Xiao-Rui Wang; Gen-Qi Xu

The anti-disturbance property of a closed-loop system of 1-d wave equation with boundary control matched disturbance

Applications of Mathematics, Vol. 64 (2019), No. 6, 695–714

Persistent URL: http://dml.cz/dmlcz/147929

THE ANTI-DISTURBANCE PROPERTY OF A CLOSED-LOOP SYSTEM OF 1-D WAVE EQUATION WITH BOUNDARY CONTROL MATCHED DISTURBANCE

Xiao-Rui Wang, Gen-Qi Xu, Tianjin

Received March 17, 2018. Published online October 30, 2019.

Abstract. We study the anti-disturbance problem of a 1-d wave equation with boundary control matched disturbance. In earlier literature, the authors always designed the controller such as the sliding mode control and the active disturbance rejection control to stabilize the system. However, most of the corresponding closed-loop systems are boundedly stable. In this paper we show that the linear feedback control also has a property of anti-disturbance, even if the disturbance includes some information of the system. By choosing suitable parameters introduced in the proof, we can ensure the solution of the closed-loop system is bounded in an admissible range. As an application, we discuss the control problem of a nonlinear system. As a result, it is shown that the bounded estimation of the solution is suitable.

Keywords: boundary control; disturbance; wave equation; anti-disturbance

MSC 2010: 35B35, 93C05, 93C20

1. Introduction

Disturbance widely exists in the boundary control of systems described by partial differential equations (PDEs). In the past two decades, many researchers have paid more attention to anti-disturbance problems. That is, some effective control strategies are adopted to reject the unknown disturbances such that the systems are stable. By now researchers have developed many different approaches to deal with disturbance issues, for example, the internal model principle for output regulation ([9], [10], [12], [15]); the robust control for systems with uncertainties from both the

This project was partially supported by the National Natural Science Foundation in China (NSFC 61773277), and partially supported by NSF of Qinghai Province (2017-ZJ-908).

DOI: 10.21136/AM.2019.0070-18

internal and external disturbance; the adaptive control for systems with unknown parameters ([3], [13]) and so on. Recently, using sliding mode control (SMC) technology, the authors in [2] designed a distributed feedback controller to stabilize the one-dimensional wave equation with distributed disturbance, while in [6] they considered the boundary stabilization of a one-dimensional Schrödinger equation subject to boundary control matched disturbance. For a general discussion of interior disturbance of conservation systems we refer to [19]. Moreover, based on the Lyapunov function method, [5] and [11] considered the wave equation and the Euler-Bernoulli equation with boundary disturbance. And the stabilization of one-dimensional and multi-dimensional wave equations with boundary control matched disturbance is discussed in [7], [8], [20], [18] based on the active disturbance rejection control (ADRC).

However, these approaches have drawbacks when dealing with practical systems. For example, under the sliding mode control, the closed-loop systems become nonlinear. The solvability and stability analysis becomes more difficult, since the input and output operators are unbounded for boundary control of PDEs. Moreover, these approaches have an implicit assumption that the disturbance is independent of the system state. If the disturbance includes the state information of the system, the results obtained by these approaches may not hold. But the earlier works always assumed that the disturbance is independent of the state of the system. Based on the above observations, on the assumption that the disturbance includes the state information of the system, we propose a simple method to reject disturbance. Our control objective is to ensure the uniform boundedness of solution of the closed-loop system. Moreover, we show that the linear feedback control also has a property of anti-disturbance. This result in our paper can also be applied to the control problem of nonlinear systems.

For the sake of simplification, in this paper we consider the anti-disturbance problem for the one-dimensional wave equation with disturbance in the boundary control. The system is governed by the partial differential equation

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \quad x \in (0,1),\ t > 0, \\ w(0,t) = 0, \quad w_x(1,t) = u(t) + d(t), \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \end{cases} \tag{1.1}$$

where $u(t)$ is the control input, $d(t)$ is the unknown disturbance of the system, which is assumed to be a uniformly bounded function, i.e., $|d(t)| \leqslant M$ for all $t \geqslant 0$; here $d(t)$ might include information of the system. (w_0, w_1) is the initial state of the system.

If there is no disturbance, the system (1.1) becomes

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \quad x \in (0,1),\ t > 0, \\ w(0,t) = 0, \quad w_x(1,t) = u(t), \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x). \end{cases}$$

Under the feedback control law $u(t) = -kw_t(1,t)$ with $k > 0$, the system (1.1) with no disturbance can be stabilized exponentially ([1], [14]). Substituting $u(t) = -kw_t(1,t)$ into the system (1.1), we obtain the closed-loop system

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t) \quad x \in (0,1),\ t > 0, \\ w(0,t) = 0, \quad w_x(1,t) = -kw_t(1,t) + d(t), \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x). \end{cases} \tag{1.2}$$

In this paper we will discuss the properties of the feedback closed-loop system (1.2). In particular, we want to know whether or not we can choose the parameter k such that the bound of the solution to (1.2) is small for any initial data.

Before the discussion, we introduce the energy space $\mathcal{H} = H^1_E[0,1] \times L^2[0,1]$, where $H^1_E[0,1] = \{f \in H^1[0,1];\ f(0) = 0\}$, and the inner product

$$((f,g),(u,v)) = \int_0^1 [f'(x)\overline{u'(x)} + g(x)\overline{v(x)}]\,\mathrm{d}x.$$

Clearly, $\mathcal{H}$ is a Hilbert space.

Define an operator $\mathcal{A}$ in $\mathcal{H}$ by

$$\mathcal{A}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} g \\ f'' \end{pmatrix} \tag{1.3}$$

with the domain

$$D(\mathcal{A}) = \{(f,g) \in (H^2[0,1] \cap H^1_E[0,1]) \times H^1[0,1];\ f'(1) = -kg(1)\}.$$

And define an operator $\mathcal{B}$: $\mathbb{C} \to \mathcal{H}_{-1}$ by

$$\mathcal{B}u = (0, \delta(x-1))^\top u, \quad u \in \mathbb{C}. \tag{1.4}$$

With help of these operators, the system (1.2) can be written as an evolutionary equation in $\mathcal{H}$

$$\begin{cases} \dfrac{\mathrm{d}W(t)}{\mathrm{d}t} = \mathcal{A}W(t) + \mathcal{B}d(t), \\ W(0) = W_0. \end{cases} \tag{1.5}$$

where $W(t) = (w(x,t), w_t(x,t))^\top$, $W_0 = (w_0(x), w_1(x))^\top$.

The rest is organized as follows. In Section 2, we discuss the well-posedness of the closed loop system (1.5) by the admissible control theory. In Section 3, we prove the uniform boundedness of the solution to system (1.5). To estimate exactly the bound of the solution, we consider an optimization problem of the parameters k, δ, and η which are introduced in the proof; the numerical results show that by suitable choices of parameters, we can ensure that the solution of the closed-loop system satisfies $\|(w, w_t)\| \leqslant M(1+\varepsilon)$, the numerical results are given in Appendix. As an application of the result of the present paper, in Section 4 we consider the control problem of a nonlinear system. Finally, in Section 5, we give a conclusion.

2. The well-posedness of the system (1.5)

In this section we discuss the well-posedness of the system (1.5) by the admissible control theory. Let $\mathcal{A}$ and $\mathcal{B}$ be defined in (1.5). Then the solvability of system (1.5) is equivalent to the admissibility of $\mathcal{B}$ for $\mathcal{A}$.

First we recall the definition of the admissible control operator [17]. Consider the control system

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \\ z(0) = z_0 \end{cases}$$

in the state Hilbert space $\mathbb{X}$ and the control Hilbert space $\mathbb{U}$, where A generates a C_0 semigroup e^{At}, $t \geqslant 0$, and $B \in \mathcal{L}(\mathbb{U}, \mathbb{X}_{-1})$. Here $\mathbb{X}_{-1}$ is the completion space of $\mathbb{X}$ under the norm $\|x\|_{-1} = \|R(\beta, A)x\|_{\mathbb{X}}$ for some $\beta \in \varrho(A)$.

Definition 2.1. The operator B is called an L^2 admissible control operator for A if the following two conditions are satisfied.

(1) For any $\kappa > 0$,

$$\Phi_\kappa(u) := \int_0^\kappa \mathrm{e}^{A(t-s)} Bu(s)\,\mathrm{d}s \in \mathbb{X} \quad \forall\, u(s) \in L^2(0,\kappa).$$

(2) For some $\kappa_0 > 0$ there exists a $K_{\kappa_0} > 0$ such that

$$\|\Phi_{\kappa_0}(u)\|_{\mathbb{X}}^2 \leqslant K_{\kappa_0} \int_0^{\kappa_0} \|u(s)\|^2\,\mathrm{d}s.$$

By the duality principle, to prove the admissibility of $\mathcal{B}$ for $\mathcal{A}$ we only need to show that $\mathcal{B}^*$ is an L^2 admissible observation operator for $\mathcal{A}^*$ (see [17], [4], or [16]).

We have the following result.

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be defined as* (1.3) *and* (1.4). *Then $\mathcal{B}$ is an L^2 admissible control operator for $\mathcal{A}$.*

P r o o f. A straightforward computation yields

$$\mathcal{A}^* \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = - \begin{pmatrix} \psi \\ \varphi'' \end{pmatrix},$$

where $D(\mathcal{A}^*) = \{(\varphi, \psi) \in (H^2[0,1] \cap H^1_E[0,1]) \times H^1[0,1];\ \varphi'(1) = k\psi(1)\}$ and

$$\mathcal{B}^* \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \psi(1) \end{pmatrix}.$$

Then the dual system of system (1.2) or (1.5) is

$$\begin{cases} w^*_{tt}(x,t) = w^*_{xx}(x,t), \\ w^*(0,t) = 0, \\ w^*_x(1,t) = -k w^*_t(1,t), \\ y_0(t) = w^*_t(1,t). \end{cases}$$

Define the energy function

$$E(t) = \frac{1}{2} \int_0^1 [w^{*2}_x(x,t) + w^{*2}_t(x,t)] \,\mathrm{d}x.$$

Differentiating $E(t)$ with respect to t, we obtain

$$\begin{aligned} \frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \int_0^1 w^*_t(x,t) w^*_{tt}(x,t) \,\mathrm{d}x + \int_0^1 w^*_x(x,t) w^*_{xt}(x,t) \,\mathrm{d}x \\ &= w^*_x(x,t) w^*_t(x,t) \Big|_0^1 - \int_0^1 w^*_x(x,t) w^*_{xt}(x,t) \,\mathrm{d}x + \int_0^1 w^*_x(x,t) w^*_{xt}(x,t) \,\mathrm{d}x \\ &= w^*_x(1,t) w^*_t(1,t) = -k w^{*2}_t(1,t). \end{aligned}$$

Integrating from 0 to T with respect to t in the above equation, we have

$$E(t) - E(0) + k \int_0^T w^{*2}_t(1,t) \,\mathrm{d}t = 0,$$

and hence

$$\int_0^T w^{*2}_t(1,t) \,\mathrm{d}t = \frac{1}{k}(E(0) - E(t)) \leqslant \frac{1}{k} E(0). \tag{2.1}$$

On the other hand,

$$\mathcal{A}^{*-1} \begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix} = \begin{pmatrix} \int_0^x \int_\xi^1 \psi(y) \,\mathrm{d}y \,\mathrm{d}\xi + k\varphi(1)x \\ -\varphi(x) \end{pmatrix} \quad \forall\, (\varphi, \psi)^\top \in \mathcal{H},$$

and

$$\mathcal{B}^* \mathcal{A}^{*-1} \begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix} = -\varphi(1).$$

Therefore $\mathcal{B}^* \mathcal{A}^{*-1}$ is bounded from $\mathcal{H}$ to $\mathbb{C}$. This together with (2.1) shows that $\mathcal{B}^*$ is admissible for $\mathcal{A}^*$. Thus, $\mathcal{B}$ is admissible for $\mathcal{A}$. $\square$

Since $\mathcal{A}$ generates an exponentially stable C_0-semigroup $\mathrm{e}^{\mathcal{A}t}$ on $\mathcal{H}$, we can suppose that $\|\mathrm{e}^{\mathcal{A}t}\| \leqslant L\mathrm{e}^{-w_0 t}$, where $L > 0$ and $w_0 > 0$. Then the following result holds.

Theorem 2.2. *For any initial value $(w_0, w_1)^\top \in \mathcal{H}$ and $d(t) \in L^2_{\rm loc}(0,\infty)$, there exists a unique mild solution $(w, w_t)^\top \in \mathcal{H}$ for system (1.5) which can be written as*

$$\begin{pmatrix} w(x,t) \\ w_t(x,t) \end{pmatrix} = \mathrm{e}^{\mathcal{A}t} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} + \int_0^t \mathrm{e}^{\mathcal{A}(t-s)} \mathcal{B} d(s)\,\mathrm{d}s. \tag{2.2}$$

R e m a r k 2.1. If the disturbance $d(t)$ includes the state information of the system, i.e., $d(t) = d(t, w, w_t)$, then the system is a nonlinear one, the state of the closed-loop system is a solution of the integral equation (2.2).

3. Estimation of the uniform boundedness of the solution (1.5)

In this section, we discuss the boundedness of solution of (1.5), which means the anti-disturbance property of system (1.5) or (1.2). In particular, we shall estimate the bound of the solution when t is large enough.

We observe that if $d(t)$ includes the state information of the system, i.e., $d(t) = d(t, w, w_t)$, we always suppose that the solution exists uniquely. In this case, we can decompose the solution into two parts:

$$\begin{pmatrix} w^{(1)}(x,t) \\ w_t^{(1)}(x,t) \end{pmatrix} = \mathrm{e}^{\mathcal{A}t} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}, \quad \begin{pmatrix} w^{(2)}(x,t) \\ w_t^{(2)}(x,t) \end{pmatrix} = \int_0^t \mathrm{e}^{\mathcal{A}(t-s)} \mathcal{B} d(s, w, w_t)\,\mathrm{d}s.$$

Then

$$\begin{pmatrix} w(x,t) \\ w_t(x,t) \end{pmatrix} = \begin{pmatrix} w^{(1)}(x,t) \\ w_t^{(1)}(x,t) \end{pmatrix} + \begin{pmatrix} w^{(2)}(x,t) \\ w_t^{(2)}(x,t) \end{pmatrix}. \tag{3.1}$$

To estimate the solution of (2.2) or (1.2), we only need to estimate the second term on the right-hand side of (3.1). So without loss of generality we can assume that $d(t)$ does not depend on the state of the system.

It is easy to see that the second term of (3.1) is a solution of system (1.2) corresponding to the initial value $(0,0)^\top$. So we consider the system

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \quad x \in (0,1),\ t > 0, \\ w(0,t) = 0, \\ w_x(1,t) = -k w_t(1,t) + d(t), \\ w(x,0) = 0, \quad w_t(x,0) = 0, \end{cases} \tag{3.2}$$

whose solution is

$$\begin{pmatrix} w(x,t) \\ w_t(x,t) \end{pmatrix} = \int_0^t \mathrm{e}^{\mathcal{A}(t-s)} \mathcal{B} d(s)\,\mathrm{d}s. \tag{3.3}$$

The following theorem gives a bound for the solution to (3.2).

Theorem 3.1. *Let* $\sup\limits_{t\geqslant 0} |d(t)| = M < \infty$. *If* $0<\delta<1$ *and* $\eta>0$ *can be chosen such that* $1-\delta k \geqslant 0$ *and* $\delta k^2/2+\delta/2-k+(1-\delta k)/\eta \leqslant 0$ *hold, then the solution* (3.2) *satisfies the inequality*

$$\left\| \begin{pmatrix} w(x,t) \\ w_t(x,t) \end{pmatrix} \right\|_{\mathcal{H}} \leqslant M \sqrt{\frac{(1+\delta)[\delta+2(1-\delta k)\eta]}{(1-\delta)\delta}}.$$

Proof. Multiplying both sides of the equation in system (3.2) by $w_t(x,t)$ and integrating on $[0,1]\times[0,t]$, we have

$$\int_0^t\int_0^1 w_{tt}(x,t)w_t(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^t\int_0^1 w_{xx}(x,t)w_t(x,t)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.4}$$

Integration by parts yields

$$\begin{aligned}\int_0^t\int_0^1 w_{tt}(x,t)w_t(x,t)\,\mathrm{d}x\,\mathrm{d}t &= \int_0^t\int_0^1 w_t(x,t)\,\mathrm{d}x\,\mathrm{d}w_t(x,t)\\ &= \int_0^1 w_t^2(x,t)\,\mathrm{d}x\Big|_0^t - \int_0^t\int_0^1 w_t(x,t)w_{tt}(x,t)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

so

$$\int_0^t\int_0^1 w_{tt}(x,t)w_t(x,t)\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\int_0^1 w_t^2(x,t)\,\mathrm{d}x\Big|_0^t = \frac{1}{2}\int_0^1 w_t^2(x,t)\,\mathrm{d}x. \tag{3.5}$$

Similarly, we have

$$\begin{aligned}&\int_0^t\int_0^1 w_{xx}(x,t)w_t(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad= \int_0^t w_t(1,t)w_x(1,t)\,\mathrm{d}t - \frac{1}{2}\int_0^1 w_x^2(x,t)\,\mathrm{d}x + \frac{1}{2}\int_0^1 w_x^2(x,0)\,\mathrm{d}x\\ &\quad= \int_0^t w_t(1,t)w_x(1,t)\,\mathrm{d}t - \frac{1}{2}\int_0^1 w_x^2(x,t)\,\mathrm{d}x.\end{aligned} \tag{3.6}$$

From (3.4), (3.5), and (3.6), we obtain

$$\frac{1}{2}\int_0^1 w_x^2(x,t)\,\mathrm{d}x + \frac{1}{2}\int_0^1 w_t^2(x,t)\,\mathrm{d}x = \int_0^t w_t(1,t)w_x(1,t)\,\mathrm{d}t. \tag{3.7}$$

Multiplying the equation in system (3.2) by $xw_x(x,t)$ and integrating on $[0,1]\times[0,t]$, we obtain

$$\int_0^t\int_0^1 xw_x(x,t)w_{tt}(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^t\int_0^1 xw_x(x,t)w_{xx}(x,t)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.8}$$

The left-hand side of the above is

$$\begin{aligned}
&\int_0^t\int_0^1 xw_x(x,t)w_{tt}(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^1 xw_x(x,t)w_t(x,t)\,\mathrm{d}x\Big|_0^t - \int_0^t\int_0^1 xw_{xt}(x,t)w_t(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^1 xw_x(x,t)w_t(x,t)\,\mathrm{d}x - \frac{1}{2}\int_0^t w_t^2(1,t)\,\mathrm{d}t + \frac{1}{2}\int_0^t\int_0^1 w_t^2(x,t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.9}$$

Since the right-hand side is

$$\begin{aligned}
\int_0^t\int_0^1 xw_x(x,t)w_{xx}(x,t)\,\mathrm{d}x\,\mathrm{d}t = &\int_0^t xw_x^2(x,t)\,\mathrm{d}t\Big|_0^1 - \int_0^t\int_0^1 w_x^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^t\int_0^1 xw_{xx}(x,t)w_x(x,t)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

it can be easily shown that

$$\int_0^t\int_0^1 xw_x(x,t)w_{xx}(x,t)\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\int_0^t w_x^2(1,t)\,\mathrm{d}t - \frac{1}{2}\int_0^t\int_0^1 w_x^2(x,t)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.10}$$

From (3.8), (3.9), and (3.10), we have

$$\begin{aligned}
\frac{1}{2}\int_0^t\int_0^1 [w_t^2(x,t)+w_x^2(x,t)]\,\mathrm{d}x\,\mathrm{d}t + \int_0^1 xw_x(x,t)w_t(x,t)\,\mathrm{d}x \\
= \frac{1}{2}\int_0^t w_x^2(1,t)\,\mathrm{d}t + \frac{1}{2}\int_0^t w_t^2(1,t)\,\mathrm{d}t.
\end{aligned} \tag{3.11}$$

Let $\delta\in(0,1)$. Then (3.7) and (3.11) multiplied by δ yields

$$\begin{aligned}
&\frac{1}{2}\int_0^1 w_x^2(x,t)\,\mathrm{d}x + \frac{1}{2}\int_0^1 w_t^2(x,t)\,\mathrm{d}x + \delta\int_0^1 xw_x(x,t)w_t(x,t)\,\mathrm{d}x \\
&\quad + \frac{\delta}{2}\int_0^t\int_0^1 [w_t^2(x,t)+w_x^2(x,t)]\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^t w_t(1,t)w_x(1,t)\,\mathrm{d}t + \frac{\delta}{2}\int_0^t w_x^2(1,t)\,\mathrm{d}t + \frac{\delta}{2}\int_0^t w_t^2(1,t)\,\mathrm{d}t.
\end{aligned} \tag{3.12}$$

Let

$$G(t)=\frac{1}{2}\int_0^1 w_x^2(x,t)\,\mathrm{d}x+\frac{1}{2}\int_0^1 w_t^2(x,t)\,\mathrm{d}x+\delta\int_0^1 xw_x(x,t)w_t(x,t)\,\mathrm{d}x.$$

Note that

$$\left|\int_0^1 xw_x(x,t)w_t(x,t)\,\mathrm{d}x\right|\leqslant\frac{1}{2}\int_0^1 w_x^2(x,t)\,\mathrm{d}x+\frac{1}{2}\int_0^1 w_t^2(x,t)\,\mathrm{d}x.$$

We have

$$\begin{aligned}\frac{1-\delta}{2}\left[\int_0^1 w_x^2(x,t)\,\mathrm{d}x+\int_0^1 w_t^2(x,t)\,\mathrm{d}x\right]&\leqslant G(t)\\ &\leqslant\frac{1+\delta}{2}\left[\int_0^1 w_x^2(x,t)\,\mathrm{d}x+\int_0^1 w_t^2(x,t)\,\mathrm{d}x\right].\end{aligned}$$

From (3.12) we get

$$\begin{aligned}G(t)=&\frac{\delta}{2}\int_0^t w_x^2(1,t)\,\mathrm{d}t+\frac{\delta}{2}\int_0^t w_t^2(1,t)\,\mathrm{d}t+\int_0^t w_t(1,t)w_x(1,t)\,\mathrm{d}t\\ &-\frac{\delta}{2}\int_0^t\int_0^1[w_x^2(x,t)+w_t^2(x,t)]\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

hence,

$$\begin{aligned}\frac{\mathrm{d}G}{\mathrm{d}t}&=\frac{\delta}{2}w_x^2(1,t)+\frac{\delta}{2}w_t^2(1,t)+w_t(1,t)w_x(1,t)-\frac{\delta}{2}\int_0^1[w_t^2(x,t)+w_x^2(x,t)]\,\mathrm{d}x\\ &=\frac{\delta}{2}(k^2w_t^2(1,t)-2kw_t(1,t)d(t)+d^2(t))+\Big(\frac{\delta}{2}-k\Big)w_t^2(1,t)+w_t(1,t)d(t)\\ &\quad-\frac{\delta}{2}\int_0^1[w_t^2(x,t)+w_x^2(x,t)]\,\mathrm{d}x\\ &=\Big(\frac{\delta k^2}{2}+\frac{\delta}{2}-k\Big)w_t^2(1,t)+(1-\delta k)w_t(1,t)d(t)+\frac{\delta}{2}d^2(t)\\ &\quad-\frac{\delta}{2}\int_0^1[w_t^2(x,t)+w_x^2(x,t)]\,\mathrm{d}x.\end{aligned}$$

Since the condition $1-\delta k\geqslant 0$ holds, for $\eta>0$ we have

$$\begin{aligned}\frac{\mathrm{d}G}{\mathrm{d}t}&\leqslant\Big(\frac{\delta k^2}{2}+\frac{\delta}{2}-k\Big)w_t^2(1,t)+\frac{1-\delta k}{\eta}w_t^2(1,t)+(1-\delta k)\eta d^2(t)+\frac{\delta}{2}d^2(t)\\ &\quad-\frac{\delta}{2}\int_0^1[w_t^2(x,t)+w_x^2(x,t)]\,\mathrm{d}x\\ &=\Big(\frac{\delta k^2}{2}+\frac{\delta}{2}-k+\frac{1-\delta k}{\eta}\Big)w_t^2(1,t)+\Big[\frac{\delta}{2}+(1-\delta k)\eta\Big]d^2(t)\\ &\quad-\frac{\delta}{2}\int_0^1[w_t^2(x,t)+w_x^2(x,t)]\,\mathrm{d}x.\end{aligned}$$

Using the condition

$$\frac{\delta k^2}{2} + \frac{\delta}{2} - k + \frac{1 - \delta k}{\eta} \leqslant 0$$

and the inequality

$$\frac{1}{2} \int_0^1 [w_t^2(x,t) + w_x^2(x,t)] \, \mathrm{d}x \geqslant \frac{1}{1+\delta} G(t),$$

we have

$$\frac{\mathrm{d}G}{\mathrm{d}t} \leqslant \Big[\frac{\delta}{2} + (1 - \delta k)\eta\Big] d^2(t) - \frac{\delta}{1+\delta} G(t).$$

From the above we can get

$$G(t) \leqslant \int_0^t \mathrm{e}^{-\delta/(1+\delta)(t-s)} \Big[\frac{\delta}{2} + (1 - \delta k)\eta\Big] d^2(s) \, \mathrm{d}s \leqslant \frac{1+\delta}{\delta} \Big[\frac{\delta}{2} + (1 - \delta k)\eta\Big] M^2.$$

Therefore,

$$\frac{1}{2} \int_0^1 w_t^2(x,t) \, \mathrm{d}x + \frac{1}{2} \int_0^1 w_x^2(x,t) \, \mathrm{d}x \leqslant \frac{1+\delta}{(1-\delta)\delta} \Big[\frac{\delta}{2} + (1 - \delta k)\eta\Big] M^2.$$

So the solution of system (3.2) satisfies the inequality

$$\left\| \begin{pmatrix} w(x,t) \\ w_t(x,t) \end{pmatrix} \right\|_{\mathcal{H}}^2 = \int_0^1 w_x^2(x,t) \, \mathrm{d}x + \int_0^1 w_t^2(x,t) \, \mathrm{d}x \leqslant \frac{1+\delta}{(1-\delta)\delta} [\delta + 2(1 - \delta k)\eta] M^2,$$

or equivalently,

$$\left\| \begin{pmatrix} w(x,t) \\ w_t(x,t) \end{pmatrix} \right\|_{\mathcal{H}} \leqslant M \sqrt{\frac{(1+\delta)[\delta + 2(1 - \delta k)\eta]}{(1-\delta)\delta}}.$$

The proof is completed. □

From (3.3) and Theorem 3.1, we see that

$$\left\| \int_0^t \mathrm{e}^{\mathcal{A}(t-s)} \mathcal{B} d(s) \, \mathrm{d}s \right\|_{\mathcal{H}} \leqslant M \sqrt{\frac{(1+\delta)[\delta + 2(1 - \delta k)\eta]}{(1-\delta)\delta}} \quad \forall\, t \geqslant 0.$$

The following theorem gives an asymptotic estimation for the solution of (1.2) or (1.5).

Theorem 3.2. *For any initial value* $(w_0, w_1)^T \in \mathcal{H}$ *and* $\sup_{t \geqslant 0} |d(t)| \leqslant M$, *the solution* (2.2) *of system* (1.5) *is uniformly bounded and satisfies*

$$\limsup_{t \to \infty} \left\| \begin{pmatrix} w(x,t) \\ w_t(x,t) \end{pmatrix} \right\|_{\mathcal{H}} \leqslant M \sqrt{\frac{(1+\delta)[\delta + 2(1 - \delta k)\eta]}{(1-\delta)\delta}}.$$

P r o o f. By Theorem 2.1, the solution (2.2) of system (1.5) satisfies

$$\begin{aligned}\left\|\begin{pmatrix} w(x,t) \\ w_t(x,t)\end{pmatrix}\right\|_{\mathcal{H}} &\leqslant \left\|\mathrm{e}^{\mathcal{A}t}\begin{pmatrix} w_0(x) \\ w_1(x)\end{pmatrix}\right\|_{\mathcal{H}} + \left\|\int_0^t \mathrm{e}^{\mathcal{A}(t-s)}\mathcal{B}d(s)\,\mathrm{d}s\right\|_{\mathcal{H}} \\ &\leqslant L\mathrm{e}^{-\omega_0 t}\left\|\begin{pmatrix} w_0(x) \\ w_1(x)\end{pmatrix}\right\|_{\mathcal{H}} + \left\|\int_0^t \mathrm{e}^{\mathcal{A}(t-s)}\mathcal{B}d(s)\,\mathrm{d}s\right\|_{\mathcal{H}} \\ &\leqslant L\mathrm{e}^{-\omega_0 t}\left\|\begin{pmatrix} w_0(x) \\ w_1(x)\end{pmatrix}\right\|_{\mathcal{H}} + M\sqrt{\frac{(1+\delta)[\delta+2(1-\delta k)\eta]}{(1-\delta)\delta}},\end{aligned}$$

so it holds that

$$\limsup_{t\to\infty}\left\|\begin{pmatrix} w(x,t) \\ w_t(x,t)\end{pmatrix}\right\|_{\mathcal{H}} \leqslant M\sqrt{\frac{(1+\delta)[\delta+2(1-\delta k)\eta]}{(1-\delta)\delta}}.$$

The desired result follows. □

4. Optimization problem of parameters

In what follows, we consider an optimization problem of parameters δ, η, and k:

$$\min \frac{(1+\delta)[\delta+2(1-\delta k)\eta]}{(1-\delta)\delta},$$

where

$$\begin{cases} 0<\delta<1,\ \eta>0,\ k>0; \\ 1-\delta k\geqslant 0, \\ \dfrac{\delta k^2}{2}+\dfrac{\delta}{2}-k+\dfrac{1-\delta k}{\eta}\leqslant 0. \end{cases}$$

Case 1. Here we consider a special case that $\eta=\delta$.

In this case, the optimization problem becomes

$$\min \frac{(1+\delta)[1+2(1-\delta k)]}{1-\delta}$$

such that

$$\begin{cases} 0<\delta<1,\ k>0; \\ 1-\delta k\geqslant 0, \\ \dfrac{\delta k^2}{2}+\dfrac{\delta}{2}-k+\dfrac{1-\delta k}{\delta}<0. \end{cases}$$

Let

$$f(\delta)=\frac{(1+\delta)[1+2(1-\delta k)]}{1-\delta}.$$

The constraint condition becomes

$$\begin{cases} 0 < \delta < 1, \ k > 0; \\ \delta \leqslant \dfrac{1}{k}, \\ 0 < \delta \leqslant \sqrt{\dfrac{2}{1+k^2}}. \end{cases}$$

If $k > 1$, then $0 < \delta \leqslant 1/k$, and we only need to consider the optimization problem in the interval $0 < \delta \leqslant 1/k$. A simple calculation gives

$$f'(\delta) = \frac{2}{(1-\delta)^2} + \frac{4 - 4\delta k + 2\delta^2 k - 2k}{(1-\delta)^2} = \frac{6 - 4\delta k + 2\delta^2 k - 2k}{(1-\delta)^2}.$$

Let $f'(\delta) = 0$, that is $6 - 4\delta k + 2\delta^2 k - 2k = 0$. Then δ and k satisfy the equation

$$k\delta^2 - 2k\delta + (3-k) = 0.$$

We consider the solvability of this equation under the constraint condition $0 < \delta \leqslant 1/k$.

(i) If $4k^2 - 4k(3-k) \geqslant 0$ and $0 < 1 - \sqrt{2k^2 - 3k}/k \leqslant 1/k$, that is $(1+\sqrt{5})/2 \leqslant k \leqslant 3$, we have a solution

$$\delta = 1 - \frac{\sqrt{2k^2 - 3k}}{k} \quad \text{and} \quad \delta \in \left(0, \frac{1}{k}\right],$$

and hence

$$\begin{aligned} f(\delta)|_{\delta = 1 - \sqrt{2k^2-3k}/k} &= \frac{(1 + 1 - \sqrt{2k^2-3k}/k)[1 + 2(1 - (1 - \sqrt{2k^2-3k}/k)k)]}{1 - 1 + \sqrt{2k^2-3k}/k} \\ &= \frac{(2k - \sqrt{2k^2-3k})(3 - 2k + 2\sqrt{2k^2-3k})}{\sqrt{2k^2-3k}} \\ &= (6k-3) - 4\sqrt{2k^2-3k}. \end{aligned}$$

Since

$$f\left(\frac{1}{k}\right) = \frac{1 + 1/k}{1 - 1/k} = 1 + \frac{2}{k-1},$$

and $f(0) = 3$, when $(1+\sqrt{5})/2 \leqslant k \leqslant 3$, we have

$$\begin{cases} 3 \leqslant f(\delta)|_{\delta = 1 - \sqrt{2k^2-3k}/k} \leqslant 3\sqrt{5} - 4\sqrt{\dfrac{3-\sqrt{5}}{2}}, \\ 2 \leqslant f\left(\dfrac{1}{k}\right) \leqslant 5. \end{cases}$$

Therefore,

$$f_{\min}(\delta) = 2 \quad \text{if } k = 3.$$

(ii) If $4k^2 - 4k(3-k) < 0$, that is $1 < k < \frac{3}{2}$, the equation has no real solutions. In this case, we have $f'(\delta) > 0$. Clearly,

$$f\left(\frac{1}{k}\right) = \frac{1 + 1/k}{1 - 1/k} = 1 + \frac{2}{k-1},$$

and $f(0) = 3$ and $f(1/k) > 5$. Thus

$$f_{\min}(\delta) = f(0) = 3.$$

From the above discussion we see that for $\eta = \delta$, the optimization problem has a solution

$$f_{\min}(\delta) = f\left(\frac{1}{k}\right) = 2 \quad \text{if } k = 3 \text{ and } \delta = \frac{1}{k} = \frac{1}{3}.$$

Case 2. The general case.

To minimize the value of

$$F(\eta, \delta, k) = \frac{(1+\delta)[\delta + 2(1-\delta k)\eta]}{(1-\delta)\delta},$$

we consider different choices of η, δ, and k.

Tables 1–7 in Appendix are the numerical solutions to the minimizing problem and separately show that

$\triangleright$ Table 1 gives the numerical results of the minimum values about δ with different values of k for $\eta = 0.2$ and $\eta = 0.4$ respectively. From Table 1 we see that for the same $k \geqslant 2.5$, the minimum points δ and the minimum values of F for $\eta = 0.2, 0.4$ are the same; the minimum value of F decreases as k is increasing.

$\triangleright$ Table 2 gives the numerical results of the minimum values about δ with different values of k for $\eta = 0.5$ and $\eta = 0.8$ respectively. From Table 2 we see that for the same $k \geqslant 1.8$, the minimum points δ and the minimum values of F for $\eta = 0.5, 0.8$ are the same; the minimum value of F decreases as k is increasing.

$\triangleright$ Table 3 gives the numerical results of the minimum values about δ with different values of k for $\eta = 1.0$ and $\eta = 1.2$ respectively. From Table 3 we see that for the same $k \geqslant 1.6$, the minimum points δ and the minimum values of F for $\eta = 1.0, 1.2$ are the same; the minimum value of F decreases as k is increasing.

$\triangleright$ Table 4 gives the numerical results of the minimum values about δ with different values of k for $\eta = 1.2$ and $\eta = 1.6$ respectively. From Table 4 we see that for the same $k \geqslant 1.4$, the minimum points δ and the minimum values of F for $\eta = 1.4, 1.6$ are the same; the minimum value of F decreases as k is increasing.

Tables 1–4 show that *for* $\eta \in [0.2, 1.6]$, *when* $k \geqslant 2.5$, *the minimum points and the minimum values are the same; and when* $k = 30$, *the minimum value approximates* 1.07.

$\triangleright$ Tables 5–7 give the numerical results of the minimum value problem for $k \in [1.2, 2.5]$ as $\eta = 1.8, 2.0, 2.2, 2.4, 2.6$ and 2.8, respectively. From these tables we see that for $k = 1.2$, the minimum points and the minimum values become larger with η increasing until $\eta = 2.4$. However, for each $k \geqslant 1.4$ and $\eta \in [1.2, 2.8]$, the minimum points and the minimum values are the same; and the minimum value decreases with k increasing.

Thus, for this optimization problem, we give the following numerical results.

From the numerical results, we can see the following facts:

1) *When* $\eta > 0$ *is fixed, the minimum value of the optimization problem tends to* 1 *with* k *increasing. When* $k = 30$, *the minimum value approximates* 1.07.
2) *For all* $\eta \in [0.2, 2]$, *there exists* k_0 *such that the minimum value points and the minimum value are the same for each fixed* $k \geqslant k_0$.

The above fact shows that we can take $\eta = 1$ in the proof of Theorem 3.1. Since the minimum value of the optimization problem determines the asymptotic bound of the solution of system (1.5), the bound of the solution of (1.5) is relatively smaller when k is larger, but the decay of the solution becomes slower; if k is smaller, the bound of the solution of (1.5) becomes relatively larger, the decay of the solution becomes faster. Therefore, for any $\varepsilon > 0$ we can choose suitable parameters k and δ such that the asymptotic bound of the solution is less than $M(1 + \varepsilon)$.

5. Application

In this section we consider the nonlinear control problem

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \quad x \in (0,1),\ t > 0, \\ w(0,t) = 0, \quad w_x(1,t) - f(w) = u(t), \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \end{cases} \tag{5.1}$$

where

$$f(w) = \operatorname{sign}(w_t(1,t))\varepsilon_0 \exp\left\{1 - \int_0^1 [w_x^2(x,t) + w_t^2(x,t)]\,\mathrm{d}x\right\}$$

and ε_0 is a positive constant.

If $u(t) \equiv 0$, then the energy function

$$E(t) = \frac{1}{2}\int_0^1 [w_x^2(x,t) + w_t^2(x,t)]\,\mathrm{d}x$$

satisfies

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = w_x(1,t)w_t(1,t) = \varepsilon_0|w_t(1,t)|\exp\{1-2E(t)\} > 0,$$

so we need to control the system.

We regard the term $f(w)$ as a disturbance of the system, i.e., $d(t) = f(w)$. Then

$$|d(t)| \leqslant \varepsilon_0 \exp\left\{1 - \int_0^1 [w_x^2(x,t) + w_t^2(x,t)]\,\mathrm{d}x\right\} \leqslant \mathrm{e}\varepsilon_0 = M.$$

We take feedback control $u(t) = -kw_t(1,t)$, then the closed-loop system is

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \quad x \in (0,1),\ t > 0, \\ w(0,t) = 0, \quad w_x(1,t) = -kw_t(1,t) + f(w), \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x). \end{cases} \tag{5.2}$$

Applying Theorem 3.2, we have the following result.

Theorem 5.1. *The energy function of* (5.2) *has estimation*

$$\limsup_{t\to\infty} E(t) \leqslant \frac{\mathrm{e}^2\varepsilon_0^2}{2}\,\frac{(1+\delta)[\delta + 2(1-\delta k)\eta]}{(1-\delta)\delta}.$$

R e m a r k 5.1. When there is a disturbance, the system may have an equilibrium point. For example, system (5.2) has an equilibrium point $\overline{w}(x) = \gamma x$, where γ is the solution of the function equation

$$\gamma \mathrm{e}^{\gamma^2/2} = \beta\varepsilon_0\mathrm{e}, \quad \beta \in [-1,1].$$

Obviously, for any $\beta \in [-1,1]$ with $\beta \neq 0$, the function equation has a unique nonzero solution $\gamma \neq 0$. So the equation (5.2) has an equilibrium point. In this case, we have

$$E = \frac{\gamma^2}{2} = \frac{\beta^2\varepsilon_0^2}{2}\exp(2-\gamma^2).$$

So the solution of the closed-loop system is only bounded.

6. Conclusion

In this paper, we consider the anti-disturbance property of a closed-loop system of 1-d wave equation with boundary control matched disturbance. Under the linear feedback control law we discussed the asymptotic bound of the solution of the closed-loop system. The earlier works on the stabilization problem always assumed that the disturbance is independent of the state of the system. Different from the earlier works, in our research we remove this assumption. For $|d(t)| \leqslant M$ we obtained the asymptotic bound of the solution of the closed-loop system by choosing the suitable parameters k and δ:

$$\limsup_{t\to\infty} \|(w, w_t)\|_{\mathcal{H}} \leqslant M(1+\varepsilon).$$

Indeed, if we take $\limsup_{t\to\infty} |d(t)| = M$, we can get a similar result. As shown in Section 4, our result can be applied to asymptotic estimation of the solution to the nonlinear system, in which we regard the nonlinear part as a disturbance.

Appendix

η	k	δ	minimum value	η	k	δ	minimum value
0.2	1.2	0.7949791	8.957852	0.4	1.2	0.7303371	7.285389
	1.4	0.6521739	5.003333		1.4	0.5445545	4.575178
	1.6	0.5466238	3.724389		1.6	0.4672514	3.944281
	1.8	0.4651163	3.122609		1.8	0.5131670	3.477893
	2.0	0.4000000	2.800000		2.0	0.5000000	3.000000
	2.2	0.3827822	2.609961		2.2	0.4545455	2.666667
	2.5	0.4000000	2.333333		2.5	0.4000000	2.333333
	2.7	0.3703704	2.176471		2.7	0.3703704	2.176471
	2.9	0.3448276	2.052632		2.9	0.3448276	2.052632
	3	0.3333333	2.000000		3	0.3333333	2.000000
	3.2	0.3125000	1.909091		3.2	0.3125000	1.909091
	3.5	0.2857143	1.800000		3.5	0.2857143	1.800000
	4.5	0.2222222	1.571429		4.5	0.2222222	1.571429
	5	0.2000000	1.500000		5	0.2000000	1.500000
	5.5	0.1818182	1.444444		5.5	0.1818182	1.444444
	6	0.1666667	1.400000		6	0.1666667	1.400000
	7	0.1428571	1.333333		7	0.1428571	1.333333
	10	0.1000000	1.222222		10	0.1000000	1.222222
	20	0.5000000E−01	1.105263		20	0.5000000E−01	1.105263
	30	0.3333333E−01	1.068966		30	0.3333333E−01	1.068966
	300	0.3333333E−02	1.006689		300	0.3333333E−02	1.006689

Table 1. The minimum points and minimum values for $\eta = 0.2, 0.4$ and different values of k.

η	k	δ	minimum value	η	k	δ	minimum value
0.5	1.2	0.6779661	6.643421	0.8	1.2	0.5203037	6.830254
	1.4	0.4772256	4.790890		1.4	0.5985084	5.706625
	1.6	0.5278640	4.188854		1.6	0.6250000	4.333333
	1.8	0.5555556	3.500000		1.8	0.5555556	3.500000
	2.0	0.5000000	3.000000		2.0	0.5000000	3.000000
	2.2	0.4545455	2.666667		2.2	0.4545455	2.666667
	2.5	0.4000000	2.333333		2.5	0.4000000	2.333333
	2.7	0.3703704	2.176471		2.7	0.3703704	2.176471
	2.9	0.3448276	2.052632		2.9	0.3448276	2.052632
	3	0.3333333	2.000000		3	0.3333333	2.000000
	3.2	0.3125000	1.909091		3.2	0.3125000	1.909091
	3.5	0.2857143	1.800000		3.5	0.2857143	1.800000
	4.5	0.2222222	1.571429		4.5	0.2222222	1.571429
	5	0.2000000	1.500000		5	0.2000000	1.500000
	5.5	0.1818182	1.444444		5.5	0.1818182	1.444444
	6	0.1666667	1.400000		6	0.1666667	1.400000
	7	0.1428571	1.333333		7	0.1428571	1.333333
	10	0.1000000	1.222222		10	0.1000000	1.222222
	20	0.5000000E−01	1.105263		20	0.5000000E−01	1.105263
	30	0.3333333E−01	1.068966		30	0.3333333E−01	1.068966
	300	0.3333333E−02	1.006689		300	0.3333333E−02	1.006689

Table 2. The minimum points and minimum values for $\eta = 0.5, 0.8$ and different values of k.

η	k	δ	minimum value	η	k	δ	minimum value
1.0	1.2	0.5635083	7.698387	1.2	1.2	0.6030343	8.479747
	1.4	0.6909830	5.988854		1.4	0.7142857	6.000000
	1.6	0.6250000	4.333333		1.6	0.6250000	4.333333
	1.8	0.5555556	3.500000		1.8	0.5555556	3.500000
	2.0	0.5000000	3.000000		2.0	0.5000000	3.000000
	2.2	0.4545455	2.666667		2.2	0.4545455	2.666667
	2.5	0.4000000	2.333333		2.5	0.4000000	2.333333
	2.7	0.3703704	2.176471		2.7	0.3703704	2.176471
	2.9	0.3448276	2.052632		2.9	0.3448276	2.052632
	3	0.3333333	2.000000		3	0.3333333	2.000000
	3.2	0.3125000	1.909091		3.2	0.3125000	1.909091
	3.5	0.2857143	1.800000		3.5	0.2857143	1.800000
	4.5	0.2222222	1.571429		4.5	0.2222222	1.571429
	5	0.2000000	1.500000		5	0.2000000	1.500000
	5.5	0.1818182	1.444444		5.5	0.1818182	1.444444
	6	0.1666667	1.400000		6	0.1666667	1.400000
	7	0.1428571	1.333333		7	0.1428571	1.333333
	10	0.1000000	1.222222		10	0.1000000	1.222222
	20	0.5000000E−01	1.105263		20	0.5000000E−01	1.105263
	30	0.3333333E−01	1.068966		30	0.3333333E−01	1.068966
	300	0.3333333E−02	1.006689		300	0.3333333E−02	1.006689

Table 3. The minimum points and minimum values for $\eta = 1.0, 1.2$ and different values of k.

η	k	δ	minimum value	η	k	δ	minimum value
1.4	1.2	0.6407743	9.179427	1.6	1.2	0.6782688	9.795789
	1.4	0.7142857	6.000000		1.4	0.7142857	6.000000
	1.6	0.6250000	4.333333		1.6	0.6250000	4.333333
	1.8	0.5555556	3.500000		1.8	0.5555556	3.500000
	2.0	0.5000000	3.000000		2.0	0.5000000	3.000000
	2.2	0.4545455	2.666667		2.2	0.4545455	2.666667
	2.5	0.4000000	2.333333		2.5	0.4000000	2.333333
	2.7	0.3703704	2.176471		2.7	0.3703704	2.176471
	2.9	0.3448276	2.052632		2.9	0.3448276	2.052632
	3	0.3333333	2.000000		3	0.3333333	2.000000
	3.2	0.3125000	1.909091		3.2	0.3125000	1.909091
	3.5	0.2857143	1.800000		3.5	0.2857143	1.800000
	4.5	0.2222222	1.571429		4.5	0.2222222	1.571429
	5	0.2000000	1.500000		5	0.2000000	1.500000
	5.5	0.1818182	1.444444		5.5	0.1818182	1.444444
	6	0.1666667	1.400000		6	0.1666667	1.400000
	7	0.1428571	1.333333		7	0.1428571	1.333333
	10	0.1000000	1.222222		10	0.1000000	1.222222
	20	0.5000000E−01	1.105263		20	0.5000000E−01	1.105263
	30	0.3333333E−01	1.068966		30	0.3333333E−01	1.068966
	300	0.3333333E−02	1.006689		300	0.3333333E−02	1.006689

Table 4. The minimum points and minimum values for $\eta = 1.4, 1.6$ and different values of k.

η	k	δ	minimum value	η	k	δ	minimum value
1.8	1.2	0.7171516	10.31972	2.0	1.2	0.7597469	10.72982
	1.4	0.7142857	6.000000		1.4	0.7142857	6.000000
	1.6	0.6250000	4.333333		1.6	0.6250000	4.333333
	1.8	0.5555556	3.500000		1.8	0.5555556	3.500000
	2.0	0.5000000	3.000000		2.0	0.5000000	3.000000
	2.2	0.4545455	2.666667		2.2	0.4545455	2.666667
	2.5	0.4000000	2.333333		2.5	0.4000000	2.333333

Table 5. The minimum points and minimum values for $\eta = 1.8, 2.0$ and different values of k.

η	k	δ	minimum value	η	k	δ	minimum value
2.2	1.2	0.8106686	10.97524	2.4	1.2	0.8333333	11.00000
	1.4	0.7142857	6.000000		1.4	0.7142857	6.000000
	1.6	0.6250000	4.333333		1.6	0.6250000	4.333333
	1.8	0.5555556	3.500000		1.8	0.5555556	3.500000
	2.0	0.5000000	3.000000		2.0	0.5000000	3.000000
	2.2	0.4545455	2.666667		2.2	0.4545455	2.666667
	2.5	0.4000000	2.333333		2.5	0.4000000	2.333333

Table 6. The minimum points and minimum values for $\eta = 2.2, 2.4$ and different values of k.

η	k	δ	minimum value	η	k	δ	minimum value
2.6	1.2	0.8333333	11.00000	2.8	1.2	0.8333333	11.00000
	1.4	0.7142857	6.000000		1.4	0.7142857	6.000000
	1.6	0.6250000	4.333333		1.6	0.6250000	4.333333
	1.8	0.5555556	3.500000		1.8	0.5555556	3.500000
	2.0	0.5000000	3.000000		2.0	0.5000000	3.000000
	2.2	0.4545455	2.666667		2.2	0.4545455	2.666667
	2.5	0.4000000	2.333333		2.5	0.4000000	2.333333

Table 7. The minimum points and minimum values for $\eta = 2.6, 2.8$ and different values of k.

References

[1] *S. Cox, E. Zuazua*: The rate at which energy decays in a damped string. Commun. Partial Differ. Equations *19* (1994), 213–143. zbl MR doi

[2] *Q. H. Fu, G. Q. Xu*: Exponential stabilization of 1-d wave equation with distributed disturbance. WSEAS Trans. Math. *14* (2015), 192–201.

[3] *W. Guo, B.-Z. Guo, Z.-C. Shao*: Parameter estimation and stabilization for a wave equation with boundary output harmonic disturbance and non-collocated control. Int. J. Robust Nonlinear Control *21* (2011), 1297–1321. zbl MR doi

[4] *B.-Z. Guo, F.-F. Jin*: The active disturbance rejection and sliding mode control approach to the stabilization of the Euler-Bernoulli beam equation with boundary input disturbance. Automatica *49* (2013), 2911–2918. zbl MR doi

[5] *B.-Z. Guo, W. Kang*: The Lyapunov approach to boundary stabilization of an anti-stable one-dimensional wave equation with boundary disturbance. Int. J. Robust Nonlinear Control *24* (2014), 54–69. zbl MR doi

[6] *B.-Z. Guo, J.-J. Liu*: Sliding mode control and active disturbance rejection control to the stabilization of one-dimensional Schrödinger equation subject to boundary control matched disturbance. Int. J. Robust. Nonlinear Control *24* (2014), 2194–2212. zbl MR doi

[7] *B.-Z. Guo, J.-J. Liu, A. S. Al-Fhaid, M. Arshad Mahmood, A. M. M. Younas, A. Asiri*: The active disturbance rejection control approach to stabilisation of coupled heat and ODE system subject to boundary control matched disturbance. Int. J. Control *88* (2015), 1554–1564. zbl MR doi

[8] *B.-Z. Guo, H.-C. Zhou*: The active disturbance rejection control to stabilization for multi-dimensional wave equation with boundary control matched disturbance. IEEE Trans. Autom. Control *60* (2015), 143–157. zbl MR doi

[9] *E. Immonen, S. Pohjolainen*: Feedback and feedforward output regulation of bounded uniformly continuous signals for infinite-dimensional systems. SIAM J. Control Optim. *45* (2006), 1714–1735. zbl MR doi

[10] *B. Jayawardhana, G. Weiss*: State convergence of passive nonlinear systems with an L^2 input. IEEE Trans. Autom. Control *54* (2009), 1723–1727. zbl MR doi

[11] *F.-F. Jin, B.-Z. Guo*: Lyapunov approach to output feedback stabilization for the Euler-Bernoulli equation with boundary input disturbance. Automatica *52* (2015), 95–102. zbl MR doi

[12] *Z. Ke, H. Logemann, R. Rebarber*: Approximate tracking and disturbance rejection for stable infinite-dimensional systems using sampled-data low-gain control. SIAM J. Control Optim. *48* (2009), 641–671. zbl MR doi

[13] *M. Krstic*: Adaptive control of an anti-stable wave PDE. Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal. *17* (2010), 853–882. zbl MR

[14] *M. Nakao*: Decay of solutions of the wave equation with a local nonlinear dissipation. Math. Ann. *305* (1996), 403–417. zbl MR doi

[15] *R. Rebarber, G. Weiss*: Internal model based tracking and disturbance rejection for stable well-posed systems. Automatica *39* (2003), 1555–1569. zbl MR doi

[16] *Y. Shang, G. Xu*: Dynamic control of an Euler-Bernoulli equation with time-delay and disturbance in the boundary control. Int. J. Control *92* (2019), 27–41. zbl MR doi

[17] *G. Weiss*: Admissibility of unbounded control operators. SIAM J. Control Optimization *27* (1989), 527–545. zbl MR doi

[18] *Y. R. Xie, G. Q. Xu*: Stabilization of a wave equation with a tip mass based on disturbance observer of time-varying gain. J. Dyn. Control Syst. *23* (2017), 667–677. zbl MR doi

[19] *G. Q. Xu*: Exponential stabilization of conservation systems with interior disturbance. J. Math. Anal. Appl. *436* (2016), 764–781. zbl MR doi

[20] *Z. Zhao, B. Guo*: Active disturbance rejection control to stabilize one-dimensional wave equation with interior domain anti-damping and boundary disturbance. Control Theory Appl. *30* (2013), 1553–1563. zbl doi

Authors' addresses: *Xiao-Rui Wang*, School of Mathematics, Tianjin University, Tianjin 300350, China; Department of Mathematics and Statistics, Qinghai Nationalities University, Xining, Qinghai 810007, China, e-mail: `xrwang2012@126.com`; *Gen-Qi Xu*, School of Mathematics, Tianjin University, Tianjin 300350, China, e-mail: `gqxu@tju.edu.cn`.