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SMOOTH IMPLICATIONS ON A FINITE CHAIN

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Mas et al. adapted the notion of smoothness, introduced by Godo and Sierra, and discussed two kinds of smooth implications (a discrete counterpart of continuous fuzzy implications) on a finite chain. This work is devoted to exploring the formal relations between smoothness and other six properties of implications on a finite chain. As a byproduct, several classes of smooth implications on a finite chain are characterized.

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1. INTRODUCTION

In fuzzy logic, one of the first problems that one has to face up is the choice, among all the fuzzy implications that are available, of the most suited one. Clearly, there is not a universal answer to this because the choice is largely context-dependent. However, the need of some logical properties (e. g. the left neutrality property, the identity principle, the order principle, the exchange principle, the strong negation principle, the law of contraposition, etc) may be as a criteria to help in making this choice. For another, these logical properties are used to characterize many interesting subclasses of fuzzy implications [1]. Using these logical properties to present characterizations of subclasses of fuzzy implications, the interdependencies of them are important. However, these logical properties are not independent, for instance, the order principle implies the identity principle. This raises a natural question: what are the formal relations among those logical properties? Shi et al. [6] answered this question.

On the other hand, the expert's reasonings are usually based on a set of linguistic terms or labels which can be represented as a finite chain. Therefore, the study of operators defined on a finite chain attracts researchers' attention. This approach is important because several disadvantages of the use of a numerical scale can be avoided [7]. It is known that the behavior of operators on the finite chain is really different from those of corresponding operators on $[0, 1]$, for example, there does not exist a counterpart of any strict t-norm (like the product) on a finite chain. The formal relations among properties of implications on finite chains remain unknown. This paper is devoted to investigating the formal relations between smoothness and other six properties of implications

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on a finite chain. As a byproduct, the characterizations of several classes of smooth implications on a finite chain are presented.

2. PRELIMINARIES

Let L_{n+1} be a finite chain denoted by $L_{n+1} = \{0 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1\}$. Such an L_{n+1} can be understood as a set of linguistic terms or "labels". For more information about t-norm, t-conorm and strong negation on L_{n+1}, see [5].

Definition 2.1. (Definitions 1 and 2 in Mas et al. [4])

(i) A function $f : L_{n+1} \to L_{n+1}$ is said to be *smooth* if it satisfies one of the following conditions:

- f is increasing and if $i \geq 1$, $f(x_i) = x_k$ and $f(x_{i-1}) = x_l$, then $k - l \leq 1$.
- f is decreasing and if $i \geq 1$, $f(x_i) = x_k$ and $f(x_{i-1}) = x_l$, then $l - k \leq 1$.

(ii) A binary operation $F : L^2_{n+1} \to L_{n+1}$ is said to be *smooth* if it is smooth in each variable.

Definition 2.2. A binary operator $I : L^2_{n+1} \to L_{n+1}$ is an implication, if

- I is decreasing in the first place and increasing in the second one. That is, if $x_i < x_j$, then $I(x_j, x_k) \leq I(x_i, x_k)$ and $I(x_k, x_i) \leq I(x_k, x_j)$ for any $x_k \in L_{n+1}$.
- $I(0, 0) = I(1, 1) = 1$ and $I(1, 0) = 0$.

There are many other required properties of implications depending on the context where they are going to be applied, the specially interesting ones for implications on L_{n+1} being:

1) the *left neutrality property*, if

$$I(1, x_i) = x_i, \quad \text{for any } x_i \in L_{n+1}; \tag{NP}$$

2) the *identity principle*, if

$$I(x_i, x_i) = 1, \quad \text{for any } x_i \in L_{n+1}; \tag{IP}$$

3) the *order principle*, if

$$I(x_i, x_j) = 1 \Leftrightarrow x_i \leq x_j, \quad \text{for any } x_i, x_j \in L_{n+1}; \tag{OP}$$

4) the *law of contraposition*, if

$$I(x_i, x_j) = I(x_{n+1-j}, x_{n+1-i}), \quad \text{for any } x_i, x_j \in L_{n+1}; \tag{CP}$$

5) the *strong negation principle*, if

$$I(x_i, 0) = x_{n+1-i} \quad \text{for any } x_i \in L_{n+1}; \tag{SN}$$

6) the *exchange principle*, if

$$I(x_i, I(x_j, x_l)) = I(x_j, I(x_i, x_l)), \quad \text{for any } x_i, x_j, x_l \in L_{n+1}. \tag{EP}$$

The dependencies among the properties above are similar to those for the $[0,1]$-case, such as,

$$\text{(EP)} \wedge \text{(OP)} \Rightarrow \text{(NP)}; \quad \text{(EP)} \wedge \text{(SN)} \Rightarrow \text{(NP)}; \quad \text{(NP)} \wedge \text{(CP)} \Rightarrow \text{(SN)};$$

$$\text{(EP)} \wedge \text{(OP)} \wedge \text{(CP)} \Rightarrow \text{(SN)}; \quad \text{(OP)} \Rightarrow \text{(IP)}; \quad \text{(EP)} \wedge \text{(SN)} \Rightarrow \text{(CP)}$$

(see [1, 6]).

Given any t-conorm S on L_{n+1}, its S-implication I_S [3] is defined by $I_S(x_i, x_j) = S(x_{n+1-i}, x_j)$ for any $x_i, x_j \in L_{n+1}$. Mas et al. [3] pointed out that I_S is smooth if and only if S is smooth. The Łukasiewicz implication defined by

$$I_{\text{Ł}}(x_i, x_j) = x_{\min(n+1-i+j, n+1)} \quad \text{for any } x_i, x_j \in L_{n+1}$$

is S-implication fulfilling smoothness, (CP), (EP), (OP), (IP), (NP) and (SN) (see [3] for details).

3. MAIN RESULTS

The implications on L_{n+1} lead to the classical implication if $n = 0$, so we only consider the case $n > 0$. Firstly, we start with the case $n = 1$, which is really different from the other cases.

Example 3.1. There exist only two smooth implications I_1 and I_2 on L_2 as follows:

I_1	0	x_1	1
0	1	1	1
x_1	x_1	x_1	1
1	0	x_1	1

I_2	0	x_1	1
0	1	1	1
x_1	x_1	1	1
1	0	x_1	1

A routine calculation shows that I_2 satisfies all six properties of implications and I_1 fulfills all expect for (OP) and (IP).

Below, we consider the case $n > 1$. Before doing it, we present a useful lemma.

Lemma 3.2. Let $x_i, x_j, x_s, x_t \in L_{n+1}$, $i < j$ and $f : L_{n+1} \to L_{n+1}$ be smooth. Suppose $j - i = t - s$.

(i) If $f(x_i) = x_s$, $f(x_j) = x_t$ and f is increasing, then $f(x_l) = x_{l-i+s}$ for any $l \in [i, j]$.

(ii) If $f(x_i) = x_t$, $f(x_j) = x_s$ and f is decreasing, then $f(x_l) = x_{t+i-l}$ for any $l \in [i, j]$.

Proof. We only prove the part (i) because part (ii) can be dealt with similarly.

A proof, similar to that of Theorem 2 in [2], can show that f is increasing and smooth if and only if the following statement holds:

- if $x_i, x_m, x_j \in L_{n+1}$ such that $i < j$ and $f(x_i) \leq x_m \leq f(x_j)$, then there exists $n \in [i, j]$ such that $x_m = f(x_n)$.

The above result implies that f is an onto function on the finite set $[x_i, x_j]$ and again it is also one-to-one on $[x_i, x_j]$. From the increasingness of f, we obtain that $f(x_l) = x_{l-i+s}$ for any $l \in [i, j]$. □

Proposition 3.3. A smooth implication I on L_{n+1} satisfies (NP) and (SN).

Proof. Example 3.1 guarantees the case $n = 1$. Consider $n > 1$. By the definition of I, $I(0,0) = 1$ and $I(1,0) = 0$. Definition 2.2 and Lemma 3.2(ii) elucidates that $I(x_i, 0) = x_{n+1-i}$ for any $x_i \in L_{n+1}$, which says that I satisfies (SN). Once again from that I is an implication, it follows that $I(1,1) = 1$ and $I(1,0) = 0$ and by Definition 2.2 and Lemma 3.2(i) $I(1, x_i) = x_i$ for any $x_i \in L_{n+1}$. Consequently, I satisfies (NP). □

Remark 3.4. The above result is really different from that of $[0,1]$-case. For $[0,1]$-case, there exists a continuous fuzzy implication I on $[0,1]$ satisfying neither (NP) nor (SN), for example (see Proposition 6.10 in [6] for details),

$$\widehat{I}(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ \sqrt{1-(x-y)} & \text{if } x > y, \end{cases} \qquad x, y \in [0,1].$$

Unfortunately, the smoothness implies none of the rest as indicated in the following example.

Example 3.5. Consider $n > 1$. The smooth implication I_3 defined by

$$I_3(x_i, x_j) = \begin{cases} x_n & \text{if } i = 2, j = n-1, \\ 1 & \text{if } i = 1, j \in \{n-1, n\}, \\ x_{\max(n+1-i,j)} & \text{otherwise,} \end{cases}$$

satisfies none of (EP), (CP), (OP) and (IP).

Similar to the $[0,1]$-case, all six properties of implications do not imply the smoothness as shown below.

Example 3.6. Consider $n > 1$. The Fodor implication I_4 on L_{n+1} defined by

$$I_4(x_i, x_j) = \begin{cases} 1 & \text{if } i \leq j, \\ x_{\max(n+1-i,j)} & \text{otherwise,} \end{cases}$$

is not smooth, and satisfies (NP), (SN), (CP), (EP), (OP) and (IP).

Below, we study smooth implications with other desirable properties. We start with smooth implications with (EP).

Proposition 3.7. Consider $n > 1$. Then the following statements are equivalent:

(i) I is a smooth implication on L_{n+1} and satisfies (EP).

(ii) I is a S-implication derived from a smooth t-conorm.

(iii) There exists a subset J of L_{n+1}:

$$J = \{0 = x_{i_0} < x_{i_1} < \cdots < x_{i_{m-1}} < x_{i_m} = 1\},$$

such that I_S can be represented as

$$I(x_i, x_j) = \begin{cases} x_{\min(n+1-i+j-i_k, i_{k+1})} & \text{if there is } x_{i_k} \in J \text{ such that} \\ & x_{i_k} \le x_{n+1-i}, x_j \le x_{i_{k+1}}, \\ x_{\max(n+1-i,j)} & \text{otherwise.} \end{cases} \quad (1)$$

Proof. (i) $\Rightarrow$ (ii) The properties of (SN), (NP) and (CP) come from Proposition 3.3 and Proposition 9.1 in [6]. By Theorem 1 in [3], there is a t-conorm S on L_{n+1} such that $I = I_S$. The smoothness of S is guaranteed from Proposition 9 in [3].

(ii) $\Rightarrow$ (iii) It is guaranteed from Proposition 5 in [3].

(iii) $\Rightarrow$ (i) A simple calculation shows that I defined by (1) is a smooth implication fulfilling (EP). □

Remark 3.8. Let I be an implication defined by (1). Obviously, $I = I_{Ł}$ if $|J| = 2$ and I satisfies neither (OP) nor (IP) if $|J| > 2$.

By virtue of Proposition 9.1 in [6], Propositions 3.3 and 3.7 and Remark 3.8, we have the following corollary.

Corollary 3.9. Consider $n > 1$.

(i) A smooth implication I on L_{n+1} satisfying (EP) fulfills (NP), (SN) and (CP).

(ii) There exists a smooth implication I on L_{n+1} satisfying (EP) but neither (OP) nor (IP).

Remark 3.10. The result obtained from Proposition 3.7 and Corollary 3.9(ii) are similar to those of the $[0, 1]$-case (see Theorem 2.4.10 in [1] and Proposition 8.3 in [6] for the $[0, 1]$-case). However, the result obtained from Corollary 3.9(i) is different from those of $[0, 1]$-case, such as, $\widetilde{I}(x, y) = \min(1 - x^2 + y, 1)$ is a continuous fuzzy implication on $[0, 1]$ fulfilling (EP) but not (CP) (see Proposition 9.5 in [6] for details).

Next, we present a characterization of smooth implications on L_{n+1} with (IP).

Proposition 3.11. Consider $n > 1$. Then a smooth implication I on L_{n+1} satisfies either (IP) or (OP) if and only if $I = I_{Ł}$. In particular, a smooth implication on L_{n+1} satisfying either (OP) or (IP) fulfills (NP), (SN), (CP) and (EP).

Proof. Obviously, $I = I_{Ł}$ satisfies (IP), (OP), (NP), (SN), (CP) and (EP).

We only prove the case for (IP) because (OP) implies (IP). Suppose that a smooth implication I satisfies (IP). $I(x_i, x_i) = 1$ and $I(1, x_i) = x_i$ for any $x_i \in L_{n+1}$ by (IP) and Proposition 3.3. Consider an arbitrary fixed $i_0 \in L_{n+1}$. Lemma 3.2 elucidates $I(x_j, x_{i_0}) = x_{n+1-j+i_0}$ for any $x_j \in [x_{i_0}, 1]$. The monotonicity implies $I(x_j, x_{i_0}) = 1$ for any $j < i_0$ and consequently, $I(x_j, x_{i_0}) = x_{\min(n+1-j+i_0, n+1)}$ for any $x_j \in L_{n+1}$. $I = I_{Ł}$ is guaranteed from the arbitrary choice of i_0. □

Remark 3.12. The results obtained from Proposition 3.11 are different from those of $[0,1]$-case, such as, $\widehat{I}$ from Remark 3.4 is a continuous fuzzy implication on $[0,1]$ satisfying (OP) but neither (NP) nor (SN).

Unfortunately, there are no favorable characterization for smooth implications. Similar to the $[0,1]$-case, the smoothness and (CP) imply none of (EP), (IP) and (OP) as demonstrated in the following example.

Example 3.13. Consider $n > 1$. The smooth implication I_5 on L_{n+1}

$$I_5(x_i, x_j) = \begin{cases} x_n & \text{if } i = 2, j = n-1, \\ 1 & \text{if } i = 1, j = n, \\ x_{\max(n+1-i,j)} & \text{otherwise,} \end{cases}$$

satisfies (CP) but none of (EP), (IP) and (OP).

4. CONCLUSION

In this paper the formal relations between smoothness and the other properties of implications on a finite chain have been investigated. We have also presented the characterizations of several classes of smooth implications on a finite chain. From our results, we deduced that smooth implications on a finite chain is quite different from the behavior of continuous fuzzy implications on $[0,1]$. The results, obtained in this paper, will be beneficial to approximate reasoning based on finite families of linguistic terms, and their consequent applications in fields in which approximate reasoning is applied.

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