Mathematica Bohemica

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Mathematica Bohemica, Vol. 144 (2019), No. 4, 373–392

Persistent URL: http://dml.cz/dmlcz/147934

BOUND SETS AND TWO-POINT BOUNDARY VALUE PROBLEMS FOR SECOND ORDER DIFFERENTIAL SYSTEMS

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Received February 7, 2019. Published online June 24, 2019.
Communicated by Jiří Šremr

Cordially dedicated to the memory of Štefan Schwabik

Abstract. The solvability of second order differential systems with the classical separated or periodic boundary conditions is considered. The proofs use special classes of curvature bound sets or bound sets together with the simplest version of the Leray-Schauder continuation theorem. The special cases where the bound set is a ball, a parallelotope or a bounded convex set are considered.

Keywords: two-point boundary value problem; curvature bound set; Leray-Schauder theorem; Bernstein-Hartman condition

MSC 2010: 34B15, 47H11

1. Introduction

In a recent paper (see [26]), Szymańska-Dębowska has introduced a generalization of the Poincaré-Miranda theorem (see e.g. [20]) to some class of set-valued mappings, and has combined it with a shooting argument to find sufficient conditions for the solvability of the mixed, Neumann and Dirichlet two-point boundary value problems for second order differential systems of the form

$$x'' = f(t, x, x'), \tag{1.1}$$

where $f\colon [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

She assumes that the components f_i of f satisfy the sign condition:

(S1) For each $i \in \{1, \ldots, n\}$, there exist $r_i > 0$ such that $u_i f_i(t, u, v) \geqslant 0$ when $(t, u, v) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^n$ and $|u_i| \geqslant r_i$.

and that f satisfies the linear growth condition:

DOI: 10.21136/MB.2019.0014-19

(S2) There exist continuous nonnegative functions a_j on $[0,1]$, $j = 1,2,3$ such that

$$|f(t,u,v)| \leqslant a_1(t)|u| + a_2(t)|v| + a_3(t),$$

for all $(t,u,v) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^n$.

For the Dirichlet problem, a supplementary condition is requested:

(S3) For each $i \in \{1,\dots,n\}$, there exist $r_i > 0$ such that $v_i f_i(t,u,v) > 0$ when $(t,u,v) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^n$ and $|v_i| \geqslant r_i$.

The first two results in [26] state that the *mixed boundary value problem*

$$x'' = f(t,x,x'), \quad x'(0) = 0 = x(1) \tag{1.2}$$

and the *Neumann boundary value problem*

$$x'' = f(t,x,x'), \quad x'(0) = 0 = x'(1) \tag{1.3}$$

have a solution when f satisfies conditions (S1) and (S2). The third result in [26] states that the *Dirichlet boundary value problem*

$$x'' = f(t,x,x'), \quad x(0) = 0 = x(1) \tag{1.4}$$

has a solution when f satisfies conditions (S2) and (S3).

Denote by $\langle \cdot | \cdot \rangle$ the usual inner product in $\mathbb{R}^n$, by $|\cdot|$ the corresponding Euclidean norm, and by DV and D^2V, respectively, the gradient vector and the Hessian matrix of a smooth real function V on $\mathbb{R}^n$.

In this paper, we replace (S1) by a more general geometrical condition, inspired by the concept of *autonomous curvature bound set for f*, introduced in [11] in the frame of periodic and Dirichlet problems.

($\mathbb{H}$1) There exists an open bounded neighborhood C of 0 in $\mathbb{R}^n$ having the property that $f\colon [0,1] \times \overline{C} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and, for each $u \in \partial C$, one can find a function $V(\cdot;u) \in C^2(\mathbb{R}^n,\mathbb{R})$ such that
- (i) $C \subset \{w \in \mathbb{R}^n\colon V(w;u) < 0\}$,
- (ii) $V(u;u) = 0$,
- (iii) $\langle DV(u;u) | u \rangle > 0$,
- (iv) $\langle D^2V(u;u)v | v \rangle \geqslant 0$ for all $v \in \mathbb{R}^n$,
- (v) $\langle D^2V(u;u)v | v \rangle + \langle DV(u;u) | f(t,u,v) \rangle \geqslant 0$ for all $t \in [0,1]$ and $v \in \mathbb{R}^n$ such that $\langle DV(u;u) | v \rangle = 0$.

The word 'curvature' mentioned above comes from the fact that, for $u \in \partial C$, i.e. such that $V(u;u) = 0$, with, say $C \subset \mathbb{R}^2$, $DV(u;u)$ is normal to the curve $V(u;u) = 0$ at u, hence the v considered in (v), orthogonal to the normal $DV(u;u)$, are tangent to the curve at u. Hence $\langle D^2V(u;u)v\,|\,v\rangle$ is proportional to the curvature of the curve $V(u;u)=0$ at u, given by the expression $|DV(u)|^{-3}\langle D^2V(u;u)T(u)\,|\,T(u)\rangle$, where $T(u)=(-\partial_2V(u;u),\partial_1V(u;u))$, see [29].

The condition (S1) corresponds to the choice of $C=\prod_{j=1}^{n}(-r_j,r_j)$ for some $r_j>0$, $j=1,\dots,n$, for which $\partial C=\bigcup_{i=1}^{n}(C_i^+\cup C^-)$, with

$$C_i^{\pm}:=\{u\in\overline{C}\colon\ u_i=\pm r_i\},\quad i=1,\dots,n.$$

The choice of

$$V(w;u)=\begin{cases} w_i-r_i & \text{if } u\in C_i^+,\\ -w_i-r_i & \text{if } u\in C_i^-\end{cases}\tag{1.5}$$

gives

$$DV(w;u)=\begin{cases} e_i & \text{if } u\in C_i^+,\\ -e_i & \text{if } u\in C_i^-,\end{cases}$$

where e_i is the ith element of the canonical basis of $\mathbb{R}^n$, $i=1,\dots,n$, and $D^2V(u;u)=0$. When u belongs to several $C_j^{\pm}$ one can choose any of the corresponding $V(\cdot;u)$. Condition ($\mathbb{H}$1)-(v) becomes

$$u_if_i(t,u,v)\geqslant 0\quad\text{when } |u_i|=r_i \text{ and } v_i=0,\ i=1,\dots,n,$$

which is less restrictive than condition (S1).

We replace (S2) by the following growth condition, first introduced by Hartman (see [12], [13]) for second order differential systems with Dirichlet conditions.

($\mathbb{H}$2) There exist $L\geqslant 0$ and $\gamma\in[0,1/R_C)$, with $R_C:=\max_{u\in\overline{C}}|u|$, such that

$$|f(t,u,v)|\leqslant\gamma|v|^2+L$$

for all $(t,u,v)\in[0,1]\times\overline{C}\times\mathbb{R}^n$.

As shown by Hartman in [12], [13], the differential inequality

$$|x''(t)|\leqslant\gamma|x'(t)|^2+L\quad\text{for all } t\in[0,1],\ \gamma R_C<1\tag{1.6}$$

implies the existence of a uniform bound for $|x'(t)|$ for the functions such that $|x(t)|\leqslant R_C$ for all $t\in[0,1]$. In Section 2, we give a much shorter new proof of this result (Lemma 2.1), valid for Hilbert space-valued functions $x(t)$ verifying a boundary condition containing all the ones considered in this paper. Before Hartman, Bass in [2] had introduced, in his study of the asymptotic behavior of the solutions of some second order systems, the stronger condition

$$|f(t,u,v)| \leqslant \beta|v| + L,$$

generalized, independently of Hartman, by Opial (see [23]) to

$$|f(t,u,v)| \leqslant \beta|v|^\alpha + L$$

with $\beta \geqslant 0$ and $\alpha \in [0,2)$. Slightly more general versions were obtained later by Schmitt-Thompson (see [25]) and Fabry (see [6]). In the scalar case ($n = 1$), Bernstein in [4] had introduced the Bernstein-Hartman condition (1.6) without the size restriction upon γ, and deduced the bound on $|x'(t)|$, a result improved by Nagumo's replacement of $\gamma v^2 + L$ by $h(|v|)$ with $\int^\infty s\,\mathrm{d}s/h(s) = \infty$ (see [22]). Hence, for $n = 1$, our results, although valid, are not optimal, and we refer to [5], [11] for a detailed study of the scalar case.

Condition (S2) implies Condition ($\mathbb{H}2$) for $\overline{C} = \prod\limits_{j=1}^{n} [-r_j, r_j]$, because, if (S2) holds, then, for all $(t,u,v) \in [0,1] \times \overline{C} \times \mathbb{R}^n$, one has, for any $\varepsilon > 0$,

$$\begin{aligned} |f(t,u,v)| &\leqslant \max_{[0,1]} a_1\sqrt{n}|r| + \max_{[0,1]} a_2|v| + \max_{[0,1]} a_3 \\ &\leqslant \max_{[0,1]} a_1\sqrt{n}|r| + \max_{[0,1]} a_3 + \frac{\max_{[0,1]} a_2^2}{2\varepsilon^2} + \frac{\varepsilon^2}{2}|v|^2, \end{aligned}$$

with $r = (r_1, \ldots, r_n)$.

Our first existence result (Theorem 3.1) shows that conditions ($\mathbb{H}1$) and ($\mathbb{H}2$) imply the existence of a solution x, such that $x(t) \in \overline{C}$ for all $t \in [0,1]$, for the problems (1.2), (1.3), (1.4), for the second mixed problem

$$x'' = f(t,x,x'), \quad x(0) = 0 = x'(1) \tag{1.7}$$

and for the periodic problem

$$x'' = f(t,x,x'), \quad x(0) - x(1) = 0 = x'(0) - x'(1). \tag{1.8}$$

This result was proved in [11] for Dirichlet and periodic boundary conditions under assumptions stronger than ($\mathbb{H}1$) and ($\mathbb{H}2$) and with a longer and more complicated proof. Notice that no condition of type (S3) is required for the Dirichlet conditions. Consequently, the special case of Theorem 3.1 for $C = (-r_1, r_1) \times \ldots \times (-r_n, r_n)$

(Corollary 3.2), improves in several directions the first two theorems of [26] mentioned above.

In the case where C is the open ball B_R of center 0 and radius $R > 0$ (Corollary 3.1), one can take $V(w; u) = \frac{1}{2}(|w|^2 - R^2)$ for all $u \in \partial B_R$, so that $DV(w; u) = w$, $D^2V(w; u)$ is the identity matrix, and ($\mathbb{H}$1)-(v) becomes

$$|v|^2 + \langle u \,|\, f(t, u, v)\rangle \geqslant 0 \quad \text{when } |u| = R \text{ and } \langle v \,|\, u\rangle = 0, \tag{1.9}$$

a condition introduced by Hartman for Dirichlet problem in [12]. Recent extensions of Hartman condition (1.9) in various other directions can be found in [1], [7], [8], [9], [10], [15], [27], [28], [30].

When C is an open bounded convex neighborhood of 0 (Corollary 3.3), one can take, for each $u \in \partial C$, $V(w; u) = \langle \nu(u) \,|\, w - u\rangle$, where $\nu(u)$ is an outer normal to ∂C at u. The corresponding condition ($\mathbb{H}$1)-(v)

$$\langle \nu(u) \,|\, f(t, u, v)\rangle \geqslant 0 \quad \text{when } u \in \partial C \text{ and } \langle v \,|\, \nu(u)\rangle = 0,$$

was first introduced by Bebernes in [3] (see also [11]) for periodic problems.

Finally, the following generalizations and variants of condition (S3), inspired by the concept of *bound set* introduced in [11], provide new existence results for the mixed boundary value problems. Let $\overline{\text{co}}\, C$ denote the convex closure of C.

($\mathbb{H}3^-$) (respectively, ($\mathbb{H}3^+$)) There exists an open, bounded, neighborhood C of 0 in $\mathbb{R}^n$ having the property that $f\colon [0,1] \times \overline{\text{co}}\, C \times \overline{C} \to \mathbb{R}^n$ is continuous and, for each $v \in \partial C$, one can find $V(\cdot; v) \in C^1(\mathbb{R}^n, \mathbb{R})$ such that
- (i) $C \subset \{w \in \mathbb{R}^n \colon V(w; v) < 0\}$,
- (ii) $V(v; v) = 0$,
- (iii) $\langle DV(v; v) | v\rangle > 0$,
- (iv) One of the followint two inequalities holds:

$$\langle DV(v; v) | f(t, u, v)\rangle \leqslant 0 \quad \text{when } (t, u, v) \in [0,1] \times \overline{\text{co}}\, C \times \partial C, \tag{1.10}$$

$$\text{(respectively, } \langle DV(v; v) | f(t, u, v)\rangle \geqslant 0 \text{ when } (t, u, v) \in [0,1] \times \overline{\text{co}}\, C \times \partial C). \tag{1.11}$$

Theorem 4.1 (respectively, Theorem 4.2) states that condition ($\mathbb{H}3^-$) (respectively, $\mathbb{H}3^+$) implies the existence of a solution to the problem (1.2) (respectively, (1.7)).

Again, conditions of the type (S3) correspond to the special case where C is a product of intervals (Corollaries 4.2 and 4.5), and the special cases where C is a ball or a boundet convex set are considered (Corollaries 4.1, 4.4, 4.3 and 4.6). In the convex case, the results are reminiscent to the ones obtained in [21] for nonlocal boundary value problems.

Theorems 3.1, 4.1, and 4.2 are proved by reducing the boundary value problems to suitable fixed point problems in suitable function spaces, to which the following simplest special case of the Leray-Schauder continuation theorem (see [18] and also [19], Corollary IV.7) is applied.

Proposition 1.1. *Let X be a real normed space, $\Omega \subset X$ an open bounded neighborhood of 0, and $F\colon \overline{\Omega} \to X$ a compact mapping. If $x \neq \lambda F(x)$ for every $(x, \lambda) \in \partial\Omega \times (0, 1)$, then F has at least one fixed point in $\overline{\Omega}$.*

Notice that, in contrast to [26], the same proof in Theorem 3.1 works for all considered boundary conditions and is technically simpler than the ones given in [26]. Furthermore, Theorems 4.2 and 4.4 cover situations which have not been considered in [26] for the mixed boundary conditions.

2. A Bernstein-Hartman type lemma

In order to obtain the C^1 a priori estimates requested by Proposition 1.1, we use the following lemma, a special case of a more general result of Hartman (see [12], [13]) for functions with values in $\mathbb{R}^n$. We give here a much shorter proof for functions with values in a real Hilbert space H with inner product $(\cdot|\cdot)$ and the corresponding norm $\|\cdot\|$.

Lemma 2.1. *Assume that $x \in C^2([0, T], H)$ is such that*

$$(x(T)|x'(T)) = (x(0)|x'(0)) \tag{2.1}$$

and satisfies the inequalities

$$\|x(t)\| \leqslant R, \tag{2.2}$$

and

$$\|x''(t)\| \leqslant \gamma\|x'(t)\|^2 + K \tag{2.3}$$

for all $t \in [0, T]$ and some $R > 0$, $K \geqslant 0$ and $\gamma \geqslant 0$ such that

$$\gamma R < 1. \tag{2.4}$$

Then there exists $M = M(R, \gamma, K, T)$ such that for all $t \in [0, T]$,

$$\|x'(t)\| \leqslant M. \tag{2.5}$$

P r o o f. For each $t \in [0,T]$ and each function $x \in C^2([0,T],H)$ verifying conditions (2.2), (2.3), one has

$$-(x(t)|x''(t)) \leqslant \|x(t)\|\|x''(t)\| \leqslant R\|x''(t)\| \leqslant \gamma R\|x'(t)\|^2 + KR,$$

and hence

$$(1-\gamma R)\|x'(t)\|^2 \leqslant (x(t)\,|\,x''(t)) + \|x'(t)\|^2 + KR = ((x(t)\,|\,x'(t)))' + KR.$$

Consequently, using (2.1) and (2.4),

$$\int_0^T \|x'(t)\|^2 \,\mathrm{d}t \leqslant \frac{KRT}{1-\gamma R} := M_0^2, \tag{2.6}$$

with $M_0 = M_0(R,\gamma,K,T)$. The mean value theorem for the integral of a continuous real function implies the existence of $\tau \in [0,T]$ such that

$$\|x'(\tau)\|^2 = \frac{1}{T}\int_0^T \|x'(t)\|^2 \,\mathrm{d}t \leqslant \frac{M_0^2}{T}. \tag{2.7}$$

Now, integrating (2.3) on $[0,T]$ and using (2.6), we obtain

$$\int_0^T \|x''(t)\| \,\mathrm{d}t \leqslant \gamma \int_0^T \|x'(t)\|^2 \,\mathrm{d}t + KT \leqslant \gamma M_0^2 + KT. \tag{2.8}$$

From (2.7) and (2.8) it follows then that, for all $t \in [0,T]$,

$$\begin{aligned}\|x'(t)\| &= \left\|x'(\tau) + \int_\tau^t x''(s)\,\mathrm{d}s\right\| \leqslant \|x'(\tau)\| + \left|\int_\tau^t \|x''(s)\|\,\mathrm{d}s\right| \\ &\leqslant \frac{M_0}{\sqrt{T}} + \int_0^T \|x''(s)\|\,\mathrm{d}s \leqslant \frac{M_0}{\sqrt{T}} + \gamma M_0^2 + KT,\end{aligned}$$

and the result follows with $M := \gamma M_0^2 + KT + M_0/\sqrt{T}$. □

R e m a r k 2.1. The condition (2.1) is satisfied for all boundary conditions $l_j(x) = 0$, $j = 1,\ldots,5$. Furthermore, for $n \geqslant 2$, the condition (2.4) is sharp, as shown by the example of the family of functions, introduced by Heinz (see [14]),

$$x\colon\ [0,2\pi] \to \mathbb{R}^2, \quad t \mapsto (\cos nt, \sin nt),\ n \in \mathbb{N},$$

for which, with $\langle\cdot|\cdot\rangle$ the usual inner product and $|\cdot|$ the Euclidean norm in $\mathbb{R}^2$,

$$|x(t)| = 1, \quad |x''(t)| = |x'(t)|^2 = n^2, \quad \langle x(0)\,|\,x'(0)\rangle = \langle x(2\pi)\,|\,x'(2\pi)\rangle,$$

so that that the conclusion of Lemma 2.1 does not hold for $\gamma R = 1$ and $T = 2\pi$, as $|x'(t)| = n$ can be arbitrarily large.

3. A first existence result

Denote by $C([0,1],\mathbb{R}^n)$ the space of all continuous functions with the uniform norm $\|x\|_0 := \max_{t\in[0,1]} |x(t)|$, and, for $k \geqslant 1$, by $C^k([0,1],\mathbb{R}^n)$ the space of k-times continuously differentiable functions with the norm $\max_{0\leqslant j\leqslant k} \|x^{(j)}\|_0$.

Let us introduce the following linear functionals $l_j\colon C^1([0,1],\mathbb{R}^n) \to \mathbb{R}^{2n}$, $j = 1,\dots,5$ associated to the classical linear two-point boundary conditions:

$$
\begin{aligned}
&l_1\colon x \mapsto (x'(0), x(1)) && \text{(first mixed)},\\
&l_2\colon x \mapsto (x(0), x'(1)) && \text{(second mixed)},\\
&l_3\colon x \mapsto (x'(0), x'(1)) && \text{(Neumann)},\\
&l_4\colon x \mapsto (x(0), x(1)) && \text{(Dirichlet)},\\
&l_5\colon x \mapsto (x(0)-x(1), x'(0)-x'(1)) && \text{(periodic)}.
\end{aligned}
$$

The following classical results are just recalled for convenience. Elementary computations imply that, for each $j \in \{1,\dots,5\}$ and for each $z \in C([0,1],\mathbb{R}^n)$, the linear two-point boundary value problem

$$x'' - x = z(t), \quad l_j(x) = 0$$

has a unique solution which can be written in the form

$$x(t) = \int_0^1 G_j(t,s) z(s)\,\mathrm{d}s,$$

where the Green functions $G_j\colon [0,1]\times[0,1] \to \mathbb{R}$, are, respectively, given by

$$G_1(t,s) = \begin{cases} -\tanh 1 \cosh t \cosh s + \sinh t \cosh s & \text{if } 0 \leqslant s \leqslant t \leqslant 1,\\ -\tanh 1 \cosh t \cosh s + \cosh t \sinh s & \text{if } 0 \leqslant t < s \leqslant 1,\end{cases}$$

$$G_2(t,s) = \begin{cases} \tanh 1 \sinh t \sinh s - \cosh t \sinh s & \text{if } 0 \leqslant s \leqslant t \leqslant 1,\\ \tanh 1 \sinh t \sinh s - \sinh t \cosh s & \text{if } 0 \leqslant t < s \leqslant 1,\end{cases}$$

$$G_3(t,s) = \begin{cases} -\coth 1 \cosh t \cosh s + \sinh t \cosh s & \text{if } 0 \leqslant s \leqslant t \leqslant 1,\\ -\coth 1 \cosh t \cosh s + \cosh t \sinh s & \text{if } 0 \leqslant t < s \leqslant 1,\end{cases}$$

$$G_4(t,s) = \begin{cases} \coth 1 \sinh t \sinh s - \cosh t \sinh s & \text{if } 0 \leqslant s \leqslant t \leqslant 1,\\ \coth 1 \sinh t \sinh s - \sinh t \cosh s & \text{if } 0 \leqslant t < s \leqslant 1,\end{cases}$$

$$G_5(t,s) = \begin{cases} -\Big(\dfrac{\mathrm{e}^{t-s}}{2(\mathrm{e}-1)} + \dfrac{\mathrm{e}^{-(t-s)}}{2(1-\mathrm{e}^{-1})}\Big) & \text{if } 0 \leqslant s \leqslant t \leqslant 1,\\ -\Big(\dfrac{\mathrm{e}^{t-s}}{2(1-\mathrm{e}^{-1})} + \dfrac{\mathrm{e}^{-(t-s)}}{2(\mathrm{e}-1)}\Big) & \text{if } 0 \leqslant t < s \leqslant 1.\end{cases}$$

Furthermore, the G_j are continuous and their first derivatives $\partial_t G_j(t,s)$ have a finite jump when $s=t$.

Consequently, for each $j \in \{1,\dots,5\}$, the nonlinear boundary value problem

$$x'' = f(t,x,x'), \quad l_j(x) = 0, \tag{3.1}$$

which can be written as

$$x'' - x = f(t,x,x') - x, \quad l_j(x) = 0,$$

is equivalent to the nonlinear integral equation

$$x(t) = \int_0^1 G_j(t,s)(f(s,x(s),s'(s)) - x(s))\,\mathrm{d}s := F_j(x)(t),$$

i.e. to the fixed point problem $x = F_j(x)$ in $C^1([0,1],\mathbb{R}^n)$. Using the Ascoli-Arzelà theorem, it is easy to show that each mapping $F_j\colon C^1([0,1],\mathbb{R}^n) \to C^1([0,1],\mathbb{R}^n)$, $j=1,\dots,5$ is compact on bounded sets.

We state and prove our first general existence result.

Theorem 3.1. *If $f\colon [0,1]\times\overline{C}\times\mathbb{R}^n \to \mathbb{R}^n$ satisfies the conditions* ($\mathbb{H}1$) *and* ($\mathbb{H}2$), *then, for each $j \in \{1,\dots,5\}$, the problem* (3.1) *has at least one solution such that $x(t) \in \overline{C}$ for all $t \in [0,1]$.*

Proof. Let $j \in \{1,\dots,5\}$. In order to apply Proposition 1.1 to F_j, we consider the family of fixed point problems

$$x = \lambda F_j(x), \quad \lambda \in (0,1),$$

or, equivalently, the family of boundary value problems

$$x'' - x = \lambda(f(t,x,x') - x), \quad l_j(x) = 0. \tag{3.2}$$

By assumption ($\mathbb{H}2$), for any $(t,u,v) \in [0,1]\times\overline{C}\times\mathbb{R}^n$ and $\lambda \in [0,1]$ we have

$$\begin{aligned}|(1-\lambda)u + \lambda f(t,u,v)| &\leqslant (1-\lambda)|u| + \lambda|f(t,u,v)| \leqslant \gamma|v|^2 + (1-\lambda)R_C + \lambda L\\ &\leqslant \gamma|v|^2 + \max\{R_C, L\}.\end{aligned}$$

Let M be associated to $\overline{C}$, γ, R_C and L by Lemma 2.1 with $R = R_C$, $K = \max\{R_C, L\}$, and let the open bounded neighborhood $\Omega \subset C^1([0,1],\mathbb{R}^n)$ of 0 be defined by

$$\Omega := \{x \in C^1([0,1],\mathbb{R}^n)\colon\ x(t) \in C,\ |x'(t)| < M+1\ \forall\, t \in [0,1]\}.$$

If $\lambda \in (0,1)$ and x is any possible solution of (3.2) such that $x \in \partial\Omega$, then $x(t) \in \overline{C}$ for all $t \in [0,1]$, which implies that $|x'(t)| \leqslant M < M+1$ for all $t \in [0,1]$, and $x(\tau) \in \partial C$ for some $\tau \in [0,1]$. Consequently, the function $t \mapsto V(x(t);x(\tau))$ achieves its maximum 0 on $[0,1]$ at τ. For $j=1$, using the boundary conditions, $\tau \in [0,1)$,

$$(V(x(t);x(\tau)))'_{t=\tau} = \langle DV(x(\tau);x(\tau))\,|\,x'(\tau)\rangle = 0, \tag{3.3}$$

and, using $(\mathbb{H}1)$,

$$\begin{aligned}
(3.4)\quad 0 &\geqslant (V(x(t);x(\tau)))''_{t=\tau} \\
&= \langle D^2V(x(\tau);x(\tau))x'(\tau)\,|\,x'(\tau)\rangle + \langle DV(x(\tau);x(\tau))\,|\,x''(\tau)\rangle \\
&= \langle D^2V(x(\tau);x(\tau))x'(\tau)\,|\,x'(\tau)\rangle \\
&\quad + \langle DV(x(\tau);x(\tau))\,|\,(1-\lambda)x(\tau) + \lambda f(\tau,x(\tau),x'(\tau))\rangle \\
&= (1-\lambda)(\langle D^2V(x(\tau);x(\tau))x'(\tau)\,|\,x'(\tau)\rangle + \langle DV(x(\tau);x(\tau))\,|\,x(\tau)\rangle) \\
&\quad + \lambda(\langle D^2V(x(\tau);x(\tau))x'(\tau)\,|\,x'(\tau)\rangle + \langle DV(x(\tau);x(\tau))\,|\,f(\tau,x(\tau),x'(\tau))\rangle) \\
&> 0,
\end{aligned}$$

a contradiction. For $j=2$, similarly, $\tau \in (0,1]$, (3.3) holds and (3.4) gives a contradiction. For $j=3$, $\tau \in [0,1]$, (3.3) holds and again (3.4) gives a contradiction. For $j=4$, $\tau \in (0,1)$, (3.3) holds and once more (3.4) gives a contradiction. Finally, for $j=5$, if the maximum of $V(x(t);x(\tau))$ is achieved at $\tau=0$, it is also achieved at $\tau=1$, so that

$$0 \geqslant (V(x(0);x(0)))' = (V(x(1);x(1)))' \geqslant 0.$$

Thus, both first derivatives vanish and $0 \geqslant (V(x(t);x(\tau)))''_{t=\tau}$ for $\tau = 0$ and 1, leading to the contradiction with (3.4).

It follows from Proposition 1.1 that, for each $j \in \{1,\dots,5\}$, F_j has a fixed point in $\overline{\Omega}$, and each problem (3.1) has a solution such that $x(t) \in \overline{C}$ for all $t \in [0,1]$. $\square$

If $C = B_R$, the open ball in $\mathbb{R}^n$ of center 0 and radius $R > 0$, one can take $V(w;u) = \frac{1}{2}(|w|^2 - R^2)$ for all $u \in \partial B_R$, and Theorem 3.1 implies the following result.

Corollary 3.1. *If there exists $R > 0$ such that $f\colon [0,1]\times\overline{B}_R\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous, satisfies condition $|v|^2 + \langle u\,|\,f(t,u,v)\rangle \geqslant 0$ when $(t,u,v) \in [0,1]\times\partial B_R\times\mathbb{R}^n$ and $\langle u\,|\,v\rangle = 0$, and if condition $(\mathbb{H}2)$ with $C = B_R$ holds, then, for each $j \in \{1,\dots,5\}$, the problem (3.1) has at least one solution such that $|x(t)| \leqslant R$ for all $t \in [0,1]$.*

The case of Dirichlet conditions goes back to [12] and the case of periodic conditions to [17] for f locally Lipschitzian with respect to u and v, and to [24] for f continuous, with proofs less direct and longer than the one of Theorem 3.1.

E x a m p l e 3.1. Let $A\colon [0,1] \to \mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$ be a symmetric and continuous matrix-valued function such that

$$\langle A(t)u\,|\,u\rangle \geqslant a|u|^2 \tag{3.5}$$

for some $a>0$ and all $(t,u)\in[0,1]\times\mathbb{R}^n$. Let $b\geqslant 0$ and let $h\colon [0,1]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ be continuous and such that

$$\begin{aligned} |h(t,u,v)| &\leqslant c \quad \text{for some } c\geqslant 0 \text{ and all } (t,u,v)\in[0,1]\times\mathbb{R}^n\times\mathbb{R}^n,\\ h(t,0,0) &\neq 0 \quad \text{for some } t\in[0,1]. \end{aligned} \tag{3.6}$$

Let us consider the differential system

$$x'' = A(t)x - b|x'|^2x + h(t,x,x'). \tag{3.7}$$

If $|u|=R$ and $v\in\mathbb{R}^n$, we have

$$\begin{aligned} |v|^2 + \langle u|A(t)u - b|v|^2u + h(t,u,v)\rangle &= (1-b\langle u|u\rangle)|v|^2 + \langle A(t)u|u\rangle + \langle u|h(t,u,v)\rangle\\ &\geqslant (1-b|u|^2)|v|^2 + a|u|^2 - c|u|\\ &= (1-bR^2)|v|^2 + R(aR-c) \geqslant 0 \end{aligned}$$

when $R=c/a$ and $bc^2\leqslant a^2$. On the other hand, when $|u|\leqslant c/a$ and $v\in\mathbb{R}^n$, we have

$$|A(t)u - b|v|^2u + h(t,u,v)| \leqslant \frac{bc}{a}|v|^2 + \max_{t\in[0,1]}|A(t)|\frac{c}{a} + c,$$

and hence the condition in Lemma 2.1 holds when

$$bc^2 < a^2. \tag{3.8}$$

Corollary 3.1 implies that the system (3.7) has at least one nontrivial solution such that $|x(t)|\leqslant c/a$ for all $t\in[0,1]$ and $l_i(x)=0$, $i\in\{1,\dots,5\}$, when conditions (3.5), (3.6), and (3.8) hold. This result does not follow from the theorems in [26].

If we take in Theorem 3.1

$$C = P := (-r_1,r_1)\times\ldots\times(-r_n,r_n)$$

and V given in (1.5), we obtain the following existence result.

Corollary 3.2. *If $f\colon [0,1]\times\overline{P}\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous, such that, for each $i\in\{1,\dots,n\}$,*

$$u_if_i(t,u,v)\geqslant 0$$

when $(t,u,v)\in[0,1]\times\overline{P}\times\mathbb{R}^n$, $|u_i|=r_i$, and $v_i=0$, and if f satisfies the condition ($\mathbb{H}2$) with $C=P$, then, for each $j\in\{1,\dots,5\}$, the problem (3.1) has at least one solution such that $|x_i(t)|\leqslant r_i$ for all $t\in[0,1]$ and all $i=1,\dots,n$.

E x a m p l e 3.2. Let us consider the differential system

$$x_i'' = (1 + a|x'|^2)b_i \sin x_i + h_i(t, x, x'), \quad i = 1, \ldots, n, \tag{3.9}$$

where $a \geqslant 0$, $b_i > 0$, $i = 1, \ldots, n$, and, with $P := (-\frac{1}{2}\pi, \frac{1}{2}\pi) \times \ldots \times (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, $h_i\colon [0,1] \times \overline{P} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and such that

$$-b_i \leqslant h_i(t, u, v) \leqslant b_i \quad \forall\, (t, u, v) \in [0,1] \times \overline{P} \times \mathbb{R}^n\colon \\ |u_i| = \tfrac{1}{2}\pi,\ v_i = 0,\ 1 \leqslant i \leqslant n. \tag{3.10}$$

For $u_i = \frac{1}{2}\pi$ and $v_i = 0$, we have

$$\frac{\pi}{2}\big((1 + a|v|^2)b_i + h_i(t, u, v)\big) \geqslant \frac{\pi}{2}(b_i - b_i) = 0,$$

and, for $u_i = -\frac{1}{2}\pi$ and $v_i = 0$, we have

$$-\frac{\pi}{2}\big(-(1 + a|v|^2)b_i + h_i(t, u, v)\big) \geqslant \frac{\pi}{2}(b_i - b_i) = 0.$$

Furthermore, for $u \in \overline{P}$ and $y \in \mathbb{R}^n$, we have

$$\begin{aligned} |(1 + a|v|^2)b_i \sin u_i + h_i(t, u, v)| &\leqslant (1 + a|v|^2)b_i + b_i \\ &= ab_i|v|^2 + 2b_i, \quad i = 1, \ldots, n, \end{aligned}$$

and hence, letting

$$f_i(t, u, v) = (1 + a|v|^2)b_i \sin u_i + h_i(t, u, v), \quad i = 1, \ldots, n,$$

we get

$$|f(t, u, v)| \leqslant \sqrt{n}(|b|(1 + a|v|^2) + |b|) = \sqrt{n}a|v|^2 + 2\sqrt{n}|b|.$$

Using Corollary 3.2, the system (3.9) has at least one solution such that $-\frac{1}{2}\pi \leqslant x_i(t) \leqslant \frac{1}{2}\pi$, $i = 1, \ldots, n$ if conditions (3.10) and $a < 2/(n\pi)$ hold, using the fact that $R_P = \sqrt{n}(\frac{1}{2}\pi)$. This result does not follow from the theorems in [26].

Corollary 3.2 can also be seen as a special case of another consequence of Theorem 3.1, based upon the following result from convex analysis (see [16]).

Proposition 3.1. *If $C \subset \mathbb{R}^n$ is an open convex neighborhood of 0, then, for each $u \in \partial C$, there exists $\nu(u) \in \mathbb{R}^n \setminus \{0\}$ such that $\langle \nu(u) \,|\, u \rangle > 0$ and*

$$C \subset \{w \in \mathbb{R}^n\colon\ \langle \nu(u) \,|\, w - u \rangle < 0\}.$$

We call $\nu\colon \partial C \to \mathbb{R}^n \setminus \{0\}$ an outer normal field to ∂C.

Proposition 3.1 immediately implies that, if C is an open bounded convex neighborhood of 0 and ν is an outer normal field to ∂C, then, for each $u \in \partial C$, $V(w;u) = \langle \nu(u), w-u \rangle$ satisfies conditions (i)–(iv) in ($\mathbb{H}1$). Hence Theorem 3.1 has the following special case.

Corollary 3.3. *If there exists an open bounded convex neighborhood C of 0, with an outer normal field ν, such that $f\colon [0,1] \times \overline{C} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, satisfies the condition*

$$\langle \nu(u) \,|\, f(t,u,v) \rangle \geqslant 0 \tag{3.11}$$

when $t \in [0,1]$, $u \in \partial C$ and $\langle \nu(u) \,|\, v \rangle = 0$, and if condition ($\mathbb{H}2$) holds, then, for each $j \in \{1,\dots,5\}$, the problem (3.1) has at least one solution such that $x(t) \in \overline{C}$ for all $t \in [0,1]$.

In contrast to Corollary 3.1, Corollary 3.2 is also a special case of Corollary 3.3.

Remark 3.1. Going back to Example 3.1, if $C = B_R$, we can take $\nu(u) = u$ for each $u \in \partial B_R$. Thus condition (3.11) becomes

$$\begin{aligned} &\langle u | -b|v|^2 u + A(t)u + h(t,u,v) \rangle \\ &\quad = -|v|^2 b \langle u \,|\, u \rangle + \langle A(t)u \,|\, u \rangle + \langle h(t,u,v) \,|\, u \rangle. \end{aligned} \tag{3.12}$$

Consequently, if $b > 0$ is positive definite for some $t \in [0,1]$, there will exist no $R > 0$ such that (3.12) is nonnegative for $|u| = R$ and all $v \in \mathbb{R}^n$. This shows that the use of curvature bound sets with positive curvature like in Corollary 3.1 can lead to statements which escape results like Corollary 3.3 where the functions V associated to the curvature bound set are affine, and have curvature zero.

4. Further existence results for the mixed boundary conditions

We now show that, for $j \in \{1,2\}$, the existence of a solution for (3.1) can also be obtained under conditions related to Assumption (S3).

We need some preliminary results on first order linear and nonlinear problems. Let $z \in C([0,1],\mathbb{R}^n)$. The solutions of the initial value problem

$$y' + y = z(t), \quad y(0) = 0$$

are given by

$$y(t) = \int_0^t \mathrm{e}^{-(t-s)} z(s) \,\mathrm{d}s.$$

Consequently, if $f\colon [0,1]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous (possibly defined only on some subset), the global initial value problem

$$y'(t)=f\Bigl(t,\int_1^t y(s)\,\mathrm{d}s,y(t)\Bigr),\quad y(0)=0, \tag{4.1}$$

is equivalent to the nonlinear functional equation

$$y(t)=\int_0^t \mathrm{e}^{-(t-s)}\Bigl(f\Bigl(s,\int_1^s y(r)\,\mathrm{d}r,y(s)\Bigr)+y(s)\Bigr)\,\mathrm{d}s:=N_1(y)(t),$$

i.e. to the fixed point problem $y=N_1(y)$ in $C([0,1],\mathbb{R}^n)$. It is easy to show that $N_1\colon C([0,1],\mathbb{R}^n)\to C([0,1],\mathbb{R}^n)$ is compact on bounded sets. We apply Proposition 1.1 to N_1.

Lemma 4.1. *If there exists an open bounded neighborhood C of 0 in $\mathbb{R}^n$ such that $f\colon [0,1]\times\overline{\mathrm{co}}C\times\overline{C}\to\mathbb{R}^n$ satisfies condition* ($\mathbb{H}3^-$), *then the problem* (4.1) *has at least one solution y such that $y(t)\in\overline{C}$ for all $t\in[0,1]$.*

Proof. Define the open bounded neighborhood Ω of 0 in $C([0,1],\mathbb{R}^n)$ by

$$\Omega=\{y\in C([0,1],\mathbb{R}^n)\colon\ y(t)\in C\ \forall\,t\in[0,1]\}. \tag{4.2}$$

According to Proposition 1.1, we must show that, for each $\lambda\in(0,1)$, no possible fixed point of λN_1, i.e. no possible solution of the problem

$$y'(t)+y(t)=\lambda\Bigl(f\Bigl(t,\int_1^t y(s)\,\mathrm{d}s,y(t)\Bigr)+y(t)\Bigr),\quad y(0)=0, \tag{4.3}$$

belongs to $\partial\Omega$. If $\lambda\in(0,1)$ and $y(t)\in\partial\Omega$ is a possible solution to (4.3), then $y(t)\in\overline{C}$ for all $t\in[0,1]$ and $y(\tau)\in\partial C$ for some $\tau\in(0,1]$. Consequently, the function $t\mapsto V(y(t);y(\tau))$ achieves its maximum 0 at τ, so that, using condition ($\mathbb{H}3^-$),

$$\begin{aligned}0&\leqslant\langle DV(y(\tau);y(\tau))\,|\,y'(\tau)\rangle\\&=\Bigl\langle DV(y(\tau);y(\tau))\Bigm|-(1-\lambda)y(\tau)+\lambda f\Bigl(\tau,\int_1^\tau y(s)\,\mathrm{d}s,y(\tau)\Bigr)\Bigr\rangle\\&=-(1-\lambda)\langle DV(y(\tau);y(\tau))\,|\,y(\tau)\rangle\\&\quad+\lambda\Bigl\langle DV(y(\tau);y(\tau))\Bigm|f\Bigl(\tau,\int_1^\tau y(s)\,\mathrm{d}s,y(\tau)\Bigr)\Bigr\rangle<0,\end{aligned}$$

a contradiction. □

Lemma 4.1 easily implies the following existence theorem for a second order system with the first mixed boundary conditions

$$x'' = f(t, x, x'), \quad x'(0) = 0 = x(1). \tag{4.4}$$

Theorem 4.1. *If there exists an open bounded neighborhood C of 0 in $\mathbb{R}^n$ such that $f\colon [0,1] \times \overline{\mathrm{co}}C \times \overline{C} \to \mathbb{R}^n$ satisfies condition $(\mathbb{H}3^-)$, then the problem (4.4) has at least one solution x such that $x(t) \in \overline{\mathrm{co}}C$ and $x'(t) \in \overline{C}$ for all $t \in [0,1]$.*

Proof. If we set $y = x'$, so that, using the boundary condition $x(1) = 0$,

$$x(t) = \int_1^t y(s)\,\mathrm{d}s, \quad t \in [0,1],$$

it is clear that the problem (4.4) is equivalent to the problem (4.1). The conclusion follows from the application of Lemma 4.1 and the fact that $\int_1^t y(s)\,\mathrm{d}s$ belongs to the closed convex hull $\overline{\mathrm{co}}C$ of C. $\square$

The special cases where C is a ball or a parallelotope or a convex set follow immediately.

Corollary 4.1. *If there exists $R > 0$ such that $f\colon [0,1] \times \overline{B_R} \times \overline{B_R} \to \mathbb{R}^n$ is continuous and satisfies the condition*

$$\langle v \,|\, f(t,u,v) \rangle \leqslant 0$$

when $t \in [0,1]$, $|u| \leqslant R$ and $|v| = R$, then the problem (4.4) has at least one solution x such that $|x(t)| \leqslant R$ and $|x'(t)| \leqslant R$ for all $t \in [0,1]$.

Example 4.1. Let $A\colon [0,1] \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ be a symmetric matrix-valued function such that

$$\langle A(t)v \,|\, v \rangle \leqslant -a|v|^2 \quad \text{for some } a > 0 \text{ and all} \in (t,v) \in [0,1] \times \mathbb{R}^n, \tag{4.5}$$

and let $p > 1$ and $h\colon [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and such that

$$\langle v \,|\, h(t,u,v) \rangle \leqslant b|v|^p + c|v| \quad \text{for all } (t,u,v) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^n, \tag{4.6}$$

for some $b \in [0,a)$, $c \geqslant 0$. We consider the boundary value problem

$$x'' = |x'|^{p-2} A(t)x' + h(t,x,x'), \quad x'(0) = 0 = x(1). \tag{4.7}$$

If $R > 0$ and $|v| = R$, then

$$\langle v \,|\, |v|^{p-2} A(t)v \rangle + \langle v \,|\, h(t,u,v)\rangle \\ \leqslant -aR^p + bR^p + cR = -R((a-b)R^{p-1} - c) \leqslant 0$$

if $R^{p-1} \geqslant c/(a-b)$. Hence, using Corollary 4.3, the system (4.7) has at least one solution when conditions (4.5) and (4.6) hold.

Corollary 4.2. *If there exists $P = \prod_{i=1}^{n} (-r_i, r_i)$ such that $f\colon\ [0,1] \times \overline{P} \times \overline{P} \to \mathbb{R}^n$ is continuous and satisfies, for each $i \in \{1, \ldots, n\}$, the condition*

$$v_i f_i(t,u,v) \leqslant 0$$

when $(t,u,v) \in [0,1] \times \overline{P} \times \overline{P}$ and $|v_i| = r_i$, then the problem (4.4) has at least one solution x such that $|x_i(t)| \leqslant r_i$ and $|x_i'(t)| \leqslant r_i$ for all $t \in [0,1]$ and all $i = 1, \ldots, n$.

Example 4.2. Let $a_1, \ldots, a_n, p > 0$ and let us consider the problem

$$x_i'' = -a_i |x_i'|^p \sin x_i' + h_i(t,x,x'), \quad x_i'(0) = 0 = x_i(1), \quad i = 1, \ldots, n, \tag{4.8}$$

where $h\colon\ [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and such that

$$-a_i \left(\tfrac{1}{2}\pi\right)^p \leqslant h_i(t,u,v) \leqslant a_i \left(\tfrac{1}{2}\pi\right)^p, \quad i = 1, \ldots, n$$

when $(t,u,v) \in [0,1] \times \overline{P} \times \overline{P}$ and $|v_i| = \frac{1}{2}\pi$, with

$$P := \left(-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi\right) \times \ldots \times \left(\tfrac{1}{2}\pi, \tfrac{1}{2}\pi\right).$$

Corollary 4.2 immediately implies that problem (4.8) has at least one solution such that $|x_i(t)| \leqslant \frac{1}{2}\pi$ and $|x_i'(t)| \leqslant \frac{1}{2}\pi$ for all $t \in [0,1]$.

Corollary 4.3. *If there exists an open bounded convex neighborhood C of 0, with an outer normal field ν, such that $f\colon\ [0,1] \times \overline{C} \times \overline{C} \to \mathbb{R}^n$ is continuous and satisfies the condition*

$$\langle \nu(v) | f(t,u,v) \rangle \leqslant 0$$

when $t \in [0,1], u \in \overline{C}$ and $v \in \partial C$, then the problem (4.4) has at least one solution x such that $x(t) \in \overline{C}$ and $x'(t) \in \overline{C}$ for all $t \in [0,1]$.

Similar results hold for a second order system with the second mixed boundary conditions

$$x'' = f(t, x, x'), \quad x(0) = 0 = x'(1). \tag{4.9}$$

We only emphasize the differences in the statement and in the proofs. For $z \in C([0,1], \mathbb{R}^n)$, the solutions of the terminal value problem

$$y' - y = z(t), \quad y(1) = 0$$

are given by

$$y(t) = \int_1^t \mathrm{e}^{(t-s)} z(s)\,\mathrm{d}s.$$

If $f\colon [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, the terminal value problem

$$y'(t) = f\Big(t, \int_0^t y(s)\,\mathrm{d}s, y(t)\Big), \quad y(1) = 0, \tag{4.10}$$

is equivalent to the nonlinear functional equation

$$y(t) = \int_1^t \mathrm{e}^{(t-s)} \Big(f\Big(s, \int_0^s y(r)\,\mathrm{d}r, y(s)\Big) - y(s)\Big)\,\mathrm{d}s := N_2(y)(t),$$

i.e. to the fixed point problem $y = N_2(y)$ in $C([0,1], \mathbb{R}^n)$, with $N_2\colon C([0,1], \mathbb{R}^n) \to C([0,1], \mathbb{R}^n)$ compact on bounded subsets.

Lemma 4.2. *If there exists an open bounded neighborhood C of 0 in $\mathbb{R}^n$ such that $f\colon [0,1] \times \overline{\mathrm{co}}C \times \overline{C} \to \mathbb{R}^n$ satisfies condition $(\mathbb{H}3^+)$, then the problem (4.10) has at least one solution y such that $y(t) \in \overline{C}$ for all $t \in [0,1]$.*

Proof. Let Ω be defined in (4.2), let $\lambda \in (0,1)$ and let $y(t) \in \partial\Omega$ be a possible fixed point of λN_2, i.e. a possible solution to

$$y'(t) - y(t) = \lambda\Big(f\Big(t, \int_0^t y(s)\,\mathrm{d}s, y(t)\Big) - y(t)\Big), \quad y(1) = 0.$$

Then there is $\tau \in [0,1)$ such that the function $t \mapsto \langle V(y(t); y(\tau))\rangle$ reaches its maximum 0 at τ, so that, using condition $(\mathbb{H}3^-)$,

$$\begin{aligned} 0 &\geqslant \langle DV(y(\tau); y(\tau)) \,|\, y'(\tau)\rangle \\ &= (1-\lambda)\langle DV(y(\tau); y(\tau)) \,|\, y(\tau)\rangle \\ &\quad + \lambda \Big\langle DV(y(\tau); y(\tau)) \,\Big|\, f\Big(\tau, \int_0^\tau y(s)\,\mathrm{d}s, y(\tau)\Big)\Big\rangle > 0, \end{aligned}$$

a contradiction. The result follows from Proposition 1.1. □

Theorem 4.2. *If there exists an open bounded neighborhood C of 0 in $\mathbb{R}^n$ such that $f\colon [0,1]\times\overline{\mathrm{co}}\,C\times\overline{C}\to\mathbb{R}^n$ satisfies condition $(\mathbb{H}3^+)$, then the problem (4.9) has at least one solution x such that $x(t)\in\overline{\mathrm{co}}\,C$ and $x'(t)\in\overline{C}$ for all $t\in[0,1]$.*

Proof. If we set $y=x'$, so that, using the boundary condition $x(0)=0$,

$$x(t)=\int_0^t y(s)\,\mathrm{d}s,\quad t\in[0,1],$$

problem (4.9) is equivalent to the terminal value problem (4.10). The result follows from Lemma 4.2. □

Corollary 4.4. *If there exists $R>0$ such that $f\colon [0,1]\times\overline{B_R}\times\overline{B_R}\to\mathbb{R}^n$ is continuous and satisfies the condition*

$$\langle v\,|\,f(t,u,v)\rangle\geqslant 0$$

when $t\in[0,1]$, $|u|\leqslant R$ and $|v|=R$, then the problem (4.9) has at least one solution x such that $|x(t)|\leqslant R$ and $|x'(t)|\leqslant R$ for all $t\in[0,1]$.

Corollary 4.5. *If there exists $P=\prod\limits_{i=1}^{n}(-r_i,r_i)$ such that $f\colon [0,1]\times\overline{P}\times\overline{P}\to\mathbb{R}^n$ is continuous and satisfies, for each $i\in\{1,\dots,n\}$, the condition*

$$v_i f_i(t,u,v)\geqslant 0$$

when $(t,u,v)\in[0,1]\times\overline{P}\times\overline{P}$ and $|v_i|=r_i$, then the problem (4.9) has at least one solution x such that $|x_i(t)|\leqslant r_i$ and $|x_i'(t)|\leqslant r_i$ for all $t\in[0,1]$ and all $i\in\{1,\dots,n\}$.

Corollary 4.6. *If there exists an open bounded convex neighborhood C of 0, with an outer normal field ν, such that $f\colon [0,1]\times\overline{C}\times\overline{C}\to\mathbb{R}^n$ is continuous and satisfies the condition*

$$\langle\nu(v)\,|\,f(t,u,v)\rangle\geqslant 0$$

when $t\in[0,1]$, $u\in\overline{C}$ and $v\in\partial C$, then the problem (4.9) has at least one solution x such that $x(t)\in\overline{C}$ and $x'(t)\in\overline{C}$ for all $t\in[0,1]$.

Remark 4.1. Theorems 4.1 and 4.2 do not require a growth condition of type $(\mathbb{H}2)$. The reason is that Assumptions $(\mathbb{H}3^-)$ or $(\mathbb{H}3^+)$ provide directly an a priori estimate upon x', from which, using the boundary conditions, the a priori estimate on x follows.

R e m a r k 4.2. The conditions (1.10) and (1.11) have opposite signs in Theorems 4.4 and 4.9. One may lose the existence if one replaces (1.10) by (1.11) in Theorem 4.1 and (1.11) by (1.10) in Theorem 4.2. It suffices of course to show the result at the level of the Lemmas 4.1 and 4.2. Indeed, the problem

$$y' = 2(1 + y|y|), \quad y(0) = 0$$

verifies condition (1.11) for any $C = (-R, R)$, has for $t \geqslant 0$ the (unique) solution $y(t) = \tan 2t$ defined on the maximal interval $[0, \frac{1}{4}\pi) \subsetneq [0, 1)$, and hence no solution on $[0, 1]$. On the other hand, the terminal value problem on $[0, 1]$

$$y' = -2(1 + y|y|), \quad y(1) = 0$$

verifies condition (1.10) for any $C = (-R, R)$. By the change of variables $\tau = 1 - t$ and $z(\tau) = y(1 - \tau)$, it is reduced to the scalar initial value $[0, 1]$

$$z' = 2(1 + z|z|), \quad z(0) = 0$$

which has no solution in $[0, 1]$.

References

[1] *P. Amster, J. Haddad*: A Hartman-Nagumo type condition for a class of contractible domains. Topol. Methods Nonlinear Anal. *41* (2013), 287–304. zbl MR
[2] *R. W. Bass*: On non-linear repulsive forces. Contributions to the Theory of Nonlinear Oscillations, Volume IV. (S. Lefschetz, ed.). Annals of Mathematics Studies 41. Princeton University Press, Princeton, 1958, pp. 201–211. zbl MR doi
[3] *J. W. Bebernes*: A simple alternative problem for finding periodic solutions of second order ordinary differential systems. Proc. Am. Math. Soc. *42* (1974), 121–127. zbl MR doi
[4] *S. N. Bernstein*: Sur les équations du calcul des variations. Ann. Sci. Éc. Norm. Supér., Sér. III. *29* (1912), 431–485. (In French.) zbl MR doi
[5] *C. De Coster, P. Habets*: Two-Point Boundary Value Problems: Lower and Upper Solutions. Mathematics in Science and Engineering 205. Elsevier, Amsterdam, 2006. zbl MR doi
[6] *C. Fabry*: Nagumo conditions for systems of second-order differential equations. J. Math. Anal. Appl. *107* (1985), 132–143. zbl MR doi
[7] *N. Fewster-Young*: A singular Hartman inequality for existence of solutions to nonlinear systems of singular, second order boundary value problems. Int. J. Differ. Equ. Appl. *14* (2015), 195–228. zbl doi
[8] *N. Fewster-Young*: A priori bounds and existence results for singular boundary value problems. Electron. J. Qual. Theory Differ. Equ. (2016), 1–15. zbl MR doi
[9] *M. Frigon*: Boundary and periodic value problems for systems of nonlinear second order differential equations. Topol. Methods Nonlinear Anal. *1* (1993), 259–274. zbl MR doi
[10] *M. Frigon*: Boundary and periodic value problems for systems of differential equations under Bernstein-Nagumo growth condition. Differ. Integral Equ. *8* (1995), 1789–1804. zbl MR
[11] *R. E. Gaines, J. L. Mawhin*: Coincidence Degree, and Nonlinear Differential Equations. Lecture Notes in Mathematics 568. Springer, Cham, 1977. zbl MR doi

[12] *P. Hartman*: On boundary value problems for systems of ordinary, nonlinear, second order differential equations. Trans. Am. Math. Soc. *96* (1960), 493–509. zbl MR doi
[13] *P. Hartman*: Ordinary Differential Equations. John Wiley & Sons, New York, 1964. zbl MR
[14] *E. Heinz*: On certain nonlinear elliptic differential equations and univalent mappings. J. Anal. Math. *5* (1956/1957), 197–272. zbl MR doi
[15] *J. Henderson, Q. Sheng, C. C. Tisdell*: Constructive existence results for solutions to systems of boundary value problems via general Lyapunov methods. Differ. Equ. Appl. *9* (2017), 57–68. zbl MR doi
[16] *J.-B. Hiriart-Urruty, C. Lemaréchal*: Convex Analysis and Minimization Algorithms I. Fundamentals. Grundlehren der mathematischen Wissenschaften 305. Springer, Berlin, 1993. zbl MR doi
[17] *H.-W. Knobloch*: On the existence of periodic solutions for second order vector differential equations. J. Differ. Equations *9* (1971), 67–85. zbl MR doi
[18] *J. Leray, J. Schauder*: Topologie et équations fonctionnelles. Ann. Sci. Éc. Norm. Supér., Sér. III. *51* (1934), 45–78. (In French.) zbl MR doi
[19] *J. Mawhin*: Topological Degree Methods in Nonlinear Boundary Value Problems. CBMS Regional Conference Series in Mathematics 40. AMS, Providence, 1979. zbl MR doi
[20] *J. Mawhin*: Variations on Poincaré-Miranda's theorem. Adv. Nonlinear Stud. *13* (2013), 209–217. zbl MR doi
[21] *J. Mawhin, K. Szymańska-Dębowska*: Convex sets and second order systems with nonlocal boundary conditions at resonance. Proc. Am. Math. Soc. *145* (2017), 2023–2032. zbl MR doi
[22] *M. Nagumo*: Über die Differentialgleichung $y'' = f(t, y, y')$. Proc. Phys.-Math. Soc. Japan, III. Ser. *19* (1937), 861–866. (In German.) zbl
[23] *Z. Opial*: Sur la limitation des dérivées des solutions bornées d'un système d'équations différentielles du second ordre. Ann. Polon. Math. *10* (1961), 73–79. (In French.) zbl MR doi
[24] *N. Rouche, J. Mawhin*: Équations différentielles ordinaires. Tome I: Théorie générale. Tome II: Stabilité et solutions périodiques. Masson et Cie, Paris, 1973. (In French.) zbl MR
[25] *K. Schmitt, R. Thompson*: Boundary value problems for infinite systems of second-order differential equations. J. Differ. Equations *18* (1975), 277–295. zbl MR doi
[26] *K. Szymańska-Dębowska*: On a generalization of the Miranda Theorem and its application to boundary value problems. J. Differ. Equations *258* (2015), 2686–2700. zbl MR doi
[27] *V. Taddei*: Two-points boundary value problems for Carathéodory second order equations. Arch. Math., Brno *44* (2008), 93–103. zbl MR
[28] *V. Taddei, F. Zanolin*: Bound sets and two-point boundary value problems for second order differential equations. Georgian Math. J. *14* (2007), 385–402. zbl MR doi
[29] *J. A. Thorpe*: Elementary Topics in Differential Geometry. Undergraduate Texts in Mathematics. Springer, New York, 1979. zbl MR doi
[30] *C. C. Tisdell, L. H. Tan*: On vector boundary value problems without growth restrictions. JIPAM, J. Inequal. Pure Appl. Math. *6* (2005), Article No. 137, 10 pages. zbl MR

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