric on $\mathbb{S}^{n-1}$. Spherically symmetric manifolds were named as *generalized space forms* by Katz and Kondo (see [15]), and a standard model for such manifolds is given by the warped product $[0,l) \times_h \mathbb{S}^{n-1}$ equipped with metric (2.7), where h is called *the warping function* and satisfies the conditions of Definition 2.3.

For a spherically symmetric manifold $M^* := [0,l) \times_h \mathbb{S}^{n-1}$ (with the base point p^*) and $r < l$, by (2.3) we have

$$\mathrm{vol}[\widetilde{B}(p^*, r)] = w_n \int_0^r h^{n-1}(t)\,\mathrm{d}t, \tag{2.8}$$

and moreover, by the co-area formula (see, for instance, [8], pages 85–86), we also know that the volume of the boundary $\partial\widetilde{B}(p^*,r)$ is given by $\mathrm{vol}[\partial\widetilde{B}(p^*,r)] = w_n h^{n-1}(r)$, where w_n denotes the $(n-1)$-volume of the unit sphere in $\mathbb{R}^n$.

For more information about the spherically symmetric manifold $M^* = [0,l) \times_h \mathbb{S}^{n-1}$ (e.g., the regularity of the metric of M^*, the asymptotically spectral properties, the first Dirichlet eigenvalues of the Laplace and p-Laplace operators on M^*, etc.), please see [11], Section 2 and [22], Section 2 in detail.

2.2. Volume comparison theorems for manifolds with radial curvature bounded. As before, for the given complete manifold M, let $d(x,\cdot)$ be the Riemannian distance to x (on M). In order to state volume comparison theorems below, we need the following concepts.

Definition 2.2. Given a continuous function $k\colon [0,l) \to \mathbb{R}$, we say that M has a radial Ricci curvature lower bound $(n-1)k$ at the point x if

$$\mathrm{Ric}(v_z,v_z) \geqslant (n-1)k(d(x,z)) \quad \forall\, z \in M \setminus \mathrm{Cut}(x) \cup \{x\},$$

where Ric is the Ricci curvature of M.

Definition 2.3. Given a continuous function $k\colon [0,l) \to \mathbb{R}$, we say that M has a radial sectional curvature upper bound k along any unit-speed minimizing geodesic starting from a point $x \in M$ if

$$K(v_z,V) \leqslant k(d(x,z)) \quad \forall\, z \in M \setminus (\mathrm{Cut}(x) \cup \{x\}),$$

where $V \perp v_z$, $V \in S_z^{n-1} \subseteq T_zM$, and $K(v_z,V)$ is the sectional curvature of the plane spanned by v_z and V.

Remark 2.4. As in Subsection 2.1, in Definitions 2.2 and 2.3, $\mathrm{Cut}(x)$ is the cut-locus of x on M, and $v_z \in S_p^{n-1} \subseteq T_zM$ is the unit tangent vector of the minimizing geodesic $\gamma_{x,z}$ emanating from x and joining x and z. Clearly, v_z is in the radial direction. In fact, the notion of having radial curvature bound has been used by the author in [11], [21], [22] to investigate some problems like eigenvalue comparisons for the Laplace and p-Laplace operators (between the given complete manifold and its model manifold), the heat kernel comparison, etc. This notion can also be found in other literature (see, for instance, [16], [27]). Let $t := d(x,\cdot)$, the inequality in Definition 2.2 (or Definition 2.3) becomes $\mathrm{Ric}(v_z,v_z) \geqslant (n-1)k(t)$ (or $K(v_z,V) \leqslant k(t)$) for any $z \in M \setminus \mathrm{Cut}(x) \cup \{x\}$. We also say that the radial Ricci (or sectional) curvature of M is bounded from below (or above) by $(n-1)k(t)$ (or $k(t)$) w.r.t. $x \in M$ if the above inequality is satisfied.

Define a function $\widetilde{\theta}(t,\xi)$ on $M \setminus \mathrm{Cut}(x)$ as

$$\widetilde{\theta}(t,\xi) = \Big[\frac{J(t,\xi)}{h(t)}\Big]^{n-1}.$$

Then we have the following volume comparison result, which corresponds to [11], Theorem 3.3 and Corollary 3.4 (equivalently, [22], Theorem 2.6 or [21], Theorem 2.2.3 and Corollary 2.2.4).

Theorem 2.5 (A generalized Bishop's volume comparison theorem I)**.** *Given $\xi \in S_x^{n-1} \subseteq T_xM$ and a model space $M^* = [0,l) \times_h \mathbb{S}^{n-1}$ with the base point p^*, under the curvature assumption on the radial Ricci tensor,* $\mathrm{Ric}(v_z,v_z) \geqslant -(n-1)h''(t)/h(t)$ *on M for $z = \gamma_\xi(t) = \exp_x(t\xi)$ with $t < \min\{d_\xi, l\}$, the function $\widetilde{\theta}$ is nonincreasing in t. In particular, for all $t < \min\{d_\xi, l\}$ we have $J(t,\xi) \leqslant h(t)$. Furthermore, this inequality is strict for all $t \in (t_0, t_1]$ with $0 \leqslant t_0 < t_1 < \min\{d_\xi, l\}$ if the above curvature assumption holds with a strict inequality for t in the same interval. Besides, for $r_0 < \min\{l(x), l\}$ with $l(x)$ defined by* (2.4) *we have*

$$\mathrm{vol}[B(x,r_0)] \leqslant \mathrm{vol}[\widetilde{B}(p^*,r_0)],$$

with equality if and only if $B(x,r_0)$ is isometric to $\widetilde{B}(p^*,r_0)$.

Similarly, we have the following volume comparison conclusion, which corresponds to [11], Theorem 4.2 (equivalently, [22], Theorem 2.7 or [21], Theorem 2.3.2).

Theorem 2.6 (A generalized Bishop's volume comparison theorem II)**.** *Assume M has a radial sectional curvature upper bound $k(t) = -h''(t)/h(t)$ w.r.t. $x \in M$ for $t < \beta \leqslant \min\{\mathrm{inj}_c(x), l\}$, where $\mathrm{inj}_c(x) = \inf_\xi c_\xi$ with $\gamma_\xi(c_\xi)$being a first conjugate point along the geodesic $\gamma_\xi(t) = \exp_x(t\xi)$. Then on $(0,\beta)$*

$$\left(\frac{\sqrt{|g|}}{h^{n-1}}\right)' \geqslant 0, \quad \sqrt{|g|}(t) \geqslant h^{n-1}(t),$$

and equality occurs in the first inequality at $t_0 \in (0,\beta)$ if and only if

$$\mathcal{R} = -\frac{h''(t)}{h(t)}, \quad \mathbb{A} = h(t)I,$$

on all of $[0,t_0]$.

2.3. A volume comparison theorem for smooth metric measure spaces with weighted Ricci curvature bounded from below. As mentioned at the beginning of this section, the following volume comparison theorem proven by Wei and Wylie (see [29], Theorem 1.2) is the key point to prove Theorem 1.3.

Theorem 2.7 ([29]). *Let $(M, g, \mathrm{e}^{-\varphi}\,\mathrm{d}v_g)$ be an n-dimensional ($n \geqslant 2$) complete smooth metric measure space with $\mathrm{Ric}_\varphi \geqslant (n-1)H$. Fix $x_0 \in M$. If $\partial_t\varphi \geqslant -a$ along all minimal geodesic segments from x_0, then for $R \geqslant r > 0$ (assume $R \leqslant \frac{1}{2}\pi\sqrt{H}$ if $H > 0$),*

$$\frac{\mathrm{vol}_\varphi[B(x_0,R)]}{\mathrm{vol}_\varphi[B(x_0,r)]} \leqslant \mathrm{e}^{aR}\frac{\mathrm{vol}_H^n(R)}{\mathrm{vol}_H^n(r)},$$

where $\mathrm{vol}_H^n(\cdot)$ is the volume of the geodesic ball with the prescribed radius in the space n-form with constant sectional curvature H, and, as before, $\mathrm{vol}_\varphi(\cdot)$ denoting the weighted (or φ-)volume of the given geodesic ball on M. Moreover, equality in the above inequality holds if and only if the radial sectional curvatures are equal to H and $\partial_t\varphi \equiv -a$. In particular, if $\partial_t\varphi \geqslant 0$ and $\mathrm{Ric} \geqslant 0$, then M has φ-volume growth of degree at most n.

Therefore, given a complete and *noncompact* smooth metric measure n-space $(M, g, \mathrm{e}^{-\varphi}\,\mathrm{d}v_g)$, if $\partial_t\varphi \geqslant 0$ (along all minimal geodesic segments from x_0) and $\mathrm{Ric}_\varphi \geqslant 0$, then by Theorem 2.7 we have

$$\frac{\mathrm{vol}_\varphi[B(x_0,R)]}{\mathrm{vol}_\varphi[B(x_0,r)]} \leqslant \mathrm{e}^{0\cdot R}\cdot\frac{V_0(R)}{V_0(r)} = \frac{V_0(R)}{V_0(r)},$$

with, as before, $V_0(\cdot)$ denoting the volume of the ball with the prescribed radius in $\mathbb{R}^n$, which is equivalent with

$$\frac{\mathrm{vol}_\varphi[B(x_0,R)]}{V_0(R)} \leqslant \frac{\mathrm{vol}_\varphi[B(x_0,r)]}{V_0(r)} \tag{2.9}$$

for $R \geqslant r > 0$. Letting $r \to 0$ on the right-hand side of the above inequality, and together with (2.2), (2.3) and (2.6), we can get

$$\begin{aligned}\frac{\mathrm{vol}_\varphi[B(x_0,R)]}{V_0(R)} &\leqslant \lim_{r\to 0}\frac{\int_{\mathbb{S}^{n-1}}\big(\int_0^{\min\{R,d_\xi\}} J^{n-1}(t,\xi)\cdot\mathrm{e}^{-\varphi}\,\mathrm{d}t\big)\,\mathrm{d}\sigma}{\int_{\mathbb{S}^{n-1}}\int_0^R t^{n-1}\,\mathrm{d}t\,\mathrm{d}\sigma}\\ &= \frac{J'(0,\xi)\cdot\mathrm{e}^{-\varphi(x_0)}}{1} = \mathrm{e}^{-\varphi(x_0)}\end{aligned}$$

by applying L'Hôpital's rule n-times. Hence, if $\partial_t\varphi \geqslant 0$ and $\mathrm{Ric}_\varphi \geqslant 0$, we have

$$\mathrm{vol}_\varphi[B(x_0,R)] \leqslant \mathrm{e}^{-\varphi(x_0)}\cdot V_0(R) \tag{2.10}$$

for $R > 0$.

3. Proof of the main conclusion

Now, by using the facts in Section 2 and a similar method to that of [32], Theorem 1.3, we can prove Theorem 1.3 as follows.

Proof of Theorem 1.3. Since $t = t(\cdot) := d(x_0, \cdot)$ is a Lipschitz continuous function from M to $\mathbb{R}$, then for any $\lambda > 0$ we can construct a function $F(\lambda)$ as

$$F(\lambda) := \int_M \frac{t^{\alpha-1+(1-1/p)\beta}}{(\lambda + t^{1-\alpha+\beta/p})^{r(p-1)/(r-p)}} \cdot \mathrm{e}^{-\varphi}\,\mathrm{d}v_g.$$

By applying the Fubini theorem (see [26]) to the above inequality, we have

$$F(\lambda) = \int_0^\infty \mathrm{vol}_\varphi\Big[x\colon \frac{t^{\alpha-1+(1-1/p)\beta}}{(\lambda + t^{1-\alpha+\beta/p})^{r(p-1)/(r-p)}}(x) > s\Big]\,\mathrm{d}s. \tag{3.1}$$

Set

$$y = 1 - \alpha + \Big(\frac{1}{p} - 1\Big)\beta, \quad z = \frac{r(p-1)}{r-p},$$

then by (1.2) we can obtain

$$n - y - 1 - z(y + \beta) < -1. \tag{3.2}$$

By making the variable change

$$s = \frac{t^{\alpha-1+(1-1/p)\beta}}{(\lambda + t^{1-\alpha+\beta/p})^{r(p-1)/(r-p)}}$$

in (3.1) and applying (2.10) because of $\partial_t\varphi \geqslant 0$ (along all minimal geodesic segments from x_0) and $\mathrm{Ric}_\varphi \geqslant 0$, we can obtain

$$\begin{aligned} F(\lambda) &= \int_0^\infty \mathrm{vol}_\varphi[B(x_0,t)] \cdot \frac{y\lambda + (y + z(y+\beta))t^{y+\beta}}{t^{y+1}\cdot(\lambda + t^{y+\beta})^{z+1}}\,\mathrm{d}t \\ &\leqslant \mathrm{e}^{-\varphi(x_0)} \cdot \int_0^\infty \frac{w_n(y\lambda + (y + z(y+\beta))t^{y+\beta})t^{n-y-1}}{n(\lambda + t^{y+\beta})^{z+1}}\,\mathrm{d}t, \end{aligned} \tag{3.3}$$

where, as before, w_n is the $(n-1)$-volume of the unit sphere $\mathbb{S}^{n-1}$. By (3.2), (3.3) and the fact that $n - y - 1 = n + r\gamma - 1 > -1$, we have $0 \leqslant F(\lambda) < \infty$ for any $\lambda > 0$. Moreover, F is differentiable and

$$F'(\lambda) = -\frac{r(p-1)}{r-p}\int_M \frac{t^{\alpha-1+(1-1/p)\beta}}{(\lambda + t^{1-\alpha+\beta/p})^{p(r-1)/(r-p)}} \cdot \mathrm{e}^{-\varphi}\,\mathrm{d}v_g.$$

Hence, from the above argument, we know that $F(\lambda)$ is a well-defined C^1 function on $(0,\infty)$. Since for every $\lambda > 0$, $(\lambda + t^{1-\alpha+\beta/p})^{-(p-1)/(r-p)}$ is a continuous function and tends to zero as $t \to \infty$, there exists at least a sequence of functions $\{f_n(t)\}$ in $C_0^\infty(M)$ such that $f_n(t) \to (\lambda + t^{1-\alpha+\beta/p})^{-p-1/(r-p)}$ as $n \to \infty$. By assumption (1.4) and an approximation procedure for the function $(\lambda + t^{1-\alpha+\beta/p})^{-(p-1)/(r-p)}$, we can get

$$\begin{aligned} F(\lambda) &= \int_M \frac{t^{\alpha-1+(1-1/p)\beta}}{(\lambda + t^{1-\alpha+\beta/p})^{r(p-1)/(r-p)}} \cdot \mathrm{e}^{-\varphi}\, \mathrm{d}v_g \\ &\leqslant \frac{r(p-1)(1-\alpha+\beta/p)}{(n+r\gamma)(r-p)} \int_M \frac{t^{\beta}}{(\lambda + t^{1-\alpha+\beta/p})^{p(r-1)/(r-p)}} \cdot \mathrm{e}^{-\varphi}\, \mathrm{d}v_g \\ &= \frac{r(p-1)(1-\alpha+\beta/p)}{(n+r\gamma)(r-p)} \Big[F(\lambda) + \frac{r-p}{r(p-1)} \lambda F'(\lambda)\Big], \end{aligned}$$

which is equivalent with

$$-\frac{1-\alpha+\beta/p}{n+r\gamma} \cdot \lambda F'(\lambda) \leqslant \Big[\frac{r(p-1)(1-\alpha+\beta/p)}{(n+r\gamma)(r-p)} - 1\Big] F(\lambda). \tag{3.4}$$

In fact, (3.4) can be rewritten as

$$-\lambda F'(\lambda) \leqslant \eta F(\lambda) \tag{3.5}$$

with

$$\eta = \frac{r(p-1)}{r-p} - \frac{n+r\gamma}{1-\alpha+\beta/p} = \frac{p(r-1)(1-\alpha+\beta/p) - (r-p)(n+\beta)}{(1-\alpha+\beta/p)(r-p)} > 0,$$

where the last inequality holds by applying relation (1.2).

Consider a function $A\colon (0,\infty) \to \mathbb{R}$ defined as

$$A(\lambda) := \mathrm{e}^{-\varphi(x_0)} \int_{\mathbb{R}^n} |x|^{\alpha-1+(1-1/p)\beta} (\lambda + |x|^{1-\alpha+\beta/p})^{(1-p)r/(r-p)}\, \mathrm{d}v_{\mathbb{R}^n}, \tag{3.6}$$

where, as before, $|x|$ is the Euclidean length of $x \in \mathbb{R}^n$, and $\mathrm{d}v_{\mathbb{R}^n}$ the Euclidean volume density related to $g_{\mathbb{R}^n}$. It is not difficult to check that $A(\lambda)$ is a well-defined function of class C^1, since, first, by direct computation we have

$$A(\lambda) = \lambda^{(n+r\gamma)/(1-\alpha+\beta/p) - r(p-1)/(r-p)} \cdot A(1) = \lambda^{-\eta} \cdot A(1)$$

and

$$\begin{aligned} A(1) &= \mathrm{e}^{-\varphi(x_0)} \int_{\mathbb{R}^n} |x|^{\alpha-1+(1-1/p)\beta} (1 + |x|^{1-\alpha+\beta/p})^{(1-p)r/(r-p)}\, \mathrm{d}v_{\mathbb{R}^n} \\ &= \mathrm{e}^{-\varphi(x_0)} \int_0^\infty w_n \cdot s^{n-1+\alpha-1+(1-1/p)\beta} (1 + s^{1-\alpha+\beta/p})^{(1-p)r/(r-p)}\, \mathrm{d}s < \infty, \end{aligned}$$

where the last integration is finite because by (3.2) the relation

$$n-1+\alpha-1+\Big(1-\frac{1}{p}\Big)\beta+\Big(1-\alpha+\frac{\beta}{p}\Big)\frac{(1-p)r}{r-p}=n-y-1-z(y+\beta)<-1$$

holds; second, $A(\lambda)$ is differentiable and

$$A'(\lambda)=-\frac{r(p-1)}{r-p}\cdot \mathrm{e}^{-\varphi(x_0)}\int_{\mathbb{R}^n}|x|^{\alpha-1+(1-1/p)\beta}(\lambda+|x|^{1-\alpha+\beta/p})^{(1-p)r/(r-p)}\,\mathrm{d}v_{\mathbb{R}^n}.$$

Clearly, for the function $A(\lambda)$ we also have

$$-\lambda A'(\lambda)=\eta A(\lambda). \tag{3.7}$$

By applying (2.2), (2.3), (2.6), and L'Hôpital's rule, we have

$$\lim_{s\to 0}\frac{\mathrm{vol}_\varphi[B(x_0,s)]}{V_0(s)}=\lim_{s\to 0}\frac{\int_{S_{x_0}^{n-1}}\big(\int_0^{\min\{s,d_\xi\}}J^{n-1}(t,\xi)\cdot\mathrm{e}^{-\varphi}\,\mathrm{d}t\big)\,\mathrm{d}\sigma}{w_n\int_0^s t^{n-1}\,\mathrm{d}t}=\mathrm{e}^{-\varphi(x_0)}.$$

So, for a fixed small $\varepsilon>0$ there exists a number $l>0$ such that $\mathrm{vol}_\varphi[B(x_0,s)]\geqslant(1-\varepsilon)\mathrm{e}^{-\varphi(x_0)}\cdot V_0(s)$ for all $s\leqslant l$. So, from the first equality of (3.3) we can obtain

$$\begin{aligned}F(\lambda)&\geqslant\int_0^l\mathrm{vol}_\varphi[B(x_0,t)]\cdot\frac{y\lambda+(y+z(y+\beta))t^{y+\beta}}{t^{y+1}\cdot(\lambda+t^{y+\beta})^{z+1}}\,\mathrm{d}t\\&\geqslant(1-\varepsilon)\mathrm{e}^{-\varphi(x_0)}\cdot\int_0^l V_0(t)\cdot\frac{y\lambda+(y+z(y+\beta))t^{y+\beta}}{t^{y+1}\cdot(\lambda+t^{y+\beta})^{z+1}}\,\mathrm{d}t\\&=(1-\varepsilon)\mathrm{e}^{-\varphi(x_0)}\cdot\lambda^{-\eta}\int_0^{l/\lambda^{1/(y+\beta)}}V_0(t)\cdot\frac{y+(y+z(y+\beta))s^{y+\beta}}{s^{y+1}(1+s^{y+\beta})^{z+1}}\,\mathrm{d}s.\end{aligned}\tag{3.8}$$

On the other hand, similarly to the derivation of the first equality of (3.3), we have

$$\begin{aligned}A(\lambda)&=\mathrm{e}^{-\varphi(x_0)}\cdot\int_0^\infty V_0(t)\cdot\frac{y\lambda+(y+z(y+\beta))t^{y+\beta}}{t^{y+1}\cdot(\lambda+t^{y+\beta})^{z+1}}\,\mathrm{d}t\\&=\mathrm{e}^{-\varphi(x_0)}\cdot\lambda^{-\eta}\int_0^\infty V_0(t)\cdot\frac{y(y+z(y+\beta))s^{y+\beta}}{s^{y+1}(1+s^{y+\beta})^{z+1}}\,\mathrm{d}s\end{aligned}\tag{3.9}$$

by (3.6). Combining (3.8) with (3.9) yields

$$\liminf_{\lambda\to 0}\frac{F(\lambda)}{A(\lambda)}\geqslant 1-\varepsilon,$$

from which we can get

$$\liminf_{\lambda\to 0}\frac{F(\lambda)}{A(\lambda)}\geqslant 1 \tag{3.10}$$

by letting $\varepsilon\to 0$.

Now, we would like to give a *claim* that if there exists some $\lambda_0>0$ such that $F(\lambda_0)<A(\lambda_0)$, then we have $F(\lambda)<A(\lambda)$ for all $\lambda\in(0,\lambda_0]$. We will prove this by contradiction. Assume that there exists some $\widetilde{\lambda}\in(0,\lambda_0)$ such that $F(\widetilde{\lambda})\geqslant A(\widetilde{\lambda})$. Then we can define λ_1 as

$$\lambda_1:=\sup\{\widetilde{\lambda}<\lambda_0\colon\ F(\widetilde{\lambda})\geqslant A(\widetilde{\lambda})\}.$$

So, for any $\lambda\in[\lambda_1,\lambda_0]$ we have $0<F(\lambda)\leqslant A(\lambda)$ and moreover, together with (3.5) and (3.7) we have

$$\lambda[F'(\lambda)-A'(\lambda)]\geqslant\eta[F(\lambda)-A(\lambda)]\geqslant 0\quad\forall\,\lambda\in[\lambda_1,\lambda_0].$$

This implies that $F(\lambda)-A(\lambda)$ is a nondecreasing function on $[\lambda_1,\lambda_0]$. So, we have

$$0\geqslant(F-A)(\lambda_1)\leqslant(F-A)(\lambda_0)<0,$$

which is a contradiction. Hence, our *claim* is true.

Clearly, by (3.10) and the above *claim*, it follows that

$$F(\lambda)\geqslant A(\lambda)\quad\forall\,\lambda>0.$$

Therefore, together with the first equality of (3.3) and the first equality of (3.9), we have

$$\int_0^\infty[\mathrm{vol}_\varphi[B(x_0,t)]-\mathrm{e}^{-\varphi(x_0)}\cdot V_0(t)]\cdot\frac{y\lambda+(y+z(y+\beta))t^{y+\beta}}{t^{y+1}\cdot(\lambda+t^{y+\beta})^{z+1}}\,\mathrm{d}t\geqslant 0. \tag{3.11}$$

However, since $\partial_t\varphi\geqslant 0$ along all minimal geodesic segments from x_0 and $\mathrm{Ric}_\varphi\geqslant 0$, we have $\mathrm{vol}_\varphi[B(x_0,t)]\leqslant\mathrm{e}^{-\varphi(x_0)}\cdot V_0(t)$ for any $t>0$ (see (2.10)). Therefore, together with (3.11) we can easily get that $\mathrm{vol}_\varphi[B(x_0,t)]=\mathrm{e}^{-\varphi(x_0)}\cdot V_0(t)$ for *almost* every $t\geqslant 0$, and thus for every $t\geqslant 0$ by continuity. So, by Theorem 2.7, we know that the *radial* sectional curvatures are 0 and $\partial_t\varphi\equiv 0$, which implies that φ is a constant function with respect to t (i.e. $\varphi\equiv\varphi(x_0)$). Besides, since the radial sectional curvatures are equal to 0, by applying Theorems 2.5 and 2.6 *simultaneously*, we can obtain

$$\mathrm{vol}[B(x_0,r)]=V_0(r)\quad\forall\,r>0,$$

and $B(x_0,r)$ is isometric to a ball of radius r in $\mathbb{R}^n$ for any $r>0$, which is equivalent to saying that (M,g) is isometric to $(\mathbb{R}^n,g_{\mathbb{R}^n})$. This completes the proof of Theorem 1.3. □

4. Appendix

Define a quantity $\varphi_{\sup}$ as

$$\varphi_{\sup} := \sup_{x\in M} \varphi(x). \tag{4.1}$$

With the help of this quantity, we can give a partial answer to *Problem* in Section 1 as follows.

Theorem 4.1. *Let $\varphi_{\sup}$ be defined as (4.1) and $\varphi_{\sup}$ be finite (i.e. $\sup_{x\in M} \varphi(x) < \infty$). Suppose that $(M, g, \mathrm{e}^{-\varphi}\,\mathrm{d}v_g)$ is an n-dimensional ($n \geqslant 2$) complete and noncompact smooth metric measure space with nonnegative weighted Ricci curvature. For a point $x_0 \in M$ at which $\varphi(x_0)$ is away from $-\infty$, assume that the radial derivative $\partial_t\varphi$ satisfies $\partial_t\varphi \geqslant 0$ along all minimal geodesic segments from x_0 with $t := d(x_0, \cdot)$ being the distance to x_0 (on M). If furthermore the Sobolev type inequality*

$$\begin{aligned}\Bigl(\int_M |u|^{n/(n-1)} \cdot \mathrm{e}^{-\varphi}\,\mathrm{d}v_g\Bigr)^{(n-1)/n} \\ \leqslant n^{-1} D_n^{-1/n} \int_M |\nabla u| \cdot \mathrm{e}^{-\varphi}\,\mathrm{d}v_g \quad \forall\, u \in C_0^\infty(M),\end{aligned} \tag{4.2}$$

holds, and

$$(n+1)\cdot\varphi(x_0) - n\varphi_{\sup} \geqslant 0, \tag{4.3}$$

then (M, g) is isometric to $(\mathbb{R}^n, g_{\mathbb{R}^n})$, and moreover, in this case, $\varphi \equiv \varphi(x_0)$ is a constant function with respect to the variable t, and $\mathrm{e}^{-\varphi}\,\mathrm{d}v_g = \mathrm{e}^{-\varphi(x_0)}\,\mathrm{d}v_{g_{\mathbb{R}^n}}$.

Proof. For a constant $\varepsilon > 0$ small enough and any $r > 0$ define a function u_ε as

$$u_\varepsilon(x) := \begin{cases} 1, & x \in B(x_0, r-\varepsilon), \\ \dfrac{d(x, \partial B(x_0, r))}{\varepsilon}, & x \in B(x_0, r) \setminus B(x_0, r-\varepsilon), \\ 0, & x \in M \setminus B(x_0, r), \end{cases}$$

where, following the convention of the usage of notations in Section 2, $\partial B(x_0, r)$ is the boundary of the geodesic ball $B(x_0, r)$, and naturally, $d(x, \partial B(x_0, r))$ stands for the Riemannian distance from x to the boundary $\partial B(x_0, r)$. Clearly, $u_\varepsilon(x) \in C_0^\infty(M)$. Applying the Sobolev type inequality (4.2) for $u_\varepsilon(x)$ and letting $\varepsilon \to 0$, and together with the assumption that $\partial_t\varphi \geqslant 0$ along all minimal geodesic segments from x_0, which implies that $\varphi(x_0) \leqslant \varphi(x)$ for any $x \in M$ and then, of course, $\mathrm{e}^{-\varphi(x_0)} \geqslant \mathrm{e}^{-\varphi(x)}$ for $x \in M$, we can obtain

$$(\mathrm{vol}_\varphi[B(x_0,r)])^{(n-1)/n} \leqslant n^{-1} D_n^{-1/n} \cdot \mathrm{e}^{-\varphi(x_0)} \cdot \mathrm{Area}[\partial B(x_0,r)], \tag{4.4}$$

where $\mathrm{Area}[\partial B(x_0,r)]$ is the area of $\partial B(x_0,r)$. Substituting the facts that $\mathrm{e}^{-\varphi_{\sup}} \leqslant \mathrm{e}^{-\varphi(x)}$ for all $x \in M$ and

$$\frac{\mathrm{d}}{\mathrm{d}r}\mathrm{vol}[B(x_0,r)] = \mathrm{Area}[\partial B(x_0,r)]$$

into (4.4) yields

$$(\mathrm{e}^{-\varphi_{\sup}} \cdot \mathrm{vol}[B(x_0,r)])^{(n-1)/n} \leqslant n^{-1} D_n^{-1/n} \cdot \mathrm{e}^{-\varphi(x_0)} \cdot \frac{\mathrm{d}}{\mathrm{d}r}\mathrm{vol}[B(x_0,r)]$$

for any $r > 0$. By solving the above differential inequality directly, we have

$$\mathrm{vol}[B(x_0,r)] \geqslant \mathrm{e}^{[n\varphi(x_0)-(n-1)\varphi_{\sup}]} \cdot V_0(r) \quad \forall\, r > 0,$$

from which it is easy to get

$$\mathrm{e}^{-\varphi_{\sup}} \cdot \mathrm{vol}_\varphi[B(x_0,r)] \geqslant \mathrm{e}^{[n\varphi(x_0)-(n-1)\varphi_{\sup}]} \cdot V_0(r) \quad \forall\, r > 0.$$

Therefore, we have

$$\mathrm{vol}_\varphi[B(x_0,r)] \geqslant \mathrm{e}^{[(n+1)\varphi(x_0)-n\varphi_{\sup}]} \cdot \mathrm{e}^{-\varphi(x_0)} \cdot V_0(r) \quad \forall\, r > 0.$$

Furthermore, combing the above inequality with (4.3), we have

$$\mathrm{vol}_\varphi[B(x_0,r)] \geqslant \mathrm{e}^{-\varphi(x_0)} \cdot V_0(r) \tag{4.5}$$

for any $r > 0$. However, since $\partial_t \varphi \geqslant 0$ along all minimal geodesic segments from x_0 and $\mathrm{Ric}_\varphi \geqslant 0$, we have

$$\mathrm{vol}_\varphi[B(x_0,r)] \leqslant \mathrm{e}^{-\varphi(x_0)} \cdot V_0(r) \tag{4.6}$$

for any $r > 0$ (see (2.10)). Hence, by (4.5) and (4.6), it follows that $\mathrm{vol}_\varphi[B(x_0,r)] = \mathrm{e}^{-\varphi(x_0)} \cdot V_0(r)$ for any $r > 0$. Applying Theorem 2.7 directly, we know that the *radial* sectional curvatures are 0, and $\partial_t \varphi \equiv 0$, which implies that φ is a constant function with respect to t (i.e. $\varphi \equiv \varphi(x_0)$). Besides, since the radial sectional curvatures are equal to 0, by applying Theorems 2.5 and 2.6 *simultaneously*, we can obtain

$$\mathrm{vol}[B(x_0,r)] = V_0(r) \quad \forall\, r > 0,$$

and $B(x_0,r)$ is isometric to a ball of radius r in $\mathbb{R}^n$ for any $r > 0$, which is equivalent to saying that (M,g) is isometric to $(\mathbb{R}^n, g_{\mathbb{R}^n})$. This completes the proof of Theorem 4.1. □

Remark 4.2. Clearly, $\varphi =$ constant satisfies all the assumptions on φ in Theorem 4.1. Especially, when $\varphi \equiv 0$, then Theorem 4.1 is totally the same as Xia's result (only *when* $q = 1$) in [30] mentioned in Section 1. So, we can equivalently say that Xia's result (only when $q = 1$) in [30], as a special case, is covered by Theorem 4.1 here.

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