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Abstract. In this work, the fractional Lie symmetry method is applied for symmetry analysis of time fractional Kupershmidt equation. Using the Lie symmetry method, the symmetry generators for time fractional Kupershmidt equation are obtained with Riemann-Liouville fractional derivative. With the help of symmetry generators, the fractional partial differential equation is reduced into the fractional ordinary differential equation using Erdélyi-Kober fractional differential operator. The conservation laws are determined for the time fractional Kupershmidt equation with the help of new conservation theorem and fractional Noether operators. The explicit analytic solutions of fractional Kupershmidt equation are obtained using the power series method. Also, the convergence of the power series solutions is discussed by using the implicit function theorem.

1 Introduction

Fractional calculus is the theory of fractional integrals and derivatives of arbitrary order which is evolved towards the end of 17th century. Many researchers are devoted to the interpretation, properties and applications of fractional order differential equations [20], [26], [34]. In the recent years, fractional differential equations (FDEs) have been studied frequently to describe various physical aspects and procedure in hydrology, visco-elasticity, mechanics, neurons, image processing, physics, control-theory, electrochemistry and finance [7], [10], [25], [27], [28]. Many efficient

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methods have been developed to obtain the analytical and numerical solutions of fractional order differential equations like (G'/G)-expansion method [31], homotopy perturbation method [39], modified trial equation method [22], exponential function method [41], sub equational method [33], functional variable method [15] and so on.

The Lie symmetry method was first introduced by Sophus Lie [14] in 1980. This method is an algorithmic procedure to obtain the point symmetry which leaves the considered differential equation invariant. Symmetry analysis provides a lot of information about the modeled partial differential equations. Therefore, symmetry analysis [1], [4], [5], [19], [43] have several applications in the field of science and engineering. Adapting the Lie group analysis method and proposing the prolongation formulas for fractional derivatives, Gazizov et al. [9] studied the symmetry properties of fractional order differential equations with the help of Riemann-Liouville and Caputo fractional derivatives [6]. Despite the importance of conservation laws in internal properties and existence and uniqueness analysis of differential equations, the conservation laws for fractional differential equations are not widely discussed. A few works about the symmetry analysis and conservation laws of fractional differential equations can be noticed in [2], [11], [29], [35], [40]. Since, some properties of fractional derivatives are different from the integer order derivatives. Therefore, obtaining the Lie symmetries and conservation laws for fractional order differential equations is a topic of great interest for researchers.

The Kupershmidt equation plays an important role in the nonlinear dispersive waves. Solitary waves propagate in nonlinear dispersive media. These waves preserve a stable form due to dynamic balance between the dispersive and nonlinear influences. However, in reality, the next state of a the physical phenomenon may depend no only on its current state but also on its historical states (non-local property), which may be successfully modelled by using the theory of derivatives of fractional order. Fractional order derivative significantly affects the properties of the equation. The time fractional Kupershmidt equation $(0 < \alpha \leq 1)$ can be considered as a generalized form of the original equation for $\alpha = 1$. Zhang et al. [42] has obtained the generalized Kupershmidt equation with the help of generalized Burgers Heirarchy equation. The time fractional generalized Kupershmidt equation obtained from the classified generalized Kupershmidt equation by replacing its time derivative with fractional derivative (α) is as follows:

$$\mathrm{D}_t^{\alpha} u = u_{xxx} + 3u^2 u_x + 3u_x^2 + 3uu_{xx} + 2\beta uu_x + \beta u_{xx} \,, \tag{1}$$

where β is the arbitrary constant and $\mathrm{D}_t^{\alpha} u$ is the Riemann-Liouville fractional derivative of u for order α, $(0 < \alpha \leq 1)$, which is defined as follows [12], [23], [30]:

$$\mathrm{D}_t^{\alpha} u(x,t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \frac{\partial^m}{\partial t^m} \int_0^t (t-\sigma)^{m-\alpha-1} u(\sigma, x)\, \mathrm{d}\sigma, & m-1 \leq \alpha \leq m, m \in N, \\ \frac{\mathrm{d}\partial^m}{\partial t^m}, & \alpha = m \in N, \end{cases}$$

where Γ denotes the Euler's gamma function.

In this work, we have applied fractional Lie group method to obtain the symmetry properties and conservation laws for the time fractional Kupershmidt equation. Using the similarity transformations, the time fractional Kupershmidt equation is reduced into fractional differential equation with Erdélyi-Kober operator. The fractional order partial differential equation is in Euler-Lagrange form. Therefore, the conservation laws of the fractional PDE has been obtained with the help of Noether's operators [3], [17], [18] by Lie symmetries. Using power series method [38], the explicit series solutions of fractional Kupershmidt equation are derived. Also, the convergence of obtained series solutions has been proved.

This paper is arranged in the following manner: A brief introduction is presented about the fractional differential equations in section 1. In section 2, some basic properties of Riemann-Liouville fractional derivative are given. Also, the basic idea of fractional Lie symmetry method is described in detail. In section 3, the proposed method is applied on fractional Kupershmidt equation for obtaining the symmetry generators. Using the symmetry generators, the fractional Kupershmidt equation is reduced into the fractional ordinary differential equation with the help of Erdélyi-Kober fractional differential operator with Riemann-Liouville fractional derivative. In section 4, the new conserved vectors are obtained for fractional Kupershmidt equation along with formal Lagrangian using new conservation theorem and fractional generalization of Noether operators. In section 5, the explicit solutions of time fractional Kupershmidt equation are obtained in the form of power series. Also, the convergence of the obtained series solutions has been proved in section 5. In section 6, conclusion is presented about the whole study.

2 Preliminaries

2.1 Some basic properties of Riemann-Liouville (RL) fractional derivative:

For an arbitrary order α, the RL derivative has some following properties:

$$\begin{aligned}
\mathrm{d}f(x) &= \frac{\mathrm{D}_x^\alpha f(x)(\mathrm{d}x)^\alpha}{\Gamma(1+\alpha)},\\
\mathrm{D}_x^\alpha(uv) &= (\mathrm{D}_x^\alpha u)v + u(\mathrm{D}_x^\alpha v)\,,\\
\mathrm{D}_t^\alpha f(x(t)) &= \frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{D}_t^\alpha x(t)\,, \quad \text{provided } \frac{\mathrm{d}f}{\mathrm{d}x} \text{ exists}\\
\mathrm{D}_t^\alpha t^\beta &= \frac{\Gamma(\beta+1)t^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}\,, \quad \beta > \alpha - 1,\\
\int(\mathrm{d}t)^\beta &= t^\beta.
\end{aligned}$$

2.2 Basic Idea of Proposed Fractional Lie Symmetry Method

We have consider the coupled time fractional non-linear PDEs with two independent variables given in the following form:

$$\mathrm{D}_t^\alpha u = f(x, t, u, u_x, u_t, u_{xx}, u_{xxx}, \dots)\,, \tag{2}$$

where $\alpha > 0$ and subscripts represent the partial derivatives.

Let us consider the following symmetry generator of one parameter Lie group of transformations under which Eq. (2) remain invariant, given as

$$
\begin{aligned}
\tilde{x} &= x + \varepsilon\xi(x,t,u) + O(\varepsilon^2),\\
\tilde{t} &= t + \varepsilon\tau(x,t,u) + O(\varepsilon^2),\\
\tilde{u} &= u + \varepsilon\eta(x,t,u) + O(\varepsilon^2),\\
\mathrm{D}_t^\alpha \tilde{u} &= \mathrm{D}_t^\alpha u + \varepsilon\eta^{\alpha,t}(x,t,u) + O(\varepsilon^2),\\
\frac{\partial \tilde{u}}{\partial \tilde{x}} &= \frac{\partial u}{\partial x} + \varepsilon\eta^{x}(x,t,u) + O(\varepsilon^2),\\
\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} &= \frac{\partial^2 u}{\partial x^2} + \varepsilon\eta^{xx}(x,t,u) + O(\varepsilon^2),\\
\frac{\partial^3 \tilde{u}}{\partial \tilde{x}^3} &= \frac{\partial^3 u}{\partial x^3} + \varepsilon\eta^{xxx}(x,t,u) + O(\varepsilon^2),
\end{aligned}
\tag{3}
$$

where ε is the group parameter and ξ, τ are η are the infinitesimals of the transformations.

The infinitesimal generator $\mathbf{X}$ can be written in following form:

$$\mathbf{X} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u.$$

The k-th order prolongation of the fractional vector field is given as

$$\mathrm{Pr}^{(\alpha,k)}\,\mathbf{X} = \mathbf{X} + \eta^{\alpha,t}\frac{\partial}{\partial u_t^\alpha} + \eta^x\frac{\partial}{\partial u_x} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \dots + \eta^{xx\dots i_k}\frac{\partial}{\partial u_{xx\dots i_k}}\,, \quad k \geq 1$$

where the operators η^i are extended infinitesimals [19] and $\eta^{\alpha,t}$, $\nu^{\alpha,t}$ are the fractional extended infinitesimals defined as follows:

$$
\begin{aligned}
\eta^{\alpha,t} &= \mathrm{D}_t^\alpha(\eta) + \xi\mathrm{D}_t^\alpha(u_x) - \mathrm{D}_t^\alpha(\xi u_x) + \mathrm{D}_t^\alpha(u(\mathrm{D}_t\tau)) - \mathrm{D}_t^{\alpha+1}(\tau u) + \tau\mathrm{D}_t^{\alpha+1}(u),\\
\eta^x &= \mathrm{D}_x(\eta) - u_x\mathrm{D}_x(\xi) - u_t\mathrm{D}_x(\tau),\\
\eta^{xx} &= \mathrm{D}_x(\eta^x) - u_{xx}\mathrm{D}_x(\xi) - u_{xt}\mathrm{D}_x(\tau),\\
\eta^{xxx} &= \mathrm{D}_x(\eta^{xx}) - u_{xxx}\mathrm{D}_x(\xi) - u_{xxtt}\mathrm{D}_x(\tau),
\end{aligned}
\tag{4}
$$

where D_x and D_t denote the total derivatives with respect to independent variables, defined as

$$
\begin{aligned}
\mathrm{D}_t &= \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{xt}\partial_{u_x} + \dots,\\
\mathrm{D}_x &= \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + u_{tx}\partial_{u_t}\dots .
\end{aligned}
$$

Now, we focus on the expressions for $\eta^{\alpha,t}$ and $\nu^{\alpha,t}$.

The generalized Leibnitz's rule is given by

$$\mathrm{D}_t^\alpha(f(t)h(t)) = \sum_{m=0}^{\infty}\binom{\alpha}{m}\mathrm{D}_t^n f(t)\mathrm{D}_t^{\alpha-n}h(t) \tag{5}$$

where

$$\binom{\alpha}{m} = \frac{\Gamma(\alpha+1)}{\Gamma(m+1)\Gamma(\alpha+1-m)}\,.$$

Now, using Leibniz's rule (5) in the expressions of $\eta^{\alpha,t}$ and $\nu^{\alpha,t}$, we have

$$\eta^{\alpha,t} = \mathrm{D}_t^\alpha(\eta) - \alpha \mathrm{D}_t^\alpha(\tau)\frac{\partial^\alpha u}{\partial t^\alpha} - \sum_{m=0}^{\infty}\binom{\alpha}{m}\mathrm{D}_t^m(\xi)\mathrm{D}_t^{\alpha-m}u_x \\ - \sum_{m=0}^{\infty}\binom{\alpha}{m+1}\mathrm{D}_t^{m+1}(\tau)\mathrm{D}_t^{\alpha-m}u\,. \quad (6)$$

The chain rule for a composite function is as follows (see [21]):

$$\frac{\mathrm{d}^\alpha f(g(t))}{\mathrm{d}t^\alpha} = \sum_{k=0}^{\infty}\sum_{r=0}^{k}\binom{k}{r}\frac{1}{k!}[-g(t)]^r\frac{\mathrm{d}^k f(g)}{\mathrm{d}f^k}\frac{\partial^\alpha}{\partial t^\alpha}[(g(t))^{k-r}]\,. \quad (7)$$

Using Eqs. (5) and (7) in Eq. (6) with $f(t) = 1$, we have

$$\mathrm{D}_t^\alpha(\eta) = \partial_t^\alpha \eta + \Big(\eta_u \partial_t^\alpha u - u\partial_t^\alpha \eta_u\Big) + \sum_{m=0}^{\infty}\binom{\alpha}{m}\partial_t^m \eta_u \mathrm{D}_t^{\alpha-m}u + \mu\,, \quad (8)$$

where

$$\mu = \sum_{m=2}^{\infty}\sum_{n=2}^{m}\sum_{j=2}^{n}\sum_{r=0}^{j-1}\binom{\alpha}{m}\binom{m}{n}\binom{j}{r}\frac{1}{j!}\frac{t^{m-\alpha}}{\Gamma(m+1-\alpha)}(-u)^r\frac{\partial^n}{\partial t^n}(u^{j-r})\frac{\partial^{m-n+j}\eta}{\partial t^{m-n}\partial u^j}\,.$$

Thus Eq. (6) yields

$$\eta^{\alpha,t} = \partial_t^\alpha \eta + \Big(\eta_u - \alpha \mathrm{D}_t(\tau)\Big)\eta_u\partial_t^\alpha u - u\partial_t^\alpha \eta_u + \mu \\ + \sum_{m=1}^{\infty}\left[\binom{\alpha}{m}\partial_t^\alpha \eta_u - \binom{\alpha}{m+1}\mathrm{D}_t^{m+1}(\tau)\right]\mathrm{D}_t^{\alpha-m}(u) \\ - \sum_{m=1}^{\infty}\binom{\alpha}{m}\mathrm{D}_t^m(\xi)\mathrm{D}_t^{\alpha-m}u_x. \quad (9)$$

The infinitesimal generator $\mathbf{X}$ must satisfy the invariance conditions [29] for Eq. (2), which are given as follows:

$$\mathrm{Pr}^n\,\mathbf{X}(\Delta u)|_{\Delta u=0} = 0,$$

where $\Delta u = \mathrm{D}_t^\alpha u - f$.

3 Time fractional Kupershmidt equation

3.1 Lie symmetries

In this work, we consider a special case of Eq. (1) for which $\beta = 0$, to study the symmetry reduction and conservation laws. So, the time fractional Kupershmidt equation is represented as follows:

$$\mathrm{D}_t^\alpha u = u_{xxx} + 3u^2u_x + 3u_x^2 + 3uu_{xx}, \quad (10)$$

where $\mathrm{D}_t^\alpha(u)$ is Riemann-Liouville fractional derivative of order α with respect to t. Applying prolongation of fractional vector field on Eq. (10), we get the following equations:

$$\eta^{\alpha,t} - \eta^{xxx} - 6u\eta u_x - 3u^2\eta^x - 6u_x\eta^{xx} - 3\eta u_{xx} - 3u\eta^{xx} = 0\,, \tag{11}$$

Now, substituting the Eqs. (4) and (9) in the Eq. (11) and equating the coefficients of various monomials to zero and then solving the over determined system of equations, we get the following set of infinitesimals for time fractional Kupershmidt equation:

$$\xi = \frac{1}{3}\alpha c_2 x + c_1\,, \quad \tau = c_2 t + c_3\,, \quad \eta = -\frac{1}{3}\alpha c_2 u\,.$$

where c_1, c_2 and c_3 are the arbitrary constants.

Since the lower limit of the Riemann integral in Riemann-Liouville fractional partial derivative is fixed. Therefore, $\tau(x,t,u,w)|_{t=0} = 0$ should be necessary to preserve its structure under the transformations (3). Therefore, c_3 must be zero (i.e. $\tau = tc_2$).

So, the symmetry generators to form a lie algebra of Eq. (11) are found as:

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \tag{12}$$

$$\mathbf{X}_2 = \frac{1}{3}\alpha x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{1}{3}\alpha u\frac{\partial}{\partial u}\,. \tag{13}$$

Therefore, the infinitesimal generator $\mathbf{X}$ for Eq. (11) can be written as

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2\,.$$

From Eqs. (12) and (13), we can see that the the vector fields $\mathbf{X}_1$ and $\mathbf{X}_2$ are closed under Lie bracket ($[\mathbf{X}_i,\mathbf{X}_j] = \mathbf{X}_i\mathbf{X}_j - \mathbf{X}_j\mathbf{X}_i$). We have

$$[\mathbf{X}_1,\mathbf{X}_1] = 0, \quad [\mathbf{X}_1,\mathbf{X}_2] = \frac{1}{3}\alpha\mathbf{X}_1, \quad [\mathbf{X}_2,\mathbf{X}_2] = 0, \quad \text{and} \quad [\mathbf{X}_2,\mathbf{X}_1] = -\frac{1}{3}\alpha\mathbf{X}_1.$$

Theorem 1. *A solution $u = \omega(x,t)$ is the invariant solution of Eq. (2) iff*

(i) $u = \omega(x,t)$ satisfies the FPDE (2), and

(ii) $u = \omega(x,t)$ is the invariant surface, i.e.

$$\mathbf{X}\omega = 0 \Longleftrightarrow \left(\xi(x,t,u)\frac{\partial}{\partial_x} + \tau(x,t,u)\frac{\partial}{\partial_t} + \eta(x,t,u)\frac{\partial}{\partial_u}\right)\omega = 0.$$

3.2 Symmetry reduction for time fractional Kupershmidt equations

In this section, we obtain the reduced equations for (11) by imposing the Lie symmetries.

For the vector field $\mathbf{X}_2$, the characteristic equations are as follows:

$$\frac{\mathrm{d}x}{\frac{1}{3}\alpha x} = \frac{\mathrm{d}t}{t} = \frac{\mathrm{d}u}{-\frac{1}{3}\alpha u}\,. \tag{14}$$

After solving the Eq. (14), we obtain the following similarity variables:

$$z = xt^{-\frac{1}{3}\alpha}, \quad u = f(z)t^{-\frac{1}{3}\alpha}. \tag{15}$$

Theorem 2. *The transformations* (15) *reduce the Eq.* (10) *in the fractional nonlinear ordinary equation given as follows:*

$$\left(\mathrm{P}_{\frac{3}{\alpha}}^{1-\frac{4\alpha}{3},\alpha} f\right)(z)-\left(f_{zzz}+3f^2 f_z+3f_z^2+3ff_{zz}\right)=0$$

with the Erdélyi-Kober fractional differential operator $\mathrm{P}_{\beta}^{\tau,\alpha}$ *[13], [16] defined as*

$$\left(\mathrm{P}_{\beta}^{\tau,\alpha} f\right):=\prod_{j=0}^{m-1}\left(\tau+j-\frac{1}{\beta}z\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(\mathrm{K}_{\beta}^{\tau+\alpha,m-\alpha} f\right)(z)\,, \tag{16}$$

where

$$m=\begin{cases}[\alpha]+1, & \alpha\in N,\\ \alpha, & \alpha\notin N,\end{cases}$$

$$\left(\mathrm{K}_{\beta}^{\tau+\alpha,m-\alpha} f\right)(z):=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_1^{\infty}(u-1)^{\alpha-1}u^{-(\tau+\alpha)}f\left(zu^{\frac{1}{\beta}}\right)\mathrm{d}u, & \alpha>0,\\ f(z), & \alpha=0,\end{cases}$$

is the Erdélyi-Kober fractional integral operator [16].

Proof. When $m-1<\alpha<m$, $m=1,2,3,\ldots$, from Riemann-Liouville fractional derivative, we have

$$\mathrm{D}_t^{\alpha}u(x,t)=\frac{\partial^m}{\partial t^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t(t-s)^{m-\alpha-1}s^{-\frac{1}{3}\alpha}f\left(xs^{-\frac{1}{3}\alpha}\right)\mathrm{d}s\right]. \tag{17}$$

Let $w=\frac{t}{s}$, then $\mathrm{d}s=-\frac{t}{w^2}\mathrm{d}w$.

So Eq. (17) can be expressed as

$$\begin{aligned}\mathrm{D}_t^{\alpha}u(x,t)&=\frac{\partial^m}{\partial t^m}\left[\frac{t^{m-\frac{4\alpha}{3}}}{\Gamma(m-\alpha)}\int_1^{\infty}(w-1)^{m-\alpha-1}w^{m+1-\alpha-\frac{\alpha}{3}}f\left(zw^{\frac{\alpha}{3}}\right)\mathrm{d}w\right]\\&=\frac{\partial^m}{\partial t^m}\left[t^{m-\frac{4\alpha}{3}}\left(\mathrm{K}_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},m-\alpha} f\right)(z)\right]\\&=\frac{\partial^{m-1}}{\partial t^{m-1}}\left[\frac{\partial}{\partial t}\left(t^{m-\frac{4\alpha}{3}}\left(\mathrm{K}_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},m-\alpha} f\right)(z)\right)\right]\end{aligned} \tag{18}$$

For $z=xt^{\frac{-\alpha}{3}}$ and a function $\phi(z)\in C^1(0,\infty)$, we get

$$t\frac{\mathrm{d}}{\mathrm{d}t}\phi(z)=tz_t\phi'(z)=tx\left(-\frac{\alpha}{3}\right)t^{\frac{-\alpha}{3}-1}\phi'(z)=-\frac{\alpha}{3}z\frac{\mathrm{d}}{\mathrm{d}z}\phi(z)\,. \tag{19}$$

From relation (19), Eq. (18) can be written as follows:

$$\mathrm{D}_t^{\alpha}u(x,t)=\frac{\partial^{m-1}}{\partial t^{m-1}}\left[\left(t^{m-1-\frac{4\alpha}{3}}\left(\left(m-\frac{4\alpha}{3}-\frac{\alpha}{3}z\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(\mathrm{K}_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3},m-\alpha} f\right)(z)\right)\right)\right]$$

Repeating the above procedure $m-1$ times, we obtain

$$\mathrm{D}_t^\alpha u(x,t) = t^{-\frac{4\alpha}{3}} \prod_{j=0}^{m-1} \left(1 - \frac{4\alpha}{3} + j - \frac{\alpha}{3} z \frac{\mathrm{d}}{\mathrm{d}z} \left(\mathrm{K}_{\frac{3}{\alpha}}^{1-\frac{\alpha}{3}, m-\alpha} f\right)(z)\right). \tag{20}$$

Using Eq. (16) in Eq. (20), we obtain

$$\mathrm{D}_t^\alpha u(x,t) = t^{-\frac{4\alpha}{3}} \left(\mathrm{P}_{\frac{3}{\alpha}}^{1-\frac{4\alpha}{3}, \alpha} f\right)(z).$$

Therefore, Eq. (10) can be written into a non-linear fractional ordinary differential equations as follows:

$$\left(\mathrm{P}_{\frac{3}{\alpha}}^{1-\frac{4\alpha}{3}, \alpha} f\right)(z) - (f_{zzz} + 3f^2 f_z + 3f_z^2 + 3f f_{zz}) = 0. \qquad \square$$

4 Conservation laws

In this section, we have found the conserved vectors for time fractional Kupershmidt equations using new conservation theorem [24], [32].

The conservation laws for Eq. (10) are defined as a vector field $T = (T^1, T^2)$, where $T^1 = T^1(x,t,u,...)$ and $T^2 = T^2(x,t,u,...)$ are called conserved vectors for Eq. (10) if it satisfies the following conservation theorem:

$$[\mathrm{D}_t T^1 + \mathrm{D}_x T^2]_{(10)} = 0.$$

The formal Lagrangian of Eq. (10) can be written in the following form:

$$L = \gamma(x,t)(\mathrm{D}_t^\alpha(u) - u_{xxx} - 3u^2 u_x - 3u_x^2 - 3u u_{xx}), \tag{21}$$

where ω_1 is the new dependent variables of x and t.

The Euler Lagrangian operator is defined as follows:

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - \mathrm{D}_x \frac{\partial}{\partial u_x} + \mathrm{D}_x^2 \frac{\partial}{\partial u_{xx}} - \mathrm{D}_x^3 \frac{\partial}{\partial u_{xxx}} + \cdots + (\mathrm{D}_t^\alpha)^* \frac{\partial}{\partial \mathrm{D}_t^\alpha u},$$

where $(\mathrm{D}_t^\alpha)*$ is the adjoint operator of fractional differential operator D_t^α, given as follows:

$$(\mathrm{D}_t^\alpha)^* = (-1)^m I_s^{m-\alpha} \mathrm{D}_t^m,$$

where $I_s^{m-\alpha}$ is the right-hand-sided fractional integral operator of order $m-\alpha$, which is defined as

$$I_s^{m-\alpha} f(x,t) = \frac{1}{\Gamma(m-\alpha)} \int_t^s \frac{f(x,p)}{(p-t)^{\alpha+1-m}} \, \mathrm{d}p,$$

where $m = [\alpha] + 1$.

So, the adjoint equations can be written as

$$\frac{\delta L}{\delta u} = 0.$$

The component of conserved vectors are obtained by applying Noether operators to the Lagrangian. The fractional Noether operator for t-component can be written by the following formula [3], [17], [18]:

$$T^1 = \tau\tilde{I} + \sum_{k=0}^{m}(-1)^k \mathrm{D}_t^{\alpha-1-k}(W)\mathrm{D}_t^k \frac{\partial L}{\partial \mathrm{D}_t^\alpha u} - (-1)^m I\left(W, \mathrm{D}_t^m \frac{\partial L}{\partial \mathrm{D}_t^\alpha u}\right). \tag{22}$$

Here

$$I(f,g) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \int_t^T \frac{f(\tau,x)g(\mu,x)}{(\mu-\tau)^{\alpha+1-m}} \,\mathrm{d}\mu\,\mathrm{d}\tau,$$

$\tilde{I}$ is the identity operator and $W = \eta - \tau u_t - \xi u_x$ is the Lie characteristic function.

The other conserved vector T^2 for x-component is represented as

$$T^2 = \xi\tilde{I} + W\frac{\delta L}{\delta u_x} + \mathrm{D}_x(W)\frac{\delta L}{\delta u_{xx}} + (\mathrm{D}_x)^2(W)\frac{\delta L}{\delta u_{xxx}} + \dots . \tag{23}$$

Now, the Lie characteristics function for the vector $\mathbf{X}_1$ is obtained as

$$W = -\frac{\alpha}{3}xu_x - tu_t - \frac{\alpha}{3}u\,. \tag{24}$$

Substituting the value of Lagrangian (21) in Eqs. (22) and (23) and using the value of W from Eq. (24), we have obtained the t-component of conserved vector for $\mathbf{X}_2$ as follows:

$$\begin{aligned} T^1 &= \tau\tilde{I} + \mathrm{D}_t^{\alpha-1}(W)\mathrm{D}_t^0\frac{\partial L}{\partial \mathrm{D}_t^\alpha u} + I\left(W, \mathrm{D}_t\frac{\partial L}{\partial \mathrm{D}_t^\alpha u}\right) \\ &= \gamma \mathrm{D}_t^{\alpha-1}\left(-\frac{\alpha}{3}xu_x - tu_t - \frac{\alpha}{3}u\right) + I\left[\left(-\frac{\alpha}{3}xu_x - tu_t - \frac{\alpha}{3}u\right), \gamma_t\right] \end{aligned}$$

Also, the x-component of conserved vector for $\mathbf{X}_2$ is obtained in the following form:

$$\begin{aligned} T^2 &= \xi\tilde{I} + W\left[\frac{\partial L}{\partial u_x} - \mathrm{D}_x\frac{\partial L}{\partial u_{xx}} + (\mathrm{D}_x)^2\frac{\partial L}{\partial u_{xxx}}\right] + \mathrm{D}_x(W)\left[\frac{\partial L}{\partial u_{xx}} - \mathrm{D}_x\frac{\partial L}{\partial u_{xxx}}\right] \\ &\quad + (\mathrm{D}_x)^2(W)\left[\frac{\partial L}{\partial u_{xxx}}\right]\Bigg] \\ &= \gamma_{xx}\left(\frac{\alpha}{3}xu_x + tu_t + \frac{\alpha}{3}u - 1\right) - \gamma_x\left[3u\left(\frac{\alpha}{3}xu_x + tu_t + \frac{\alpha}{3}u\right)\right. \\ &\quad \left. + \left(\frac{2\alpha}{3}u_x + tu_{xt} + \frac{\alpha}{3}xu_{xx}\right)\right] + \gamma\left[3u\left(\frac{2\alpha}{3}u_x + tu_{xt} + \frac{\alpha}{3}xu_{xx}\right)\right. \\ &\quad \left. + (3u^2 + 3u_x)\left(\frac{\alpha}{3}xu_x + tu_t + \frac{\alpha}{3}u\right)\right]. \end{aligned}$$

5 Explicit power series solution

In this section, we investigate the exact analytic solutions of Eq. (10) using power series method [8], [36], [37], [38] and analyze the convergence of power series solution. Let

$$u(x,t) = u(\omega), \quad \omega = kx - \frac{\varepsilon t^\alpha}{\Gamma(1+\alpha)}\,, \tag{25}$$

where ε and k are arbitrary constants.

From Eqs. (10) and (25), we obtained

$$\varepsilon u' + k^3 u''' + 3ku^2 u' + 3k^2 (u')^2 + 3k^2 uu'' = 0\,. \tag{26}$$

Now, we seek a solution of equation (26) in the form of power series as

$$u(\omega) = \sum_{s=0}^{\infty} c_s \omega^s. \tag{27}$$

From equations (26) and (27), we have

$$\begin{aligned}
\varepsilon c_1 &+ \sum_{s=1}^{\infty} (s+1) c_{s+1} \omega^s + 6k^3 c_3 \\
&+ k^3 \sum_{s=1}^{\infty} (s+3)(s+2)(s+1) c_{s+3} \omega^s + 3c_0^2 c_1 k \\
&+ 3k \sum_{s=1}^{\infty} \sum_{j=0}^{s} \sum_{k=0}^{j} (s+1-k)\, c_k\, c_{j-k}\, c_{s+1-k} \omega^s \\
&+ 3k^2 \sum_{s=1}^{\infty} \sum_{k=0}^{s} (s+1-k)(k+1) c_{k+1} c_{s+1-k} \omega^s + 6k^2 c_0 c_2 + 3k^2 c_1^2 \\
&+ 3k^2 \sum_{n=1}^{\infty} \sum_{k=0}^{s} (s+2-k)(s+1-k) c_k c_{s+2-k} \omega^n = 0\,.
\end{aligned} \tag{28}$$

Comparing the coefficients for $n = 0$ in equation (28), we get

$$3c_0^2 c_1 k + \varepsilon c_1 + 6k^3 c_3 + 3k^2 c_1^2 + 6k^2 c_0 c_2 = 0\,, \tag{29}$$

and for $n \geq 1$ in equation (28), we have

$$\begin{aligned}
c_{s+3} = \frac{-1}{(s+3)(s+2)(s+1)} \Bigg[&(s+1) c_{s+1} \\
&+ 3k \sum_{j=0}^{s} \sum_{k=0}^{j} (s+1-k) c_k c_{j-k} c_{s+1-k} \\
&+ 3k^2 \sum_{k=0}^{s} (s+1-k)(k+1) c_{k+1} c_{s+1-k} \\
&+ 3k^2 \sum_{k=0}^{s} (s+2-k)(s+1-k)\, c_k\, c_{s+2-k} \Bigg], \quad s = 1, 2, \ldots\,.
\end{aligned} \tag{30}$$

Thus, for arbitrary chosen values of constants c_0, c_1 and c_2 the other terms of the sequence $\{c_s\}_{s=4}^{\infty}$ can be determined successively from equations (29) and (30) in a unique manner. This implies that, there exists a power series solution of equation (10), given by equation (27) in the form of power series with the coefficients given by equations (29) and (30). It can be easily shown that the power series (27) converges.

Hence, the solution of equation (10) in the form of power series can be written as follows:

$$
\begin{aligned}
u(\omega) &= c_0 + c_1\omega + c_2\omega^2 + c_3\omega^3 + \sum_{s=1}^{\infty} c_{s+3}\,\omega^{s+3} \\
&= c_0 + c_1\left(kx - \frac{\varepsilon t^\alpha}{\Gamma(1+\alpha)}\right) + c_2\left(kx - \frac{\varepsilon t^\alpha}{\Gamma(1+\alpha)}\right)^2 + c_3\left(kx - \frac{\varepsilon t^\alpha}{\Gamma(1+\alpha)}\right)^3 \\
&\quad + \frac{-1}{(s+3)(s+2)(s+1)} \sum_{s=0}^{\infty} \Bigg[(s+1)\,c_{s+1} \\
&\quad + 3k \sum_{j=0}^{s}\sum_{k=0}^{j} (s+1-k)\,c_k\,c_{j-k}\,c_{s+1-k} \\
&\quad + 3k^2 \sum_{k=0}^{s} (s+1-k)(k+1)\,c_{k+1}\,c_{s+1-k} \\
&\quad + 3k^2 \sum_{k=0}^{s} (s+2-k)(s+1-k)\,c_k\,c_{s+2-k}\Bigg]\left(kx - \frac{\varepsilon t^\alpha}{\Gamma(1+\alpha)}\right)^{s+3}, \qquad (31)
\end{aligned}
$$

where c_{s+3} $(s = 1, 2, \dots)$ are given by equation (30). The above result can be summarised as follows:

Theorem 3. *Equation* (10) *admits the power series solution in the form*

$$
u(x,t) = \sum_{s=0}^{\infty} c_s \left(kx - \frac{\varepsilon t^\alpha}{\Gamma(1+\alpha)}\right)^s,
$$

where c_0, c_1, c_2, k and $\varepsilon \neq 0$ are arbitrary constants and $c_{s+3}\,(s = 1, 2, \dots)$ can be determined by equation (30).

Convergence analysis

Now, we will prove that the power series solution of Eq. (10) is convergent by using implicit function theorem.

From Eq. (30), we can write

$$
\begin{aligned}
|c_{s+3}| \le M\Bigg[&|c_{s+1}| + \sum_{k=0}^{j} |c_k||c_{j-k}||c_{s+1-k}| \\
&+ \sum_{k=0}^{s} |c_{k+1}||c_{s+2-k}| + \sum_{k=0}^{s} |c_k||c_{s+1-k}|\Bigg],
\end{aligned}
$$

where $M = \max\{\frac{1}{|k^3|}, \frac{3}{|k^2|}, \frac{3}{|k|}\}$.

If we define a power series $\mu = Q(\omega) = \sum_{s=0}^{\infty} q_s\omega^s$ by $q_0 = |c_0|$, $q_1 = |c_1|$, $q_2 = |c_2|$, $q_3 = |c_3|$, and

$$q_{s+3} = M\Big[q_{s+1}+\sum_{k=0}^{j} c_k c_{j-k} c_{s+1-k}+\sum_{k=0}^{s} c_{k+1}c_{s+2-k}+\sum_{k=0}^{s} c_k c_{s+1-k}\Big], \quad s = 1, 2, \ldots,$$

then it is easily seen that $|c_s| \leq q_s$, $s = 0, 1, \ldots$.

In other words, the series $\mu = Q(\xi) = \sum_{s=0}^{\infty} q_s\omega^s$ is a majorant series of equation (31). Now, we will show that this series $\mu = Q(\xi)$ is convergent and has a positive radius of convergence. We have

$$\begin{aligned}
Q(\omega) &= q_0 + q_1\xi + q_2\xi^2 + q_3\xi^3 + \sum_{s=1}^{\infty} q_{s+3}\omega^{s+3} \\
&= q_0 + q_1\omega + q_2\omega^2 + q_3\omega^3 \\
&\quad + M\Bigg[\sum_{s=1}^{\infty} c_{n+1}\,\omega^{n+3} \sum_{s=1}^{\infty}\sum_{j=0}^{s}\sum_{k=0}^{j}(s+1-k)\, c_k\, c_{j-k}\, c_{s+1-k}\, \omega^{s+3} \\
&\quad + \sum_{s=1}^{\infty}\sum_{k=0}^{s} c_{k+1}\, c_{s+1-k}\, \omega^{s+3} + \sum_{s=1}^{\infty}\sum_{k=0}^{s} c_k\, c_{s+2-k}\, \omega^{s+3}\Bigg] \\
&= q_0 + q_1\omega + q_2\omega^2 + q_3\omega^3 + M\big[\omega^2 Q^3(\omega) + (2\omega - q_0\omega^2)Q^2(\omega) \\
&\quad + (\omega^2 - 3q_0\omega - q_1\omega^2)Q(\omega) + q_0^2\omega\big]\,.
\end{aligned}$$

Consider the implicit functional system with respect to the independent variable ω as follows:

$$\begin{aligned}
Q(\omega, \mu) = mu - q_0 + q_1\omega + q_2\omega^2 + q_3\omega^3 + M\Big[&\omega^2 Q^3(\omega) + (2\omega - q_0\omega^2)Q^2(\omega) \\
&+ (\omega^2 - 3q_0\omega - q_1\omega^2)Q(\omega) + q_0^2\omega\Big]\,.
\end{aligned}$$

We can see that Q is analytic in the neighborhood of $(0, q_0)$ with $Q(0, q_0) = 0$, $Q'_\mu(0, q_0) = 1 \neq 0$, so by using implicit function theorem, we see that in a neighborhood of the point $(0, q_0)$, $Q(\omega)$ is analytic with a positive radius. Thus, in the neighborhood of the point $(0, q_0)$ of the plane, the power series (31) is convergent. This completes the proof.

6 Conclusion

In this work, the fractional Lie symmetry technique has been successfully applied to study the similarity reductions of time fractional Kupershmidt equation with Riemann-Liouville fractional derivative. Using Erdelyi-Kober differential operator, the fractional PDE has been reduced into the fractional ordinary differential equation. With the help of Noether operators and new conservation theorem, the new conserved vectors have been obtained successfully along with formal Lagrangian, which are used in the study of global behaviour and the stability of solutions of time fractional Kupershmidt equation. Also, the explicit solutions for the time fractional Kupershmidt equation are determined in the form of power series. The convergence of the the power series solution is also discussed. The obtained solutions may be useful in various areas of applied mathematics in interpolating some physical phenomena, accuracy testing, comparison of numerical results and so on. There are some possible extensions of this study, e.g. symmetry analysis for space-time fractional systems of non-linear PDEs with or without variable coefficients. Some of the extension work is in progress and will be discussed in the future work.

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References

[1] R. Arora, A. Chauhan: Lie Symmetry Analysis and Some Exact Solutions of $(2+1)$-dimensional KdV-Burgers Equation. *International Journal of Applied and Computational Mathematics* 5 (1) (2019) 15.

[2] D. Baleanu, M. Inc, A. Yusuf, A.I. Aliyu: Lie symmetry analysis, exact solutions and conservation laws for the time fractional modified Zakharov-Kuznetsov equation. *Nonlinear Analysis: Modelling and Control* 22 (6) (2017) 861–876.

[3] D. Baleanu, A. Yusuf, A.I. Aliyu: Space-time fractional Rosenou-Haynam equation: Lie symmetry analysis, explicit solutions and conservation laws. *Advances in Difference Equations* 2018 (1) (2018) 46.

[4] G.W. Bluman, J.D. Cole: The general similarity solution of the heat equation. *Journal of Mathematics and Mechanics* 18 (11) (1969) 1025–1042.

[5] G.W. Bluman, S. Kumei: Use of group analysis in solving overdetermined systems of ordinary differential equations. *Journal of Mathematical Analysis and Applications* 138 (1) (1989) 95–105.

[6] K. Diethelm, N.J. Ford, A.D. Freed: A predictor-corrector approach for the numerical solution of fractional differential equations. *Nonlinear Dynamics* 29 (1-4) (2002) 3–22.

[7] R.A. El-Nabulsi: Fractional functional with two occurrences of integrals and asymptotic optimal change of drift in the Black-Scholes model. *Acta Mathematica Vietnamica* 40 (4) (2015) 689–703.

[8] L.L. Feng, S.F. Tian, X.B. Wang, T.T. Zhang: Lie Symmetry Analysis, Conservation Laws and Exact Power Series Solutions for Time-Fractional Fordy-Gibbons Equation. *Communications in Theoretical Physics* 66 (3) (2016) 321.

[9] R.K. Gazizov, A.A. Kasatkin, S.Y. Lukashchuk: Symmetry properties of fractional diffusion equations. *Physica Scripta* 2009 (T136) (2009) 014016.

[10] R. Hilfer: *Applications of fractional calculus in physics.* World Scientific (2000).

[11] M. Inc, A. Yusuf, A.I. Aliyu, D. Baleanu: Lie symmetry analysis, explicit solutions and conservation laws for the space-time fractional nonlinear evolution equations. *Physica A: Statistical Mechanics and its Applications* 496 (2018) 371–383.

[12] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo: Fractional differential equations: A emergent field in applied and mathematical sciences. In: *Factorization, Singular Operators and Related Problems.* Springer (2003) 151–173.

[13] V.S. Kiryakova: *Generalized fractional calculus and applications*. CRC Press (1993).

[14] S. Lie: Theorie der Transformationsgruppen I. *Mathematische Annalen* 16 (4) (1880) 441–528.

[15] W. Liu, K. Chen: The functional variable method for finding exact solutions of some nonlinear time-fractional differential equations. *Pramana* 81 (3) (2013) 377–384.

[16] Y. Luchko, R. Gorenflo: Scale-invariant solutions of a partial differential equation of fractional order. *Fractional Calculus and Applied Analysis* 3 (1) (1998) 63–78.

[17] S.Y. Lukashchuk: Conservation laws for time-fractional subdiffusion and diffusion-wave equations. *Nonlinear Dynamics* 80 (1–2) (2015) 791–802.

[18] E. Noether: Invariant variation problems. *Transport Theory and Statistical Physics* 1 (3) (1971) 186–207.

[19] P.J. Olver: *Applications of Lie groups to differential equations*. Springer Science & Business Media (2000).

[20] M.D. Ortigueira, J.A.T. Machado: What is a fractional derivative? *Journal of computational Physics* 293 (2015) 4–13.

[21] T.J. Osler: Leibniz rule for fractional derivatives generalized and an application to infinite series. *SIAM Journal on Applied Mathematics* 18 (3) (1970) 658–674.

[22] Y. Pandir, Y. Gurefe, E. Misirli: New exact solutions of the time-fractional nonlinear dispersive KdV equation. *International Journal of Modeling and Optimization* 3 (4) (2013) 349–351.

[23] I. Podlubny: *Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Elsevier (1998).

[24] Ch.Y. Qin, Sh.F. Tian, X.B. Wang, T.T. Zhang: Lie symmetries, conservation laws and explicit solutions for time fractional Rosenau-Haynam equation. *Communications in Theoretical Physics* 67 (2) (2017) 157.

[25] S.S. Ray, S. Sahoo, S. Das: Formulation and solutions of fractional continuously variable order mass-spring-damper systems controlled by viscoelastic and viscous-viscoelastic dampers. *Advances in Mechanical Engineering* 8 (5) (2016) 1–17.

[26] H. Richard: *Fractional Calculus: an introduction for physicists*. World Scientific (2014).

[27] Y.A. Rossikhin, M.V. Shitikova: Analysis of dynamic behaviour of viscoelastic rods whose rheological models contain fractional derivatives of two different orders. *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik: Applied Mathematics and Mechanics* 81 (6) (2001) 363–376.

[28] Y.A. Rossikhin, M.V. Shitikova: Application of fractional calculus for dynamic problems of solid mechanics: novel trends and recent results. *Applied Mechanics Reviews* 63 (1) (2010) 010801(1–52).

[29] R. Sahadevan, T. Bakkyaraj: Invariant analysis of time fractional generalized Burgers and Korteweg-de Vries equations. *Journal of Mathematical Analysis and Applications* 393 (2) (2012) 341–347.

[30] S.G. Samko, A.A. Kilbas, O.I. Marichev: Fractional integrals and derivatives: theory and applications. Gordon and Breach, Switzerland (1993)

[31] N. Shang, B. Zheng: Exact solutions for three fractional partial differential equations by the (G'/G) method. *Int. J. Appl. Math* 43 (3) (2013) 114–119.

[32] K. Singla, R.K. Gupta: Space-time fractional nonlinear partial differential equations: symmetry analysis and conservation laws. *Nonlinear Dynamics* 89 (1) (2017) 321–331.

[33] B. Tang, Y. He, L. Wei, X. Zhang: A generalized fractional sub-equation method for fractional differential equations with variable coefficients. *Physics Letters A* 376 (38–39) (2012) 2588–2590.

[34] V.E. Tarasov: On chain rule for fractional derivatives. *Communications in Nonlinear Science and Numerical Simulation* 30 (1–3) (2016) 1–4.

[35] G.W. Wang, X.Q. Liu, Y.Y. Zhang: Lie symmetry analysis to the time fractional generalized fifth-order KdV equation. *Communications in Nonlinear Science and Numerical Simulation* 18 (9) (2013) 2321–2326.

[36] X.B. Wang, S.F. Tian, Ch.Y. Qin, T.T. Zhang: Lie symmetry analysis, conservation laws and exact solutions of the generalized time fractional Burgers equation. *EPL (Europhysics Letters)* 114 (2) (2016) 20003.

[37] X.B. Wang, S.F. Tian, Ch.Y. Qin, T.T. Zhang: Lie symmetry analysis, conservation laws and analytical solutions of a time-fractional generalized KdV-type equation. *Journal of Nonlinear Mathematical Physics* 24 (4) (2017) 516–530.

[38] X.B. Wang, S.F. Tian: Lie symmetry analysis, conservation laws and analytical solutions of the time-fractional thin-film equation. *Computational and Applied Mathematics* (2018) 1–13.

[39] A. Yıldırım: An algorithm for solving the fractional nonlinear Schrödinger equation by means of the homotopy perturbation method. *International Journal of Nonlinear Sciences and Numerical Simulation* 10 (4) (2009) 445–450.

[40] A. Yusuf, A.I. Aliyu, D. Baleanu: Lie symmetry analysis and explicit solutions for the time fractional generalized Burgers-Huxley equation. *Optical and Quantum Electronics* 50 (2) (2018) 94.

[41] S. Zhang: A generalized Exp-function method for fractional Riccati differential equations. *Communications In Fractional Calculus* 1 (2010) 48–51.

[42] Y. Zhang, J. Mei, X. Zhang: Symmetry properties and explicit solutions of some nonlinear differential and fractional equations. *Applied Mathematics and Computation* 337 (2018) 408–418.

[43] R.Z. Zhdanov: Conditional Lie-Backlund symmetry and reduction of evolution equations. *Journal of Physics A: Mathematical and General* 28 (13) (1995) 3841.

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