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INVERSE EIGENVALUE PROBLEM FOR CONSTRUCTING A KIND OF ACYCLIC MATRICES WITH TWO EIGENPAIRS

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Abstract. We investigate an inverse eigenvalue problem for constructing a special kind of acyclic matrices. The problem involves the reconstruction of the matrices whose graph is an m-centipede. This is done by using the $(2m-1)$st and $(2m)$th eigenpairs of their leading principal submatrices. To solve this problem, the recurrence relations between leading principal submatrices are used.

Keywords: inverse eigenvalue problem; leading principal submatrices; graph of a matrix; eigenpair

MSC 2010: 65F18, 05C50

1. Introduction

An inverse eigenvalue problem (IEP) concerns the reconstruction of a matrix from prescribed spectral data. Determinant factors of the level of difficulty of an IEP are the structure of the matrices which are to be reconstructed and the type of eigen information available. In [3] detailed characterization of inverse eigenvalue problems is mentioned. Special types of inverse eigenvalue problems have been studied in [4], [5], [6], [8], [9], [11], [13], [14], [17]. Inverse eigenvalue problems are important in many applications such as mechanical system simulation, control theory, structural analysis, mass spring vibrations and graph theory [3], [9], [10]. In this paper, we investigate an IEP, namely the IEPC (inverse eigenvalue problem for matrices whose graph is a m-centipede). Similar problems were studied in [11], [15], [16]. The usual process of solving such problems involves the use of recurrence relations between the leading principal submatrices of $\lambda I - A$ where A is the required matrix. Some applications of the acyclic matrix discussed in this paper are in chemistry, energy and

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graph theory [1], [2]. In addition, the idea in this paper may provide some insights into other acyclic matrix inverse eigenvalue problems.

The rest of the paper is organized as follows. In Section 2, we begin to present some preliminaries and lemmas that will be used throughout the paper. In Section 3, we discuss some properties of A_{2m}. In Section 4, we discuss the solution of IEPC and present an algorithm. In Section 5, we report numerical examples to illustrate the solution of IEPC. In Section 6 the conclusion is presented.

2. Preliminaries

Let G be a simple undirected graph on n vertices, whose vertices are positive integers. A real symmetric matrix $A = (a_{ij})$ is said to have a graph G provided $a_{ij} \neq 0$ if and only if the vertices i and j are adjacent in G.

Given an $n \times n$ symmetric matrix A, the graph of A, denoted by $G(A)$, has the vertex set $V(G) = \{1, 2, 3, \ldots, n\}$ and the edge set $\{ij\colon\ i \neq j, a_{ij} \neq 0\}$. For a graph G with n vertices, we denote by $S(G)$ the set of all real symmetric matrices whose graph is G. A matrix whose graph is a tree is called an acyclic matrix. Some simple examples of acyclic matrices are the matrices whose graphs are paths or m-centipedes.

Definition 2.1. The m-centipede is the tree on $2m$ nodes obtained by joining the bottoms of m copies of the path graph P_2 laid in a row with edges (Figure 1).

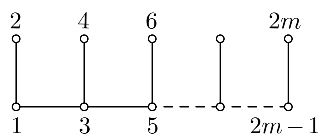

Figure 1. m-centipede C_m.

Throughout this paper, we use the following notation:

1. The matrix of a m-centipede is

$$
A_{2m} = \begin{pmatrix}
a_1 & b_1 & c_1 & 0 & \ldots & \ldots & 0 & 0 & 0 \\
b_1 & a_2 & 0 & 0 & \ldots & \ldots & 0 & 0 & 0 \\
c_1 & 0 & a_3 & b_3 & c_3 & 0 & \ldots & 0 & 0 \\
0 & 0 & b_3 & a_4 & 0 & \ldots & \ldots & \ldots & 0 \\
0 & 0 & c_3 & 0 & a_5 & b_5 & c_5 & \vdots & 0 \\
0 & 0 & 0 & 0 & b_5 & a_6 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & c_{2m-3} & 0 & a_{2m-1} & b_{2m-1} \\
0 & 0 & \ldots & \ldots & \ldots & 0 & 0 & b_{2m-1} & a_{2m}
\end{pmatrix}, \tag{2.1}
$$

where b_{2j+1} and c_{2k+1} are nonzero for all $j = 0, 1, 2, \ldots, m-1$ and $k = 0, 1, 2, \ldots, m-2$.

2. A_j is a $j \times j$ matrix that will denote the jth leading principal submatrix of the matrix A_{2m} for all $j = 1, 2, \ldots, 2m$.
3. $P_j(\lambda) = \det(\lambda I_j - A_j)$, i.e., the jth leading principal submatrix of $\lambda I_{2m} - A_{2m}$, I_j being the identity matrix of order j. For convenience of discussion, we define $P_0(\lambda) = 1$, $b_{-1} = 0$, $c_{-1} = 0$.

In this paper, we solve the following IEP:

IEPC: Given two real numbers $\lambda_{2m}^{(2m)}$, $\lambda_{2m-1}^{(2m-1)}$, real vectors $X_{2m} = (x_1, x_2, \ldots, x_{2m})^\top$ and $X'_{2m-1} = (x'_1, x'_2, \ldots, x'_{2m-1})^\top$, the problem is to find a $2m \times 2m$ matrix $A_{2m} \in S(C_m)$ such that $\lambda_{2m}^{(2m)}$ and $\lambda_{2m-1}^{(2m-1)}$ are the maximal eigenvalues of A_{2m} and A_{2m-1}, respectively, $(\lambda_{2m}^{(2m)}, X_{2m})$ is an eigenpair of A_{2m} and $(\lambda_{2m-1}^{(2m-1)}, X'_{2m-1})$ is an eigenpair of A_{2m-1}.

The following lemmas will be necessary for solving the problem in this paper.

Lemma 2.2 ([12])**.** *Let $P(\lambda)$ be a monic polynomial of degree n with all real zeroes. If λ_1 and λ_n are, respectively, the minimal and the maximal zero of $P(\lambda)$, then:*

(i) *If $x < \lambda_1$, we have that $(-1)^n P(x) > 0$.*
(ii) *If $x > \lambda_n$, we have that $P(x) > 0$.*

Lemma 2.3 ([7], Cauchy's Interlacing Theorem)**.** *Let $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ be the eigenvalues of an $n \times n$ real symmetric matrix A and $\mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_{n-1}$ the eigenvalues of an $(n-1) \times (n-1)$ principal submatrix B of A, then*

$$\lambda_1 \leqslant \mu_1 \leqslant \ldots \leqslant \mu_{n-1} \leqslant \lambda_n.$$

An immediate consequence of Cauchy's Interlacing Theorem is

Corollary 2.4. *Let A be an $n \times n$ real symmetric matrix and $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, the minimal and maximal eigenvalues of the leading principal submatrix A_j, $j = 1, 2, \ldots, n$, of A, respectively. Then*

$$\lambda_1^{(n)} \leqslant \lambda_1^{(n-1)} \leqslant \ldots \leqslant \lambda_1^{(2)} \leqslant \lambda_1^{(1)} \leqslant \lambda_2^{(2)} \leqslant \ldots \leqslant \lambda_{n-1}^{(n-1)} \leqslant \lambda_n^{(n)}, \tag{2.2}$$

and

$$\lambda_1^{(j)} \leqslant a_i \leqslant \lambda_j^{(j)}, \quad i = 1, 2, \ldots, j, \ j = 2, \ldots, n. \tag{2.3}$$

In the next section we present some properties of the matrix A_{2m} that we need to prove the problem IEPC.

3. Properties of the matrix A_{2m}

In the following, we investigate the relation between successive leading principal submatrices of $\lambda I_{2m} - A_{2m}$.

Lemma 3.1. *Let A be a $2m \times 2m$ matrix of the form* (2.1). *Then the sequence $\{P_l(\lambda) = \det(\lambda I_l - A_l)\}_{l=1}^{2m}$ satisfies the following recurrence relations:*

(i) $P_1(\lambda) = (\lambda - a_1)$,

(ii) $P_{2j}(\lambda) = (\lambda - a_{2j})P_{2j-1}(\lambda) - b_{2j-1}^2 P_{2j-2}(\lambda)$, $j = 1, 2, \ldots, m$,

(iii) $P_{2j+1}(\lambda) = (\lambda - a_{2j+1})P_{2j}(\lambda) - c_{2j-1}^2(\lambda - a_{2j})P_{2j-2}(\lambda)$, $j = 1, 2, \ldots, m-1$.

P r o o f. The result follows by expanding the determinant. $\square$

Lemma 3.2. *Let A_{2m} be a matrix of the form* (2.1) *and $\lambda_j^{(j)}$ the maximal eigenvalue of the leading principal submatrix A_j of A_{2m}, $j = 1, 2, \ldots, 2m$. Then*

$$\lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_j^{(j)}, \tag{3.1}$$

and

$$a_k < \lambda_j^{(j)}, \quad k = 1, 2, \ldots, j, \tag{3.2}$$

for each $j = 2, \ldots, 2m$.

P r o o f. From Corollary 2.4, (2.2) and (2.3) we have

$$\lambda_1^{(1)} \leqslant \lambda_2^{(2)} \leqslant \ldots \leqslant \lambda_j^{(j)},$$

and

$$a_k \leqslant \lambda_j^{(j)}, \quad k = 1, 2, \ldots, j,$$

for each $j = 2, \ldots, 2m$. Now it remains to prove that inequalities (2.2) and (2.3) are strict for $j = 2, \ldots, 2m$. By the inductive hypothesis and contradiction, the discussion shows as follows.

(a) When $j = 2$, by Lemma 3.1 we have

$$P_2(\lambda) = (\lambda - a_2)P_1(\lambda) - b_1^2. \tag{3.3}$$

Let $\lambda_2^{(2)} = \lambda_1^{(1)}$, by equation (3.3) we obtain

$$P_2(\lambda_1^{(1)}) = -b_1^2 = 0.$$

Then we obtain $b_1 = 0$, but this contradicts the restriction on A_{2m} that $b_1 \neq 0$. Hence $\lambda_1^{(1)} < \lambda_2^{(2)}$. The same occurs if we assume that $\lambda_2^{(2)} = a_2$, then we have $a_2 < \lambda_2^{(2)}$.

For $j = 3$ by Lemma 3.1 we have

$$P_3(\lambda) = (\lambda - a_3)P_2(\lambda) - c_1^2(\lambda - a_2). \tag{3.4}$$

Let $\lambda_3^{(3)} = \lambda_2^{(2)}$, by equation (3.4) we have

$$P_3(\lambda_2^{(2)}) = (\lambda_2^{(2)} - a_3)P_2(\lambda_2^{(2)}) - c_1^2(\lambda_2^{(2)} - a_2) = -c_1^2(\lambda_2^{(2)} - a_2) = 0.$$

Since $\lambda_2^{(2)} - a_2 > 0$ we obtain then $c_1 = 0$, but this contradicts the restriction on A_{2m} that $c_1 \neq 0$. Hence $\lambda_2^{(2)} < \lambda_3^{(3)}$.

If $\lambda_3^{(3)} = a_3$ then by equation (3.4) we know

$$P_3(a_3) = (a_3 - a_3)P_2(a_3) - c_1^2(a_3 - a_2) = -c_1^2(a_3 - a_2) = -c_1^2(\lambda_3^{(3)} - a_2).$$

From the above results we have $a_2 < \lambda_2^{(2)} < \lambda_3^{(3)}$, then $-c_1^2(\lambda_3^{(3)} - a_2) \neq 0$ and we get $P_3(a_3) \neq 0$, which contradicts $P_3(\lambda_3^{(3)}) = 0$. Hence, we obtain $a_3 < \lambda_3^{(3)}$.

(b) Now we assume that (3.1), (3.2) hold for $j = 4, \ldots, 2m-2$ and consider

$$P_{2m-1}(\lambda) = (\lambda - a_{2m-1})P_{2m-2}(\lambda) - c_{2m-3}^2(\lambda - a_{2m-2})P_{2m-4}(\lambda).$$

We know

$$\lambda_{2m-4}^{(2m-4)} < \lambda_{2m-2}^{(2m-2)}, \ \ c_{2m-3}^2 \neq 0, \ \ a_i < \lambda_{2m-2}^{(2m-2)}, \quad i = 2, \ldots, 2m-2,$$

then

$$-c_{2m-3}^2(\lambda_{2m-2}^{(2m-2)} - a_{2m-2})P_{2m-4}(\lambda_{2m-2}^{(2m-2)}) \neq 0,$$

hence $\lambda_{2m-2}^{(2m-2)}$ is not a zero of $P_{2m-1}(\lambda)$ and when $j = 2m-1$, we have $\lambda_{2m-2}^{(2m-2)} < \lambda_{2m-1}^{(2m-1)}$.

If $\lambda_{2m-1}^{(2m-1)} = a_{2m-1}$ then by Lemma 3.1 we have

$$P_{2m-1}(\lambda_{2m-1}^{(2m-1)}) = P_{2m-1}(a_{2m-1}) = -c_{2m-3}^2(a_{2m-1} - a_{2m-2})P_{2m-4}(a_{2m-1}).$$

From the above verified results, we know

$$a_k < \lambda_j^{(j)} < \lambda_{2m-1}^{(2m-1)}, \quad k = 1, \ldots, j;\ j = 2, \ldots, 2m-2.$$

Then $-c_{2m-3}^2(a_{2m-1} - a_{2m-2})P_{2m-4}(a_{2m-1}) \neq 0$ and we get

$$P_{2m-1}(a_{2m-1}) \neq 0.$$

But this contradicts $P_{2m-1}(\lambda_{2m-1}^{(2m-1)}) = 0$. Therefore, we obtain

$$a_{2m-1} < \lambda_{2m-1}^{(2m-1)}.$$

Finally, if $j = 2m$, by Lemma 3.1 we have

$$P_{2m}(\lambda) = (\lambda - a_{2m})P_{2m-1}(\lambda) - b_{2m-1}^2 P_{2m-2}(\lambda). \tag{3.5}$$

Since $\lambda_{2m-2}^{(2m-2)} < \lambda_{2m-1}^{(2m-1)}$, we have $-b_{2m-1}^2 P_{2m-2}(\lambda_{2m-1}^{(2m-1)}) \neq 0$, and $\lambda_{2m-1}^{(2m-1)}$ is not a root of $P_{2m}(\lambda)$. Hence, we get $\lambda_{2m-1}^{(2m-1)} < \lambda_{2m}^{(2m)}$.

If $\lambda_{2m}^{(2m)} = a_{2m}$ then

$$P_{2m}(a_{2m}) = -b_{2m-1}^2 P_{2m-2}(a_{2m}) = -b_{2m-1}^2 P_{2m-2}(\lambda_{2m}^{(2m)}) = 0,$$

contradicting $\lambda_{2m-2}^{(2m-2)} < \lambda_{2m}^{(2m)}$. Then from (2.3)

$$a_k < \lambda_{2m}^{(2m)}, \quad k = 1, \ldots, j;\ j = 2, \ldots, 2m.$$

(c) In conclusion, inequalities (3.1) and (3.2) hold for any positive integer j when $2 \leqslant j \leqslant 2m$. $\square$

From Lemma 3.2 we get the following result.

Corollary 3.3. *Let A_{2m} be a matrix of the form (2.1) and $\lambda_{2m}^{(2m)}$, $\lambda_{2m-1}^{(2m-1)}$ the maximal eigenvalues of A_{2m} and A_{2m-1}, respectively. Then we have*

(i) $\prod_{i=1}^{j} (a_{2i} - \lambda_{2m}^{(2m)}) \neq 0$, $j = 1, \ldots, m$,

(ii) $\prod_{i=1}^{j} (a_{2i} - \lambda_{2m-1}^{(2m-1)}) \neq 0$, $j = 1, \ldots, m-1$.

In the next lemma we show that every component x_l of the eigenvector X_{2m}, $l = 2, 3, \ldots, 2m$, is the coefficient of x_1 and every component x'_k of the eigenvector X'_{2m-1}, $k = 2, 3, \ldots, 2m-1$, is the coefficient of x'_1.

Lemma 3.4. *Let $X_{2m} = (x_1, x_2, \ldots, x_{2m})^\top$ and $X'_{2m-1} = (x'_1, x'_2, \ldots, x'_{2m-1})^\top$ be respectively eigenvectors of A_{2m} and A_{2m-1} corresponding to eigenvalues $\lambda_{2m}^{(2m)}$, $\lambda_{2m-1}^{(2m-1)}$. Then $x_1 \neq 0$, $x'_1 \neq 0$ and the components of these eigenvectors are given by*

$$x_{2j+1} = \frac{(-1)^j P_{2j}(\lambda_{2m}^{(2m)}) x_1}{\prod\limits_{i=1}^{j} (a_{2i} - \lambda_{2m}^{(2m)}) c_{2i-1}}, \quad j = 1, 2, \ldots, m-1, \tag{3.6}$$

$$x_{2j} = \frac{(-1)^j b_{2j-1} P_{2j-2}(\lambda_{2m}^{(2m)}) x_1}{(a_{2j} - \lambda_{2m}^{(2m)}) \prod\limits_{i=1}^{j-1} (a_{2i} - \lambda_{2m}^{(2m)}) c_{2i-1}}, \quad j = 1, 2, \ldots, m, \tag{3.7}$$

$$x'_{2j+1} = \frac{(-1)^j P_{2j}(\lambda_{2m-1}^{(2m-1)}) x'_1}{\prod\limits_{i=1}^{j} (a_{2i} - \lambda_{2m-1}^{(2m-1)}) c_{2i-1}}, \quad j = 1, 2, \ldots, m-1, \tag{3.8}$$

$$x'_{2j} = \frac{(-1)^j b_{2j-1} P_{2j-2}(\lambda_{2m-1}^{(2m-1)}) x'_1}{(a_{2j} - \lambda_{2m-1}^{(2m-1)}) \prod\limits_{i=1}^{j-1} (a_{2i} - \lambda_{2m-1}^{(2m-1)}) c_{2i-1}}, \quad j = 1, 2, \ldots, m-2. \tag{3.9}$$

P r o o f. Because $(\lambda_{2m}^{(2m)}, X_{2m})$ is an eigenpair of A_{2m}, we have

$$A_{2m} X_{2m} = \lambda_{2m}^{(2m)} X_{2m},$$

which can be transformed into the form

$$(a_1 - \lambda_{2m}^{(2m)}) x_1 + b_1 x_2 + c_1 x_3 = 0, \tag{3.10}$$

$$b_{2j-1} x_{2j-1} + (a_{2j} - \lambda_{2m}^{(2m)}) x_{2j} = 0, \quad j = 1, 2, \ldots, m, \tag{3.11}$$

$$\begin{aligned} &c_{2j-1} x_{2j-1} + (a_{2j+1} - \lambda_{2m}^{(2m)}) x_{2j+1} \\ &\qquad + b_{2j+1} x_{2j+2} + c_{2j+1} x_{2j+3} = 0, \quad j = 1, 2, \ldots, m-2. \end{aligned} \tag{3.12}$$

$$c_{2m-3} x_{2m-3} + (a_{2m-1} - \lambda_{2m}^{(2m)}) x_{2m-1} + b_{2m-1} x_{2m} = 0. \tag{3.13}$$

We define the values $v_1, v_3, \ldots, v_{2m-1}$ as

$$v_1 = x_1, \quad v_{2j+1} = x_{2j+1} \prod_{i=1}^{j} (a_{2i} - \lambda_{2m}^{(2m)}) c_{2i-1}, \quad j = 1, 2, \ldots, m-1.$$

Multiplying (3.12) by $\prod_{i=1}^{j}(a_{2i}-\lambda_{2m}^{(2m)})c_{2i-1}$, we have

$$\begin{aligned}c_{2j-1}x_{2j-1}\prod_{i=1}^{j}(a_{2i}-\lambda_{2m}^{(2m)})c_{2i-1}+(a_{2j+1}-\lambda_{2m}^{(2m)})x_{2j+1}\prod_{i=1}^{j}(a_{2i}-\lambda_{2m}^{(2m)})c_{2i-1}\\+b_{2j+1}x_{2j+2}\prod_{i=1}^{j}(a_{2i}-\lambda_{2m}^{(2m)})c_{2i-1}+c_{2j+1}x_{2j+3}\prod_{i=1}^{j}(a_{2i}-\lambda_{2m}^{(2m)})c_{2i-1}=0.\end{aligned}$$

By (3.11) we have

$$x_{2j+2}=\frac{-b_{2j+1}x_{2j+1}}{a_{2j+2}-\lambda_{2m}^{(2m)}},$$

by replacing x_{2j+2} in the above expression we get

$$\begin{aligned}c_{2j-1}x_{2j-1}\prod_{i=1}^{j}(a_{2i}-\lambda_{2m}^{(2m)})c_{2i-1}+(a_{2j+1}-\lambda_{2m}^{(2m)})x_{2j+1}\prod_{i=1}^{j}(a_{2i}-\lambda_{2m}^{(2m)})c_{2i-1}\\-\frac{b_{2j+1}^2x_{2j+1}}{a_{2j+2}-\lambda_{2m}^{(2m)}}\prod_{i=1}^{j}(a_{2i}-\lambda_{2m}^{(2m)})c_{2i-1}+c_{2j+1}x_{2j+3}\prod_{i=1}^{j}(a_{2i}-\lambda_{2m}^{(2m)})c_{2i-1}=0.\end{aligned}$$

Thus

$$\begin{aligned}&(a_{2j+2}-\lambda_{2m}^{(2m)})(a_{2j}-\lambda_{2m}^{(2m)})c_{2j-1}^2v_{2j-1}\\&\quad+((a_{2j+2}-\lambda_{2m}^{(2m)})(a_{2j+1}-\lambda_{2m}^{(2m)})-b_{2j+1}^2)v_{2j+1}+v_{2j+3}=0,\end{aligned}$$

which for $j=1,2,\ldots,m-3$ gives

$$\begin{aligned}v_{2j+3}=(b_{2j+1}^2-(a_{2j+2}-\lambda_{2m}^{(2m)})(a_{2j+1}-\lambda_{2m}^{(2m)}))v_{2j+1}\\-(a_{2j+2}-\lambda_{2m}^{(2m)})(a_{2j}-\lambda_{2m}^{(2m)})c_{2j-1}^2v_{2j-1}.\end{aligned}\tag{3.14}$$

Now, from (3.10) and (3.11) we have

$$v_3=(b_1^2-(a_2-\lambda_{2m}^{(2m)})(a_1-\lambda_{2m}^{(2m)}))x_1=-P_2(\lambda_{2m}^{(2m)})x_1.$$

From (3.14) we have

$$v_5=(b_3^2-(a_4-\lambda_{2m}^{(2m)})(a_3-\lambda_{2m}^{(2m)}))v_3-(a_4-\lambda_{2m}^{(2m)})(a_2-\lambda_{2m}^{(2m)})c_1^2v_1=P_4(\lambda_{2m}^{(2m)})x_1,$$

and following this way, we see that

$$v_{2j+1}=(-1)^jP_{2j}(\lambda_{2m}^{(2m)})x_1,\quad j=1,2,\ldots,m-1.$$

We have

$$x_{2j+1} = \frac{(-1)^j P_{2j}(\lambda_{2m}^{(2m)})x_1}{\prod_{i=1}^{j}(a_{2i}-\lambda_{2m}^{(2m)})c_{2i-1}}, \quad j = 1, 2, \ldots, m-1.$$

From (3.11), we obtain

$$x_{2j} = \frac{-b_{2j-1}x_{2j-1}}{(a_{2j}-\lambda_{2m}^{(2m)})} = \frac{(-1)^j b_{2j-1}P_{2j-2}(\lambda_{2m}^{(2m)})x_1}{(a_{2j}-\lambda_{2m}^{(2m)})\prod_{i=1}^{j-1}(a_{2i}-\lambda_{2m}^{(2m)})c_{2i-1}}, \quad j = 1, 2, \ldots, m.$$

Since X_{2m} is an eigenvector, we have $X_{2m} \neq 0$. If $x_1 = 0$, then from (3.6) and (3.7) we see that all of the other components of X_{2m} are zero and we have a contradiction. Thus $x_1 \neq 0$. The formulas (3.8) and (3.9) can be proved analogously. □

4. The solution of IEPC

The following theorem solves the problem IEPC.

Theorem 4.1. *The IEPC has a unique solution if the following conditions are satisfied:*

(i) $x_l \neq 0$ *for* $l = 1, 2, \ldots, 2m$ *and* $x'_k \neq 0$ *for* $k = 1, 2, \ldots, 2m-1$.

(ii) $E_j = \begin{vmatrix} x_{2j+1} & x'_{2j+1} \\ x_{2j-1} & x'_{2j-1} \end{vmatrix} \neq 0$, $j = 1, 2, \ldots, m-1$.

The elements of the matrix A_{2m} *are:*

$$\begin{aligned}
b_{2j-1} &= \frac{(\lambda_{2m}^{(2m)} - \lambda_{2m-1}^{(2m-1)})x'_{2j}x_{2j}}{x_{2j-1}x'_{2j} - x'_{2j-1}x_{2j}}, \\
a_{2j} &= \lambda_{2m}^{(2m)} - \frac{b_{2j-1}x_{2j-1}}{x_{2j}}, \\
c_{2j-1} &= (\lambda_{2m}^{(2m)} - \lambda_{2m-1}^{(2m-1)}) \sum_{i=1}^{2j} x_i x'_i / E_j, \\
a_{2j-1} &= \lambda_{2m}^{(2m)} - \frac{c_{2j-3}x_{2j-3} + b_{2j-1}x_{2j} + c_{2j-1}x_{2j+1}}{x_{2j-1}}
\end{aligned}$$

for $j = 1, 2, \ldots, m-1$, *and*

$$
\begin{aligned}
a_{2m-1} &= \lambda_{2m-1}^{(2m-1)} - \frac{c_{2m-3}x'_{2m-3}}{x'_{2m-1}},\\
b_{2m-1} &= (\lambda_{2m}^{(2m)} - \lambda_{2m-1}^{(2m-1)}) \sum_{i=1}^{2m-1} x_i x'_i / x_{2m} x'_{2m-1},\\
a_{2m} &= \lambda_{2m}^{(2m)} - \frac{b_{2m-1}x_{2m-1}}{x_{2m}}.
\end{aligned}
$$

P r o o f. We assume that $x_l \neq 0$ for $l = 1, 2, \ldots, 2m$ and $x'_k \neq 0$ for $k = 1, 2, \ldots, 2m-1$.

Here $(\lambda_{2m}^{(2m)}, X_{2m})$ and $(\lambda_{2m-1}^{(2m-1)}, X'_{2m-1})$ are eigenpairs of matrices A_{2m} and A_{2m-1}, respectively, so for $j = 1, 2, \ldots, m-1$ we have

$$
\begin{cases}
b_{2j-1}x_{2j-1} + (a_{2j} - \lambda_{2m}^{(2m)})x_{2j} = 0,\\
b_{2j-1}x'_{2j-1} + (a_{2j} - \lambda_{2m-1}^{(2m-1)})x'_{2j} = 0.
\end{cases}
$$

Let D_j denote the determinant of the coefficient matrix of the above system of linear equations in a_{2j} and b_{2j-1}. Then

$$
D_j = x_{2j-1}x'_{2j} - x'_{2j-1}x_{2j}.
$$

If $D_j \neq 0$, then the system will have a unique solution, given by

$$
\begin{aligned}
b_{2j-1} &= \frac{(\lambda_{2m}^{(2m)} - \lambda_{2m-1}^{(2m-1)})x'_{2j}x_{2j}}{x_{2j-1}x'_{2j} - x'_{2j-1}x_{2j}},\\
a_{2j} &= \lambda_{2m}^{(2m)} - \frac{b_{2j-1}x_{2j-1}}{x_{2j}}.
\end{aligned}
$$

We claim that the expression $D_j \neq 0$. This follows from Lemma 3.4.

By Lemma 3.4 we have

$$
D_j = \frac{(-1)^{2j-1} b_{2j-1} P_{2j-2}(\lambda_{2m}^{(2m)}) P_{2j-2}(\lambda_{2m-1}^{(2m-1)}) x_1 x'_1 (\lambda_{2m-1}^{(2m-1)} - \lambda_{2m}^{(2m)})}{(a_{2j} - \lambda_{2m}^{(2m)})(a_{2j} - \lambda_{2m-1}^{(2m-1)}) \prod_{i=1}^{j-1} (a_{2i} - \lambda_{2m-1}^{(2m-1)}) c_{2i-1} \prod_{i=1}^{j-1} (a_{2i} - \lambda_{2m}^{(2m)}) c_{2i-1}}.
$$

By Lemma 2.2 and (3.1) we get

$$
P_{2j-2}(\lambda_{2m}^{(2m)}) P_{2j-2}(\lambda_{2m-1}^{(2m-1)})(\lambda_{2m-1}^{(2m-1)} - \lambda_{2m}^{(2m)}) \neq 0,
$$

then $D_j \neq 0$. Since $(\lambda_{2m}^{(2m)} - \lambda_{2m-1}^{(2m-1)})x'_{2j}x_{2j} \neq 0$, we obtain $b_{2j-1} \neq 0$.

For finding the values of c_{2j-1} and a_{2j-1}, $j = 1, 2, \dots, m-1$, we have

$$\begin{cases} c_{2j-3}x_{2j-3} + (a_{2j-1} - \lambda_{2m}^{(2m)})x_{2j-1} + b_{2j-1}x_{2j} + c_{2j-1}x_{2j+1} = 0, \\ c_{2j-3}x'_{2j-3} + (a_{2j-1} - \lambda_{2m-1}^{(2m-1)})x'_{2j-1} + b_{2j-1}x'_{2j} + c_{2j-1}x'_{2j+1} = 0. \end{cases}$$

Because the values of c_{2j-3} and b_{2j-1} are known, so by solving the above system the values c_{2j-1}, a_{2j-1} will be obtained. Since $E_j \neq 0$ the system will have a unique solution, given by

$$c_{2j-1} = (\lambda_{2m}^{(2m)} - \lambda_{2m-1}^{(2m-1)}) \sum_{i=1}^{2j} x_i x'_i / E_j,$$

$$a_{2j-1} = \lambda_{2m}^{(2m)} - \frac{c_{2j-3}x_{2j-3} + b_{2j-1}x_{2j} + c_{2j-1}x_{2j+1}}{x_{2j-1}}.$$

By Lemma 3.4 we have

$$x_{2j+1}x'_{2j+1} = \frac{(-1)^{2j} P_{2j}(\lambda_{2m}^{(2m)}) P_{2j}(\lambda_{2m-1}^{(2m-1)}) x'_1 x_1}{\prod_{i=1}^{j} (a_{2i} - \lambda_{2m}^{(2m)}) c_{2i-1} \prod_{i=1}^{j} (a_{2i} - \lambda_{2m-1}^{(2m-1)}) c_{2i-1}}.$$

Also, by Lemma 2.2, Corollary 3.3, and (3.1) we obtain

$$\frac{(-1)^{2j} P_{2j}(\lambda_{2m}^{(2m)}) P_{2j}(\lambda_{2m-1}^{(2m-1)})}{\prod_{i=1}^{j} (a_{2i} - \lambda_{2m}^{(2m)}) c_{2i-1} \prod_{i=1}^{j} (a_{2i} - \lambda_{2m-1}^{(2m-1)}) c_{2i-1}} > 0,$$

therefore, the sign of $x_{2j+1}x'_{2j+1}$ and $x_1 x'_1$ is the same. Similarly, we can show that the sign of $x_{2j}x'_{2j}$ and $x_1 x'_1$ is the same. Hence, $\sum_{i=1}^{2j} x_i x'_i \neq 0$, and $c_{2j-1} \neq 0$.

For finding the value of a_{2m-1} we have

$$c_{2m-3}x'_{2m-3} + (a_{2m-1} - \lambda_{2m-1}^{(2m-1)})x'_{2m-1} = 0$$
$$\Rightarrow a_{2m-1} = \lambda_{2m-1}^{(2m-1)} - \frac{c_{2m-3}x'_{2m-3}}{x'_{2m-1}}.$$

By (3.13) we have

$$b_{2m-1} = \frac{-(c_{2m-3}x_{2m-3} + (a_{2m-1} - \lambda_{2m}^{(2m)})x_{2m-1})}{x_{2m}}$$
$$= (\lambda_{2m}^{(2m)} - \lambda_{2m-1}^{(2m-1)}) \sum_{i=1}^{2m-1} x_i x'_i / x_{2m} x'_{2m-1},$$

and by (3.11) we have

$$a_{2m} = \lambda_{2m}^{(2m)} - \frac{b_{2m-1}x_{2m-1}}{x_{2m}}.$$

□

From the discussion of Theorem 4.1, we propose Algorithm 1 for solving the IEPC.

Algorithm 1 (To solve problem IEPC)

Input: $\lambda_{2m-1}^{(2m-1)}$, $\lambda_{2m}^{(2m)}$, ε,
$X_{2m} = (x_1, \ldots, x_{2m})$,
$X'_{2m-1} = (x'_1, \ldots, x'_{2m-1})$.
Output: $A_{2m} \in S(C_m)$.

1: **For** $j = 1$ **to** $m - 1$ **do**

2: **If** $|x_{2j-1}x'_{2j} - x'_{2j-1}x_{2j}|$ and $|x_{2j+1}x'_{2j-1} - x'_{2j+1}x_{2j-1}| < \varepsilon$ problem **IEPC** can not be solved by this algorithm.

3: **End If**

4: $b_{2j-1} = \dfrac{(\lambda_{2m}^{(2m)} - \lambda_{2m-1}^{(2m-1)})x'_{2j}x_{2j}}{x_{2j-1}x'_{2j} - x'_{2j-1}x_{2j}}$,

5: $a_{2j} = \lambda_{2m}^{(2m)} - \dfrac{b_{2j-1}x_{2j-1}}{x_{2j}}$,

6: $c_{2j-1} = (\lambda_{2m}^{(2m)} - \lambda_{2m-1}^{(2m-1)}) \sum\limits_{i=1}^{2j} x_i x'_i / E_j$,

7: $a_{2j-1} = \lambda_{2m}^{(2m)} - \dfrac{c_{2j-3}x_{2j-3} + b_{2j-1}x_{2j} + c_{2j-1}x_{2j+1}}{x_{2j-1}}$,

8: **End For**

9: $a_{2m-1} = \lambda_{2m-1}^{(2m-1)} - \dfrac{c_{2m-3}x'_{2m-3}}{x'_{2m-1}}$,

10: $b_{2m-1} = (\lambda_{2m}^{(2m)} - \lambda_{2m-1}^{(2m-1)}) \sum\limits_{i=1}^{2m-1} x_i x'_i / x_{2m}x'_{2m-1}$,

11: $a_{2m} = \lambda_{2m}^{(2m)} - \dfrac{b_{2m-1}x_{2m-1}}{x_{2m}}$.

---

5. Numerical examples

To illustrate the results of the previous section, some numerical examples are given which have been carried out using Matlab software.

Example 5.1. Given are two distinct real numbers

$$\lambda_7^{(7)} = 3,\ \lambda_8^{(8)} = 5,\ \varepsilon = 10^{-4},$$

and real vectors

$$X_8 = (1, 0.6, 3.2, 3.2, -6.1, 0.4, 2.8, -1.7)^\top$$

and

$$X'_7 = (1, 0.7, 3, 3.5, -5.5, 0.4, 1.8)^\top,$$

find a matrix $8 \times 8 \in S(C_4)$ such that $(\lambda_8^{(8)}, X_8)$ is an eigenpair of A_8 and $(\lambda_7^{(7)}, X_7')$ is an eigenpair of A_7.

Solution: By applying Algorithm 1, we get the unique solution

$$A_8 = \begin{bmatrix} -45.4800 & 8.4000 & 14.2000 & 0 & 0 & 0 & 0 & 0 \\ 8.4000 & -9.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 14.2000 & 0 & -134.4571 & 14.0000 & -63.4857 & 0 & 0 & 0 \\ 0 & 0 & 14.0000 & -9.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & -63.4857 & 0 & -40.0081 & -1.3333 & -25.3077 & 0 \\ 0 & 0 & 0 & 0 & -1.3333 & -15.3333 & 0 & 0 \\ 0 & 0 & 0 & 0 & -25.3077 & 0 & -74.3291 & -39.8497 \\ 0 & 0 & 0 & 0 & 0 & 0 & -39.8497 & -60.6348 \end{bmatrix}.$$

From the above matrix A_8 we compute the spectra of A_7 , A_8 and obtain

$$\begin{aligned} \sigma(A_7) &= \{-169.9934, -80.4461, -46.4863, -15.4582, -11.0080, -7.2156, \underline{3.0000}\}, \\ \sigma(A_8) &= \{-170.2365, -109.7715, -47.3047, -33.5653, -15.3924, \\ &\quad -9.7731, -7.1962, \underline{5.0000}\}. \end{aligned}$$

The data which is obtained shows that the algorithm is correct.

Example 5.2. Given are two distinct real numbers

$$\lambda_5^{(5)} = 10,\ \lambda_6^{(6)} = 13,\ \varepsilon = 10^{-4},$$

and real vectors

$$X_6 = (0.5, 0.25, 1.4, 0.9, 8, 7.1)^\top$$

and

$$X_5' = (-0.25, -0.2, -0.35, -0.3, -0.075)^\top,$$

find a matrix $6 \times 6 \in S(C_3)$ such that (λ_6, X_6) is an eigenpair of A_6 and (λ_5, X_5') is an eigenpair of A_5.

Solution: By applying Algorithm 1, we get the unique solution

$$A_6 = \begin{bmatrix} 2.6000 & 4.0000 & 3.0000 & 0 & 0 & 0 \\ 4.0000 & 5.0000 & 0 & 0 & 0 & 0 \\ 3.0000 & 0 & 1.0219 & 7.7143 & 1.0408 & 0 \\ 0 & 0 & 7.7143 & 1.0000 & 0 & 0 \\ 0 & 0 & 1.0408 & 0 & 5.1429 & 8.6479 \\ 0 & 0 & 0 & 0 & 8.6479 & 3.2559 \end{bmatrix}.$$

From the above matrix A_6 we compute the eigenvalues of A_5, A_6 and obtain

$$\begin{aligned} \sigma(A_5) &= \{-7.2909, -0.2524, 5.0414, 7.2666, \underline{10.0000}\}, \\ \sigma(A_6) &= \{-7.3471, -4.4329, -0.2518, 7.1986, 9.8539, \underline{13.0000}\}. \end{aligned}$$

The underlined eigenvalues are in consonance with the maximal eigenvalues and the algorithm is correct.

6. Conclusions

The inverse eigenvalue problem for graphs has been previously solved only for special classes of graphs, such as trees, paths and brooms. In this paper, we solved the inverse eigenvalue problem for the construction of matrices whose graphs are m-centipedes by using mixed eigen data. The results obtained in this paper provide an efficient method for constructing such matrices from two eigenpairs of leading principal submatrices of the desired matrix. The problem IEPC is important in the sense that it partially describes the inverse eigenvalue problem while it constructs matrices from partial information of the prescribed eigenvalues and eigenvectors. Such partially described problems may occur in computations involving a complex physical system such that it is difficult to obtain its entire spectrum. It would be interesting to consider such IEPs for other acyclic matrices as well.

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