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A DIEUDONNÉ THEOREM FOR LATTICE GROUP-VALUED MEASURES

Giuseppina Barbieri

A version of Dieudonné theorem is proved for lattice group-valued modular measures on lattice ordered effect algebras. In this way we generalize some results proved in the real-valued case.

Keywords: effect algebra, Dieudonné theorem, modular measures, lattice group

Classification: 28A12, 28A33, 28B15

1. INTRODUCTION

In 1933 Nikodým proved the well-known Vitali-Hahn-Saks theorem, namely if a sequence of Borel measures converges pointwise to a map μ, then μ is a Borel measure. In 1951 Dieudonné proved the following more general theorem: "If a sequence of regular measures defined on Borel sets of a compact metrizable space converges on every open set, then it converges on every Borel sets. In this case, the sequence is uniformly regular". This theorem generalizes Nikodým's theorem if one substitutes the pointwise convergence on the Borel σ-algebra for the analogous condition on open sets provided a regularity assumption and a topological condition on the space are satisfied. In this note we furnish a general version of Dieudonné's theorem valid for lattice group-valued modular measures defined on lattice ordered effect algebras. For we use an abstract concept of convergence in lattice-group, namely (D)-convergence. Effect algebras (alias D-posets) have been independently introduced in 1994 by D. J. Foulis and M. K. Bennett in [4] and by F. Chovanec and F. Kôpka in [9] for modelling unsharp measurement in a quantum mechanical system. They are a generalization of many structures which arise in Quantum Physics [10] and in Mathematical Economics [8, 11], in particular they are a generalization of orthomodular posets and MV-algebras and therefore of Boolean algebras.

2. PRELIMINARIES

First we recall some basic facts on lattice groups.

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Definition 2.1. Let R be a lattice group.

R is said to be *Dedekind complete* if every nonempty subset of R, bounded from above, has a supremum in R. We recall that a nonempty subset S of R *bounded* if there exists an element $u \in R$ with $|x| \le u$ for every $x \in S$.

A sequence $(\sigma_p)_{p\in\mathbb{N}}$ in R is an (O)*-sequence* if and only if it is decreasing and $\wedge_{p\in\mathbb{N}}\sigma_p = 0$.

A bounded double sequence $(a_{t,l})_{t,l}$ in R is a (D)*-sequence* or a *regulator* if and only if $(a_{t,l})_l$ is an (O)-sequence for any $t \in \mathbb{N}$. R is *weakly σ-distributive* if and only if $\wedge_{\varphi\in\mathbb{N}^\mathbb{N}} \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} = 0$ for every (D)-sequence $(a_{t,l})_{t,l}$ in R. A sequence (x_n) in R is order convergent (briefly, (O)-convergent) to x if and only if there is a (O)-sequence (σ_p) in R such that for every $p \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ with $|x_n - x| \le \sigma_p$ whenever $n \ge n_0$ and we write $(O) - \lim x_n = x$.

A sequence (x_n) in R is (O)-Cauchy if and only if there is an (O)-sequence (τ_p) in R such that for every $p \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ with $|x_n - x_q| \le \tau_p$ whenever $n, q \ge n_0$. A sequence (x_n) in R is (D)-convergent to x if and only if there is a (D)-sequence $(a_{t,l})$ in R such that for every $\varphi \in \mathbb{N}^\mathbb{N}$ there is $n_0 \in \mathbb{N}$ with $|x_n - x| \le \bigvee_{t=1}^{\infty}(a_{t,\varphi(t)})$ whenever $n \ge n_0$ and we write $(D) - \lim x_n = x$.

A sequence (x_n) in R is (D)-Cauchy if and only if there is a (D)-sequence $(b_{t,l})_{t,l}$ in R such that for every $\varphi \in \mathbb{N}^\mathbb{N}$ there is $n_0 \in \mathbb{N}$ with $|x_n - x_q| \le \bigvee_{t=1}^{\infty}(b_{t,\varphi(t)})$ whenever $n, q \ge n_0$.

R is said to be (O)-complete (respectively, (D)-complete) if and only if every (O)-Cauchy (respectively (D)-Cauchy) sequence is (O)-convergent (respectively, (D)-convergent).

Remark 2.2. Every Dedekind complete lattice group is both (O)-complete and (D)-complete. Moreover, every (O)-convergent sequence is also (D)-convergent to the same limit, the converse is true if and only if R is weakly σ-distributive.

We recall the Fremlin lemma which is fundamental in what follows.

Lemma 2.3. ([13, Lemma 1C], [14, Theorem 3.2.3]) Let R be Dedekind complete and $(a^n_{t,l})_{t,l}$, $n \in \mathbb{N}$, be a sequence of regulators in R. Then for every $u \in R$, $u \ge 0$ there is a (D)-sequence $(a_{t,l})_{t,l}$ in R with

$$u \wedge (\sum_{n=1}^{q} \bigvee_{t=1}^{\infty} a^n_{t,\varphi(t+n)}) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

for every $q \in \mathbb{N}$ and $\varphi \in \mathbb{N}^\mathbb{N}$.

Now we define a D-lattice, or in other words a lattice ordered effect algebra.

From now on, let R be a Dedekind complete lattice group.

Definition 2.4. Let $(L, \le)$ be a poset with a smallest element 0 and a greatest element 1 and let $\ominus$ be a partial operation on L such that $b \ominus a$ is defined if and only if $a \le b$ and for all a, b, $c \in L$:

If $a \le b$ then $b \ominus a \le b$ and $b \ominus (b \ominus a) = a$.

If $a \leq b \leq c$ then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.
Then $(L, \leq, \ominus)$ is called a *difference poset (D-poset* for short), or a *difference lattice (D-lattice* for short) if L is a lattice.

One defines in L a partial operation $\oplus$ as follows:

$a \oplus b$ is defined and $a \oplus b = c$ if and only if $c \ominus b$ is defined and $c \ominus b = a$.

The operation $\oplus$ is well-defined by the cancellation law [10, page 13] ($a \leq b, c$ and $b \ominus a = c \ominus a$ implies $b = c$), and $(L, \oplus, 0, 1)$ is an effect algebra (see [10, Theorem 1.3.4]), that is the following conditions are satisfied for all $a, b, c \in L$:

If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$;

If $b \oplus c$ is defined and $a \oplus (b \oplus c)$ is defined, then $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$;

There exists a unique $a' \in E$ such that $a \oplus a'$ is defined and $a \oplus a' = 1$;

If $a \oplus 1$ is defined, then $a = 0$.

We say that a and b are orthogonal if $a \leq b'$ and we write $a \perp b$. Therefore $a \oplus b$ is defined if and only if $a \perp b$, and in this case $a \oplus b = (a' \ominus b)'$ by [10, Lemma 1.2.5].

Definition 2.5. A D-lattice L is called σ-complete if it is closed under countable suprema and infima.

Proposition 2.6. Assume that a, b, c are elements of an effect algebra L. Then

(i) If $a \leq b \leq c$, then $b \ominus a \leq c \ominus a$ and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$

(ii) $a \ominus a = 0$, $a \ominus 0 = a$ and $a \oplus 0 = a$

(iii) If $a \leq b$, then $b \ominus a = 0$ if and only if $b = a$

(iv) If $a \leq b$, then $b = a \oplus (b \ominus a)$

(v) If $a \perp b$, then $a \leq a \oplus b$ and $(a \oplus b) \ominus a = b$

(vi) If $a \leq b \leq c'$, then $a \oplus c \leq b \oplus c$

(vii) If $a \perp b$ and $a \perp c$, then $a \oplus b = a \oplus c$ if and only if $b = c$

(viii) If $a \leq b \leq c$, then $a \perp c \ominus b$ and $a \oplus (c \ominus b) = c \ominus (b \ominus a)$

(ix) If L is a D-lattice, $\leq c$ and $b \leq c$, then $c \ominus (a \wedge b) = (c \ominus a) \vee (c \ominus b)$

From now on, let L be a D-lattice.

Our source of information about D-lattices is the book by Dvurečenskij and Pulmannova [10].

Definition 2.7. A lattice group-valued map μ on L is called a *measure* if it satisfies

$$\mu(a \oplus b) = \mu(a) + \mu(b)$$

whenever a, b are orthogonal elements of L.

By [12] a *modular function* μ on a lattice is a map which satisfies

$$\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b).$$

Definition 2.8. A lattice group-valued map μ on L is called a *modular measure* it it is both a modular function and a measure.

The semivariation of μ is defined by

$$\tilde{\mu}(a) = \bigvee\{|\mu(b)| : b \in L, b \leq a\}$$

where $a \in L$.

For the rest of the paper, let $\mathcal{F}$, $\mathcal{G} \subseteq L$ be two lattices closed under the sum and $g \ominus f \in \mathcal{G}$ for each $f \in \mathcal{F}$ and $g \in \mathcal{G}$.

We make use of an abstract concept of regularity where $\mathcal{F}$ and $\mathcal{G}$ play the role of compact sets and open sets, respectively. (See Definition 3.3)

3. REGULARITY AND EXHAUSTIVITY

Definition 3.1. A lattice group-valued modular measure μ on L is called $\mathcal{G}$-exhaustive if for every orthogonal sequence (g_n) in $\mathcal{G}$ we have (D)-$\lim_n \mu(g_n) = 0$

Let (μ_i) be a sequence of lattice group-valued modular measures on L. It is called uniformly $\mathcal{G}$-exhaustive if for for every orthogonal sequence (g_n) in $\mathcal{G}$ we have (D)-$\lim_n \mu_i(g_n) = 0$ uniformly with respect to $i \in \mathbb{N}$; more precisely, there is a (D)-sequence $(a_{t,l})_{t,l}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $j \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ with $\tilde{\mu}_j(g_n) \leq \vee_{t=1}^{\infty} a_{t,\varphi(t)}$ for every $n \geq n_0$.

Proposition 3.2. A lattice group-valued measure μ on L is exhaustive if and only if every increasing sequence is a Cauchy sequence.

Proof. Let (a_n) be an orthogonal sequence in L. Set $b_n = \oplus_{i \leq n} a_i$. Then b_n is an increasing sequence, hence (D)-$\lim_n \mu(b_n) = 0$. In particular, there is a (D)-sequence $(a_{t,l})_{t,l}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $n_0 \in \mathbb{N}$ with $\mu(b_{n+1} \ominus b_n) \leq \vee_{t=1}^{\infty} a_{t,\varphi(t)}$ for every $n \geq n_0$. Then $b_{n+1} = b_n \oplus (b_{n+1} \ominus b_n)$. On the other hand, we have $b_{n+1} = b_n \oplus a_n$, therefore $a_n = b_{n+1} \ominus b_n$. Hence (D)-$\lim_n \mu(a_n) = 0$.

We now prove that every increasing sequence is Cauchy. By way of contradiction, let (a_n) be an increasing sequence in L and suppose that it is not Cauchy. Then we can inductively obtain subsequences (b_n) and (c_n) of (a_n) with $b_n \leq c_n \leq b_{n+1}$ and for every D-sequence $a_{t,l}$ there are $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$ with $\mu(c_n \ominus b_n) \not\leq \vee_{t=1}^{\infty} a_{t,\varphi(t)}$. Set $(d_1, d_2, \ldots d_n, \ldots) = (b_1, c_1, b_2, c_2, \ldots)$. Then (d_n) is not Cauchy, a contradiction. $\square$

Definition 3.3. A lattice group-valued modular measure μ on L is regular if it satisfies the following properties:

(i) For every $a \in L$ there exists a couple of sequences $g_n \downarrow$ in $\mathcal{G}$ and $f_n \uparrow$ in $\mathcal{F}$ with

$$f_n < f_{n+1} < a < g_{n+1} < g_n \text{ for every } n \in \mathbb{N} \tag{1}$$

and

$$(D) - \lim \tilde{\mu}(g_n \ominus f_n) = 0. \tag{2}$$

(ii) For every $b \in \mathcal{F}$ there exists a couple of sequences $g_n \downarrow$ in $\mathcal{G}$ and $f_n \downarrow$ in $\mathcal{F}$ with

$$b < f_{n+1} < g_n < f_n \text{ for every } n \in \mathbb{N} \tag{3}$$

and

$$(D) - \lim \tilde{\mu}(g_n \ominus b) = 0. \tag{4}$$

Let (μ_i) be a sequence of lattice group-valued modular measures. We say that it is uniformly regular if the same above holds for the map $\sup_i \tilde{\mu}_i$; more precisely, if it satisfies the following properties:

(i) For every $a \in L$ there exists a couple of sequences $g_n \downarrow$ in $\mathcal{G}$ and $f_n \uparrow$ in $\mathcal{F}$ with

$$f_n < f_{n+1} < a < g_{n+1} < g_n \text{ for every } n \in \mathbb{N}$$

and there is a (D)-sequence $(a_{t,l})_{t,l}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $j \in \mathbb{N}$

$$\text{there exists } n_0 \in \mathbb{N} \text{ with } \sup_i \tilde{\mu}_i(g_n \ominus f_n) \leq \vee_{t=1}^{\infty} a_{t,\varphi(t)} \text{ for every } n \geq n_0. \tag{5}$$

(ii) For every $b \in \mathcal{F}$ there exists a couple of sequences $g_n \downarrow$ in $\mathcal{G}$ and $f_n \downarrow$ in $\mathcal{F}$ with

$$b < f_{n+1} < g_n < f_n \text{ for every } n \in \mathbb{N}$$

and there is a (D)-sequence $(a_{t,l})_{t,l}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $j \in \mathbb{N}$

$$\text{there exists } n_0 \in \mathbb{N} \text{ with } \sup_i \tilde{\mu}_i(g_n \ominus b) \leq \vee_{t=1}^{\infty} a_{t,\varphi(t)} \text{ for every } n \geq n_0. \tag{6}$$

4. THE MAIN RESULTS

We make use of the following theorem by Boccuto and Dimitriou:

Theorem 4.1. [7, Theorem 3.3] Let E be a subset of a lattice L, satisfying property (E), and $m_j : L \to R$, be a sequence of equibounded finitely additive measures, whose restrictions on E are exhaustive on E. If the limit (D)-$\lim_j m_j(e)$ exists in R for every $e \in E$ with respect to a single regulator, then the m_js are uniformly exhaustive on E.

We recall that a lattice E of subsets is said to satisfy property (E) iff every disjoint sequence $(C_h)_h$ in E has a subsequence $(C_{h_r})_r$, such that E contains the σ-algebra generated by the sets $(C_{h_r})_r$.

Theorem 4.2. Suppose that $\mathcal{G}$ is closed under countable sums. Let (μ_i) be a sequence of equibounded pointwise convergent $\mathcal{G}$-exhaustive modular measures on L. Then (μ_i) is uniformly $\mathcal{G}$-exhaustive.

Proof. By way of contradiction there are strictly increasing sequence i_n in $\mathbb{N}$ and an orthogonal sequence (g_n) in $\mathcal{G}$ such that $(D)\text{-}\lim_n \mu_{i_n}(g_n) \neq 0$. Define $\nu_n(A) := \mu_{i_n}(\oplus_{h\in A} g_h)$. With the aid of [2, 2.5] one can check that it is a sequence of finitely additive measures on the power set of $\mathbb{N}$. We can apply [7, Theorem 3.3]. So these restrictions form a uniformly exhaustive sequence. This contradicts the assumptions and it completes the proof. □

Lemma 4.3. Suppose that (μ_i) is a sequence of equibounded regular uniformly $\mathcal{G}$-exhaustive modular measures on L. Let $b \in \mathcal{F}$ such that (3) and (4) hold for μ_i $(i \in \mathbb{N})$. Then (6) holds true.

Proof. Let $a \in L$ with $a \leq g_n \ominus b$.

CLAIM: $\mu_i(a) = (D)\text{-}\lim_m \mu_i(a \ominus (f_m \ominus b) \wedge a)$.

PROOF: For every $m > n$ we have

$$\mu_i(a) - \mu_i(a \ominus (f_m \ominus b) \wedge a) = \mu_i((f_m \ominus b) \wedge a) \leq \mu_i(g_{m-1} \ominus b) \leq \tilde{\mu}_i(g_{m-1} \ominus b).$$

It suffices to show that $(D)\text{-}\lim_n \sup_i \tilde{\lambda}_i(g_n \ominus b) = 0$, where λ_i is the restriction of μ_i to $\mathcal{G}$. By way of contradiction this limit is different from 0, then there exist $(e_{t,l})_{t,l}$ a (D)-sequence and $\varphi \in \mathbb{N}^{\mathbb{N}}$ with the property that for every $p \in \mathbb{N}$, $i \in \mathbb{N}$, there exist $n > p$, $i \in \mathbb{N}$ and an element $a \in \mathcal{G}$ such that $a \leq g_n \ominus b$ and $\mu_i(a) \not\leq \bigvee_{t=1}^{\infty} e_{t,\varphi(t)}$ and so there exist $(e_{t,l})_{t,l}$ a (D)-sequence and $\varphi \in \mathbb{N}^{\mathbb{N}}$ with the property that for every $p \in \mathbb{N}$, $i \in \mathbb{N}$, there exist $m > p$, $i \in \mathbb{N}$ $\mu_i(a \ominus (f_m \ominus b) \wedge a) \not\leq \bigvee_{t=1}^{\infty} e_{t,\varphi(t)}$.

Hence we can construct by induction four sequences (n_k), (i_k), (m_k) and (a_k) in $\mathcal{G}$ with $a_k \leq g_{n_k} \ominus b \leq g_{n_{k-1}} \ominus b$ and $\mu_i(a_k \ominus (f_{m_k} \ominus b) \wedge a_k)) \not\leq \bigvee_{t=1}^{\infty} e_{t,\varphi(t)}$. Since

$$g_{n_{k-1}} \ominus g_{m_k} = (g_{n_{k-1}} \ominus b) \ominus (g_{n_k} \ominus b) \geq (f_{m_k} \ominus b) \vee a_k) \ominus (f_{m_k} \ominus b).$$

We get

$$\mu_{i_k}(g_{n_{k-1}}) - \mu_{i_k}(g_{m_k}) \geq \mu_{i_k}((f_{m_k} \ominus b) \vee a_k) - \mu_{i_k}(f_{m_k} \ominus b)$$

$$= \mu_{i_k}(a_k \ominus ((f_{m_k} \ominus b) \wedge a_k)))$$

with $n_{k-1} < n_k < m_k$. As (g_n) is a monotone sequence which is not a Cauchy sequence, this contradicts the uniform $\mathcal{G}$-exhaustivity of (μ_i). □

Applying Lemma 4.3 for $\mathcal{F} = \mathcal{G} = L$ and $b = 0$ we have:

Corollary 4.4. Let (μ_i) be a sequence of equibounded uniformly exhaustive modular measures on L and let (a_n) be a decreasing sequence of elements of L such that (D)-$\lim_n \tilde{\mu}_i(a_n) = 0$ for every $i \in \mathbb{N}$. Then the limit is uniformly with respect to $i \in \mathbb{N}$.

Theorem 4.5. Let (μ_i) be a sequence of equibounded regular uniformly $\mathcal{G}$-exhaustive modular measures on L. Then it is uniformly exhaustive and uniformly regular.

Proof. We have to show that (μ_i) is uniformly regular. The last item defining the uniform regularity is fulfilled thanks to Lemma 4.3. For the first: Let $a \in L$ and f_n, g_n satisfying (1) and (2). To prove (5), apply Corollary 4.4 to $a_n = g_n \ominus f_n$.

Now we prove that (μ_i) is uniformly exhaustive. By way of contradiction there are a sequence (a_n) of orthogonal elements of L and a sequence i_n such that $(D)-\lim_n \mu_{i_n}(a_n) \neq 0$. Thanks to the regularity we may suppose that $a_n \in \mathcal{F}$. Put $b_n = a_1 \oplus \cdots \oplus a_n$.

CLAIM: There exists a couple of sequences $g_n \uparrow$ in $\mathcal{G}$ and $f_n \uparrow$ in $\mathcal{F}$ such that $b_n \leq g_n \leq f_n \leq g_{n+1}$ and

$$\bigvee_{i\in\mathbb{N}} \tilde{\mu}_i(f_n \ominus b_n) \leq \bigvee_{t=1}^{\infty} a^n_{t,\varphi(t+n)}$$

for each $n \in \mathbb{N}$.

PROOF of the CLAIM: Proceed by induction.

(a) The base case: By uniform regularity we can pick $g_1 \in \mathcal{G}$, $f_1 \in \mathcal{F}$ with $b_1 \leq g_1 \leq f_1$ and

$$\bigvee_{i\in\mathbb{N}} \tilde{\mu}_i(f_1 \ominus b_1) \leq \bigvee_{t=1}^{\infty} a^1_{t,\varphi(t+1)}.$$

(b) The inductive step: Again, by uniform regularity we can construct the $(h+1)$th step in such a way $g_{h+1} \in \mathcal{G}$, $f_{h+1} \in \mathcal{F}$ with $b_{h+1} \vee f_h \leq g_{h+1} \leq f_{h+1}$ and

$$\bigvee_{i\in\mathbb{N}} \tilde{\mu}_i(f_{h+1} \ominus (b_{h+1} \vee f_h)) \leq \bigvee_{t=1}^{\infty} a^{h+1}_{t,\varphi(t+h+1)}.$$

Thanks to Lemma 2.3 let $(a_{t,l})_{t,l}$ be with

$$u \wedge (\sum_{n=1}^{h} \bigvee_{t=1}^{\infty} a^n_{t,\varphi(t+n)}) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

for every $h \in \mathbb{N}$ and $\varphi \in \mathbb{N}^{\mathbb{N}}$. Hence, for each $h \in \mathbb{N}$, as $f_{h+1} \ominus b_{h+1} \leq (f_{h+1} \ominus (b_{h+1} \vee f_h)) \oplus ((b_{h+1} \vee f_h) \ominus b_{h+1})$ and $\tilde{\mu}_i((b_{h+1} \vee f_h) \ominus b_{h+1}) \leq \tilde{\mu}_i(f_h \ominus b_h)$ we get

$$\tilde{\mu}_i(f_{h+1} \ominus b_{h+1}) \leq \tilde{\mu}_i(f_{h+1} \ominus (b_{h+1} \vee f_h)) + \tilde{\mu}_i(f_h \ominus b_h) \leq \vee_{t=1}^{\infty} a_{t,\varphi(t)} + \vee_{t=1}^{\infty} a_{t,\varphi(t)}$$

for each $h \in \mathbb{N}$ and $\varphi \in \mathbb{N}^{\mathbb{N}}$.

Then we get $\tilde{\mu}_i(a_{n+1}) = \tilde{\mu}_i(b_{n+1} \ominus b_n) \leq \tilde{\mu}_i(g_{n+1} \ominus f_n) + \tilde{\mu}_i(f_n \ominus b_n)$ as

$$b_{n+1} \ominus b_n \leq (g_{n+1} \ominus f_n) \oplus (f_n \ominus b_n).$$

Since by regularity (D)-$\lim_n \tilde{\mu}_i(g_{n+1} \ominus f_n) = 0$, and by above (D)-$\lim_n \tilde{\mu}_i(f_n \ominus b_n) = 0$, we get a contradiction with the hypothesis by absurd. $\square$

Theorem 4.6. Let $\mathcal{G}$ be closed under countable sums. Let (μ_i) be a sequence of equibounded $\mathcal{G}$-exhaustive regular modular measures on L converging on every element of $\mathcal{G}$. Then the sequence is converging on every element of L and it is uniformly exhaustive. Therefore its limit is exhaustive and regular.

P r o o f. Thanks to 4.2 the sequence is uniformly $\mathcal{G}$-exhaustive.

We shall show that for each element $a \in L$, the sequence $(\mu_i(a))_{i\in\mathbb{N}}$ is a Cauchy sequence and then we can apply Theorem 4.5 to complete the proof. Let $f \in \mathcal{F}$, $g \in \mathcal{G}$ with $f \leq a \leq g$, then there exists a (D)-sequence $a_{t.l}$ such that $\sup \tilde{\mu}_i(g\ominus f) \leq \vee_{t=1}^{\infty} a_{t,\varphi(t)}$ for every $\varphi \in \mathbb{N}^{\mathbb{N}}$.

Now we shall use the following equality:

$$\mu_i(a) - \mu_j(a) = \mu_i(g) - \mu_j(g) + \mu_j(g \ominus a) - \mu_i(g \ominus a).$$

Since $\mu_i(g)$ is a Cauchy sequence there exist a (D)-sequence $(b_{t.l})_{t.l}$ and $n_0 \in \mathbb{N}$ such that $|\mu_i(g) - \mu_j(g)| \leq \vee_{t=1}^{\infty} b_{t,\varphi(t)}$ for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and every $i, j \geq n_0$. Hence $d_{t.l} = a_{t.l} + b_{t.l}$ is a (D)-sequence. Therefore there exists $n_0 \in \mathbb{N}$ such that $|\mu_i(a) - \mu_j(a)| \leq \vee_{t=1}^{\infty} d_{t,\varphi(t)}$ for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and every $i, j \geq n_0$, whence the thesis follows. □

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