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ON RATIO IMPROVEMENT OF PRODI-SERRIN-LADYZHENSKAYA TYPE REGULARITY CRITERIA FOR THE NAVIER-STOKES SYSTEM

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Abstract. This paper concerns improving Prodi-Serrin-Ladyzhenskaya type regularity criteria for the Navier-Stokes system, in the sense of multiplying certain negative powers of scaling invariant norms.

Keywords: regularity criteria; Navier-Stokes equations

MSC 2010: 35B65, 35Q30, 76D03

1. Introduction

We continue our study (see [18]) of regularity criteria for the incompressible Navier-Stokes equations (NSE) in $\mathbb{R}^3$:

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{0}, \\ \nabla\cdot\boldsymbol{u} = 0, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \end{cases} \tag{1.1}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$ and π denote the unknown velocity field and scalar pressure of the fluid, respectively, and $\boldsymbol{u}_0$ is the prescribed initial data satisfying $\nabla\cdot\boldsymbol{u}_0 = 0$. Here and in what follows, we shall use the following notations:

$$\partial_t\boldsymbol{u} = \frac{\partial\boldsymbol{u}}{\partial t}, \quad \partial_i = \frac{\partial}{\partial x_i}, \quad (\boldsymbol{u}\cdot\nabla) = \sum_{i=1}^3 u_i\partial_i, \quad \Delta = \sum_{i=1}^3 \partial_i^2.$$

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The existence of a weak solution

$$\boldsymbol{u} \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3)) \tag{1.2}$$

of (1.1) has been established in the pioneer works of Leray (see [7]) and Hopf (see [5]) (for the case of bounded domains). However, the issue of regularity and uniqueness of such a weak solution remains an open problem up to now. The classical Prodi-Serrin conditions (see [3], [8], [11]) state that if

$$\boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=1,\ 3 \leqslant q \leqslant \infty, \tag{1.3}$$

then the solution is smooth on $(0,T]$.

From the scaling point of view, the above condition (1.3) is important in the sense that for solution $\boldsymbol{u}$ of (1.1),

$$\|\boldsymbol{u}_\lambda\|_{L^p(0,T;L^q(\mathbb{R}^3))} = \|\boldsymbol{u}\|_{L^p(0,\lambda^2 T;L^q(\mathbb{R}^3))}, \tag{1.4}$$

where

$$\frac{2}{p}+\frac{3}{q}=1, \quad \boldsymbol{u}_\lambda(x,t)=\lambda\boldsymbol{u}(\lambda x,\lambda^2 t), \quad \lambda>0.$$

Regularity criterion (1.3) was later extended by Beirão da Veiga (see [1]) to

$$\nabla\boldsymbol{u}\ (\text{or } \boldsymbol{\omega}=\nabla\times\boldsymbol{u}) \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=2,\ \frac{3}{2} \leqslant q \leqslant \infty. \tag{1.5}$$

Interested readers can also locate in [17] a refined version of (1.3) and (1.5) in the homogeneous Besov spaces.

Recently, Tran-Yu in [13] and [14] established a series of regularity criteria involving the ratio of some physical quantities. Among others, Tran-Yu in [13], Corollary 2 established the regularity condition

$$\int_0^T \frac{\|\boldsymbol{\omega}(\tau)\|_{L^2}^4}{1+\|\boldsymbol{u}(\tau)\|_{L^3}^2}\,\mathrm{d}\tau < \infty, \tag{1.6}$$

which was improved in [14], Theorem 1 to

$$\int_0^T \frac{\|\boldsymbol{\omega}(\tau)\|_{L^2}^4}{1+\|\boldsymbol{u}(\tau)\|_{L^3}^3}\,\mathrm{d}\tau < \infty. \tag{1.7}$$

We remark that in [13], [14], there is no 1 in the denominator; however, checking the proof shows that regularity criteria (1.6) and (1.7) are both correct. We do state the result in the above manner to be consistent with recent progress as (1.9)–(1.11). Similar consideration applies to (1.8) below.

Further extension as

$$(1.8)\qquad \int_0^T \frac{\|\boldsymbol{\omega}(\tau)\|_{L^s}^{2s/(2s-3)}}{1+\|\boldsymbol{u}(\tau)\|_{L^3}^{f(s)}}\,\mathrm{d}\tau<\infty \quad \text{with } f(s)=\begin{cases} 3, & \frac{3}{2}<s\leqslant\frac{15}{8},\\ \frac{2(3-s)}{2s-3}, & \frac{15}{8}<s<2,\\ \frac{3}{2s-3}, & 2\leqslant s<\infty \end{cases}$$

can then be found in [18]. Notice that in (1.8), $f(s)$ is not continuous at $s=2$. This is because different estimation techniques are invoked below and above 2. On the other hand, Tran-Yu in [15] treated many aspects, and showed the following three regularity conditions:

$$(1.9)\qquad \int_0^T \frac{\|\boldsymbol{u}(\tau)\|_{L^s}^{2s/(s-3)}}{1+\|\boldsymbol{u}(\tau)\|_{\dot H^{1/2}}^{\kappa}}\,\mathrm{d}\tau<\infty \quad \text{with } \kappa=\begin{cases} 2, & 3<s\leqslant 5,\\ \frac{4}{s-3}, & 5<s<\infty; \end{cases}$$

$$(1.10)\qquad \int_0^T \frac{\|\boldsymbol{u}(\tau)\|_{L^s}^{2s/(s-3)}}{1+\|\boldsymbol{u}(\tau)\|_{L^3}^{\kappa}}\,\mathrm{d}\tau<\infty \quad \text{with } \kappa=\begin{cases} 3, & 3<s\leqslant 5,\\ \frac{6}{s-3}, & 5<s<\infty; \end{cases}$$

$$(1.11)\qquad \int_0^T \frac{\|\pi(\tau)\|_{L^s}^{2s/(2s-3)}}{1+\|\boldsymbol{u}(\tau)\|_{L^3}^{\kappa}}\,\mathrm{d}\tau<\infty \quad \text{with } \kappa=\begin{cases} 3, & \frac{3}{2}<s\leqslant\frac{9}{4},\\ \frac{2s}{2s-3}, & \frac{9}{4}\leqslant s\leqslant 3,\\ \frac{6}{2s-3}, & s\geqslant 3. \end{cases}$$

These results are very interesting. Firstly, in the numerator, the physical quantities are in the Serrin's class. Secondly, in the denominator, the velocity has its critical norm.

The aim of this paper is two-fold. First, some other ratio improvement of Prodi-Serrin-Ladyzhenskaya type regularity criteria is considered, see Theorem 1.1 and Theorem 1.2. Second, motivated by [2], [16], [21], [20], we shall also consider the regularity criterion for (1.1) via $\boldsymbol{\omega}/|\boldsymbol{u}|^\gamma$ (or equivalently $\nabla\boldsymbol{u}/|\boldsymbol{u}|^\gamma$) for suitable γ, see Theorem 1.3. Notice that when $\gamma=1$, $\nabla\boldsymbol{u}/|\boldsymbol{u}|$ is the gradient estimate, whose bound is crucial in geometric analysis, see [10].

Before stating the main result, let us recall the weak solution of (1.1) in the sense of Leray and Hopf, see [9], Definitions 3.3 and 4.9 for instance.

Definition 1.1. Let $\boldsymbol{u}_0\in L^2(\mathbb{R}^3)$ with $\nabla\cdot\boldsymbol{u}_0=0$, $T>0$. A measurable $\mathbb{R}^3$-valued function $\boldsymbol{u}$ defined in $[0,T]\times\mathbb{R}^3$ is said to be a *weak solution* to (1.1) if the following three conditions hold,

(1) $\boldsymbol{u} \in L^\infty(0,T; L^2(\mathbb{R}^3) \cap L^2(0,T; H^1(\mathbb{R}^3))$;

(2) $(1.1)_1$ and $(1.1)_2$ hold in the sense of distributions, i.e.

$$\int_0^t \int_{\mathbb{R}^3} \boldsymbol{u} \cdot [\partial_t \boldsymbol{\varphi} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\varphi}] \, \mathrm{d}x \, \mathrm{d}s + \int_{\mathbb{R}^3} \boldsymbol{u}_0 \cdot \boldsymbol{\varphi}(0) \, \mathrm{d}x = \int_0^T \int_{\mathbb{R}^3} \nabla \boldsymbol{u} : \nabla \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t$$

for each $\boldsymbol{\varphi} \in C_c^\infty([0,T) \times \mathbb{R}^3)$ with $\nabla \cdot \boldsymbol{\varphi} = 0$, where $A : B = \sum\limits_{i,j=1}^{3} a_{ij} b_{ij}$ for 3×3 matrices $A = (a_{ij})$, $B = (b_{ij})$, and

$$\int_0^T \int_{\mathbb{R}^3} \boldsymbol{u} \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}t = 0$$

for each $\psi \in C_c^\infty(\mathbb{R}^3 \times [0,T))$;

(3) the *strong energy inequality*, that is,

$$\|\boldsymbol{u}(t)\|_{L^2}^2 + 2 \int_s^t \|\nabla \boldsymbol{u}(\tau)\|_{L^2}^2 \, \mathrm{d}\tau \leqslant \|\boldsymbol{u}(s)\|_{L^2}^2 \quad \forall\, s < t < T, \tag{1.12}$$

for $s = 0$ and almost all times $s \in (0,T)$.

Now, our main results read:

Theorem 1.1. *Assume $\boldsymbol{u}_0 \in L^2(\mathbb{R}^3)$. Let $\boldsymbol{u}$ be a weak solution of* (1.1) *in the sense of Leray and Hopf. If*

$$\int_0^T \frac{\|\boldsymbol{\omega}(\tau)\|_{L^s}^{2s/(2s-3)}}{1 + \|\boldsymbol{u}(\tau)\|_{\dot{H}^{1/2}}^{\kappa}} \, \mathrm{d}\tau < \infty \quad \textit{with } \kappa = \begin{cases} 2, & \dfrac{3}{2} < s \leqslant \dfrac{9}{4}, \\[2mm] \dfrac{3}{2s-3}, & \dfrac{9}{4} < s \leqslant 3, \end{cases} \tag{1.13}$$

then the solution is smooth on $(0,T]$.

Remark 1.1. By (1.9) and Sobolev imbedding theorems, we have the regularity criterion

$$\int_0^T \frac{\|\boldsymbol{\omega}(\tau)\|_{L^s}^{2s/(2s-3)}}{1 + \|\boldsymbol{u}(\tau)\|_{\dot{H}^{1/2}}^{\kappa}} \, \mathrm{d}\tau < \infty \quad \text{with } \kappa = \begin{cases} 2, & \dfrac{3}{2} < s \leqslant \dfrac{15}{8}, \\[2mm] \dfrac{4(3-s)}{3(2s-3)}, & \dfrac{15}{8} < s < 3. \end{cases} \tag{1.14}$$

Thus (1.13) is better than (1.14) for $15/8 < s < 3$. Moreover, we can treat the limiting case $s = 3$ of the Sobolev inequality.

Theorem 1.2. *Assume* $\boldsymbol{u}_0 \in L^2(\mathbb{R}^3)$. *Let* $\boldsymbol{u}$ *be a weak solution of* (1.1) *in the sense of Leray and Hopf. If*

$$\int_0^T \frac{\|\nabla\pi(\tau)\|_{L^s}^{2s/3(s-1)}}{1+\|\boldsymbol{u}(\tau)\|_{L^3}^{\kappa}}\,\mathrm{d}\tau < \infty \quad \textit{with } \kappa = \begin{cases} 3, & 1 < s \leqslant \dfrac{9}{7}, \\ \dfrac{2s}{3(s-1)}, & \dfrac{9}{7} < s < 3, \end{cases} \tag{1.15}$$

then the solution is smooth on $(0,T]$.

Remark 1.2. By (1.11) and Sobolev imbeddding theorem, we have the regularity criterion:

$$\int_0^T \frac{\|\nabla\pi(\tau)\|_{L^s}^{2s/3(s-1)}}{1+\|\boldsymbol{u}(\tau)\|_{L^3}^{\kappa}}\,\mathrm{d}\tau < \infty \quad \text{with } \kappa = \begin{cases} 3, & 1 < s \leqslant \dfrac{9}{7}, \\ \dfrac{2s}{3(s-1)}, & \dfrac{9}{7} < s \leqslant \dfrac{3}{2}, \\ \dfrac{2(3-s)}{3(s-1)}, & \dfrac{3}{2} < s < 3. \end{cases} \tag{1.16}$$

Thus (1.15) is better than (1.16) for $3/2 < s < 3$. Moreover, we provide a simple proof under our strategy.

Theorem 1.3. *Assume* $\boldsymbol{u}_0 \in L^2(\mathbb{R}^3)$. *Let* $\boldsymbol{u}$ *be a weak solution of* (1.1) *in the sense of Leray and Hopf. If*

$$\frac{\boldsymbol{\omega}}{|\boldsymbol{u}|^{\gamma}} \in L^{\alpha}(0,T;L^{\beta}(\mathbb{R}^3)), \quad \frac{2}{\alpha}+\frac{3}{\beta} = 2-\gamma,\ \frac{3}{2-\gamma} \leqslant \beta < 3,\ 0<\gamma<1, \tag{1.17}$$

then the solution is smooth on $(0,T]$.

Remark 1.3. In a bounded domain, regularity criteria in terms of $\pi/(1+|\boldsymbol{u}|^{\delta})$ or $\nabla\pi/(1+|\boldsymbol{u}|^{\delta})$ are established by the third author in [19].

2. Proof of Theorem 1.1

For any $\varepsilon \in (0,T)$, since $\nabla\boldsymbol{u} \in L^2(0,T;L^2(\mathbb{R}^3))$ and (1.12) holds for almost all times $s \in (0,T)$, we may find a $\delta \in (0,\varepsilon)$ such that

$$\boldsymbol{u}(\delta) = \boldsymbol{u}(\cdot,\delta) \in L^2(\mathbb{R}^3), \quad \nabla\boldsymbol{u}(\delta) \in L^2(\mathbb{R}^3) \Rightarrow \boldsymbol{u}(\delta) \in \dot{H}^{1/2}(\mathbb{R}^3)\cap L^3(\mathbb{R}^3),$$

as well as

$$\|\boldsymbol{u}(t)\|_{L^2}^2 + 2\int_\delta^t \|\nabla\boldsymbol{u}(\tau)\|_{L^2}^2\,\mathrm{d}\tau \leqslant \|\boldsymbol{u}(\delta)\|_{L^2}^2 \quad \forall\,\delta<t<T.$$

Take this $\boldsymbol{u}(\delta)$ as initial data, there exists $\tilde{\boldsymbol{u}} \in C([\delta, \Gamma^*), \dot{H}^{1/2}(\mathbb{R}^3) \cap L^3(\mathbb{R}^3))$, where $[\delta, \Gamma^*)$ is the life span of the unique strong solution, see [4], [6] for the local unique solvability. Moreover, by the margin case of (1.3) in [3], $\tilde{\boldsymbol{u}} \in C^\infty(\mathbb{R}^3 \times (\delta, \Gamma^*))$. According to the uniqueness result in [12], $\tilde{\boldsymbol{u}} = \boldsymbol{u}$ on $[\delta, \Gamma^*)$.

(1) If $\Gamma^* > T$, we have already that $\boldsymbol{u} \in C^\infty(\mathbb{R}^3 \times (0, T])$ due to the arbitrariness of $\varepsilon \in (0, T)$.

(2) In the case $\Gamma^* \leqslant T$, our strategy is to show that $\|\boldsymbol{u}(t)\|_{L^3}$ (or a strong norm $\|\boldsymbol{u}(t)\|_{\dot{H}^{1/2}}$) remains uniform bounded as $t \nearrow \Gamma^*$. Standard continuation argument based on [6] then yields that Γ^* is not the maximal existence time of $\tilde{\boldsymbol{u}}$. The contradiction shows that this case is impossible.

Multiplying $(1.1)_1$ by $\Lambda \boldsymbol{u}$ with the operator Λ being defined through the Fourier transform as

$$\widehat{\Lambda f}(\xi) = |\xi| \hat{f}(\xi),$$

and then integrating over $\mathbb{R}^3$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_{\dot{H}^{1/2}}^2 + \|\boldsymbol{u}\|_{\dot{H}^{3/2}}^2 = -\int_{\mathbb{R}^3} [(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}] \cdot \Lambda \boldsymbol{u} \, \mathrm{d}x \equiv I. \tag{2.1}$$

If $3/2 < s \leqslant 9/4$, invoking the fact that for any $1 < p < \infty$ there exist positive constants $c(p)$, $C(p) > 0$ such that

$$\begin{aligned} (2.2)\quad -\Delta f &= \Lambda\Lambda f \Rightarrow -\Delta \partial_i f = \partial_i \Lambda\Lambda f \\ &\Rightarrow \partial_i f = \frac{\partial_i}{\Lambda}\Lambda f = \mathcal{R}_i \Lambda f \left(\mathcal{R}_i \text{ is the Riesz transformation: } \widehat{\mathcal{R}_i f}(\xi) = \frac{\xi_i}{|\xi|}\hat{f}(\xi)\right) \\ &\Rightarrow \|\nabla f\|_{L^p} \leqslant c(p)\|\Lambda f\|_{L^p}; \\ -\Delta f &= -\sum_{i=1}^3 \partial_i\partial_i f \Rightarrow -\Delta\Lambda f = -\sum_{i=1}^3 \Lambda \partial_i\partial_i f \\ &\Rightarrow \Lambda f = -\sum_{i=1}^3 \frac{\partial_i}{\Lambda}\partial_i f = -\sum_{i=1}^3 \mathcal{R}_i \partial_i f \\ &\Rightarrow \|\Lambda f\|_{L^p} \leqslant C(p)\|\nabla f\|_{L^p}, \end{aligned}$$

we bound I as

$$\begin{aligned} (2.3)\quad I &\leqslant C\|\boldsymbol{u}\|_{L^{3s/(3-s)}}\|\nabla \boldsymbol{u}\|_{L^{6s/(4s-3)}}^2 \\ &\qquad \text{(by Hölder's inequality and classical elliptic estimates)} \\ &\leqslant C\|\nabla \boldsymbol{u}\|_{L^s}\left[\|\nabla \boldsymbol{u}\|_{L^s}^{(2s-3)/2(3-s)}\|\boldsymbol{u}\|_{\dot{H}^{3/2}}^{(9-4s)/2(3-s)}\right]^2 \\ &\qquad \text{(by Sobolev and Gagliardo-Nirenberg inequalities)} \\ &\leqslant C\|\boldsymbol{\omega}\|_{L^s}^{s/(3-s)}\|\boldsymbol{u}\|_{\dot{H}^{3/2}}^{(9-4s)/(3-s)} \\ &\leqslant C\|\boldsymbol{\omega}\|_{L^s}^{2s/(2s-3)} + \frac{1}{2}\|\boldsymbol{u}\|_{\dot{H}^{3/2}}^2. \end{aligned}$$

Plugging (2.3) into (2.1), absorbing the last term, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}(1+\|\boldsymbol{u}\|_{\dot{H}^{1/2}}^2) \leqslant C\|\boldsymbol{\omega}\|_{L^s}^{2s/(2s-3)} = C\frac{\|\boldsymbol{\omega}\|_{L^s}^{2s/(2s-3)}}{1+\|\boldsymbol{u}\|_{\dot{H}^{1/2}}^2}(1+\|\boldsymbol{u}\|_{\dot{H}^{1/2}}^2).$$

Applying the Gronwall inequality, we deduce that

$$\begin{aligned}\sup_{\delta\leqslant t<\Gamma^*}(1+\|\boldsymbol{u}(t)\|_{\dot{H}^{1/2}}^2) &\leqslant (1+\|\boldsymbol{u}(\delta)\|_{\dot{H}^{1/2}}^2)\exp\left[C\int_\delta^{\Gamma^*}\frac{\|\boldsymbol{\omega}(\tau)\|_{L^s}^{2s/(2s-3)}}{1+\|\boldsymbol{u}(\tau)\|_{\dot{H}^{1/2}}^2}\,\mathrm{d}\tau\right]\\ &\leqslant (1+\|\boldsymbol{u}(\delta)\|_{\dot{H}^{1/2}}^2)\exp\left[C\int_0^{T}\frac{\|\boldsymbol{\omega}(\tau)\|_{L^s}^{2s/(2s-3)}}{1+\|\boldsymbol{u}(\tau)\|_{\dot{H}^{1/2}}^2}\,\mathrm{d}\tau\right]<\infty,\end{aligned}$$

as desired. We complete the proof of Theorem 1.1 for $3/2 < s \leqslant 9/4$.

If, however, $9/4 < s \leqslant 3$, we may dominate I in the following manner:

$$\begin{aligned}I &\leqslant C\|\boldsymbol{u}\|_{L^{s/(s-2)}}\|\nabla\boldsymbol{u}\|_{L^s}^2 \quad \text{(by Hölder's inequality and (2.2))}\\ &\leqslant C\|\boldsymbol{u}\|_{\dot{H}^{1/2}}^{(4s-9)/(2s-3)}\|\nabla\boldsymbol{u}\|_{L^s}^{2(3-s)/(2s-3)}\|\nabla\boldsymbol{u}\|_{L^s}^2\\ &\leqslant C\|\nabla\boldsymbol{u}\|_{L^s}^{2s/(2s-3)}\|\boldsymbol{u}\|_{\dot{H}^{1/2}}^{(4s-9)/(2s-3)}.\end{aligned}\tag{2.4}$$

Putting (2.4) into (2.1) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_{\dot{H}^{1/2}}^{3/(2s-3)} \leqslant C\|\nabla\boldsymbol{u}\|_{L^s}^{2s/(2s-3)} = C\frac{\|\nabla\boldsymbol{u}\|_{L^s}^{2s/(2s-3)}}{1+\|\boldsymbol{u}\|_{\dot{H}^{1/2}}^{3/(2s-3)}}(1+\|\boldsymbol{u}\|_{\dot{H}^{1/2}}^{3/(2s-3)}).$$

Applying the Gronwall inequality, we deduce

$$\begin{aligned}&\sup_{\delta\leqslant t<\Gamma^*}(1+\|\boldsymbol{u}(t)\|_{\dot{H}^{1/2}}^{3/(2s-3)})\\ &\qquad\leqslant (1+\|\boldsymbol{u}(\delta)\|_{\dot{H}^{1/2}}^{3/(2s-3)})\exp\left[C\int_0^T\frac{\|\nabla\boldsymbol{u}(\tau)\|_{L^s}^{2s/(2s-3)}}{1+\|\boldsymbol{u}(\tau)\|_{\dot{H}^{1/2}}^{3/(2s-3)}}\,\mathrm{d}\tau\right]<\infty,\end{aligned}$$

as desired. We finish the proof of Theorem 1.1 for $9/4 < s \leqslant 3$.

3. Proof of Theorem 1.2

As in Section 1, it suffices to show that $\|\boldsymbol{u}(t)\|_{L^3}$ is uniformly bounded as $t \nearrow \Gamma^*$. First, let us recall a well-known representation of π via $\boldsymbol{u}$ as

$$\begin{aligned}
&\text{taking divergence of}(1.1)_1\\
&\Rightarrow -\Delta\pi = \nabla\cdot[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\\
&\Rightarrow -\Delta(\nabla\pi) = \nabla\nabla\cdot[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\\
&\Rightarrow \|\nabla\pi\|_{L^s} \leqslant C\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|_{L^s} \leqslant C\||\boldsymbol{u}|\cdot|\nabla\boldsymbol{u}|\|_{L^s}, \quad 1 < s < \infty.
\end{aligned}\tag{3.1}$$

Multiplying $(1.1)_1$ by $|\boldsymbol{u}|\boldsymbol{u}$ and integrating over $\mathbb{R}^3$, we get (see [16], page 50)

$$\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_{L^3}^3 + \frac{4}{9}\|\nabla(|\boldsymbol{u}|^{3/2})\|_{L^2}^2 + \||\boldsymbol{u}|^{1/2}\cdot|\nabla\boldsymbol{u}|\|_{L^2}^2 = -\int_{\mathbb{R}^3}\nabla\pi\cdot|\boldsymbol{u}|\boldsymbol{u}\,\mathrm{d}x \equiv J.\tag{3.2}$$

If $1 < s \leqslant 9/7$, we dominate J as

$$\begin{aligned}
J &\leqslant \int_{\mathbb{R}^3}|\nabla\pi|^{2s/(9-5s)}|\nabla\pi|^{(9-7s)/(9-5s)}\cdot|\boldsymbol{u}|^2\,\mathrm{d}x\\
&\leqslant \|\nabla\pi\|_{L^s}^{2s/(9-5s)}\|\nabla\pi\|_{L^{9/5}}^{(9-7s)/(9-5s)}\|\boldsymbol{u}\|_{L^9}^2\\
&\leqslant C\|\nabla\pi\|_{L^s}^{2s/(9-5s)}\||\boldsymbol{u}|\cdot|\nabla\boldsymbol{u}|\|_{L^{9/5}}^{(9-7s)/(9-5s)}\|\boldsymbol{u}\|_{L^9}^2 \quad \text{(by (3.1))}\\
&\leqslant C\|\nabla\pi\|_{L^s}^{2s/(9-5s)}\||\boldsymbol{u}|^{1/2}(|\boldsymbol{u}|^{1/2}\cdot|\nabla\boldsymbol{u}|)\|_{L^{9/5}}^{(9-7s)/(9-5s)}\|\boldsymbol{u}\|_{L^9}^2\\
&\leqslant C\|\nabla\pi\|_{L^s}^{2s/(9-5s)}\||\boldsymbol{u}|^{1/2}\|_{L^{18}}^{(9-7s)/(9-5s)}\||\boldsymbol{u}|^{1/2}\cdot|\nabla\boldsymbol{u}|\|_{L^2}^{(9-7s)/(9-5s)}\|\boldsymbol{u}\|_{L^9}^2\\
&\leqslant C\|\nabla\pi\|_{L^s}^{2s/(9-5s)}\||\boldsymbol{u}|^{3/2}\|_{L^6}^{(9-7s)/3(9-5s)}\\
&\qquad\times\||\boldsymbol{u}|^{1/2}\cdot|\nabla\boldsymbol{u}|\|_{L^2}^{(9-7s)/(9-5s)}\||\boldsymbol{u}|^{3/2}\|_{L^6}^{4/3}\\
&\leqslant C\|\nabla\pi\|_{L^s}^{2s/(9-5s)}\|\nabla(|\boldsymbol{u}|^{3/2})\|_{L^2}^{3(5-3s)/(9-5s)}\||\boldsymbol{u}|^{1/2}\cdot|\nabla\boldsymbol{u}|\|_{L^2}^{(9-7s)/(9-5s)}\\
&\leqslant C\|\nabla\pi\|_{L^s}^{2s/3(s-1)} + \frac{2}{9}\|\nabla(|\boldsymbol{u}|^{3/2})\|_{L^2}^2 + \frac{1}{2}\||\boldsymbol{u}|^{1/2}\cdot|\nabla\boldsymbol{u}|\|_{L^2}^2.
\end{aligned}\tag{3.3}$$

Putting (3.3) into (3.2), we find

$$\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_{L^3}^3 \leqslant C\|\nabla\pi\|_{L^s}^{2s/3(s-1)} = C\frac{\|\nabla\pi\|_{L^s}^{2s/3(s-1)}}{1+\|\boldsymbol{u}\|_{L^3}^3}(1+\|\boldsymbol{u}\|_{L^3}^3).\tag{3.4}$$

Applying Gronwall inequality, we obtain

$$\sup_{\delta\leqslant t<\Gamma^*}(1+\|\boldsymbol{u}(t)\|_{L^3}^3) \leqslant (1+\|\boldsymbol{u}(\delta)\|_{L^3}^3)\exp\left[C\int_0^T\frac{\|\nabla\pi(\tau)\|_{L^s}^{2s/3(s-1)}}{1+\|\boldsymbol{u}(\tau)\|_{L^3}^3}\,\mathrm{d}\tau\right] < \infty,$$

as desired. We complete the proof of Theorem 1.2 for $1 < s \leqslant 9/7$.

If, however, $9/7 < s < 3$, we bound J in a different manner:

$$J \leqslant \|\nabla\pi\|_{L^s}\|\boldsymbol{u}\|^2_{L^{2s/(s-1)}} \leqslant \|\nabla\pi\|_{L^s}\||\boldsymbol{u}|^{3/2}\|^{4/3}_{L^{4s/3(s-1)}}$$
$$\leqslant C\|\nabla\pi\|_{L^s}[\||\boldsymbol{u}|^{3/2}\|^{(7s-9)/4s}_{L^2}\|\nabla(|\boldsymbol{u}|^{3/2})\|^{3(3-s)/4s}_{L^2}]^{4/3}$$
$$\leqslant C\|\nabla\pi\|_{L^s}\|\boldsymbol{u}\|^{(7s-9)/2s}_{L^3}\|\nabla(|\boldsymbol{u}|^{3/2})\|^{(3-s)/s}_{L^2}$$
$$\leqslant C\|\nabla\pi\|^{2s/3(s-1)}_{L^s}\|\boldsymbol{u}\|^{(7s-9)/3(s-1)}_{L^3} + \frac{2}{9}\|\nabla(|\boldsymbol{u}|^{3/2})\|^2_{L^2}. \tag{3.5}$$

Putting (3.5) into (3.2), we find

$$\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|^3_{L^3} \leqslant C\|\nabla\pi\|^{2s/3(s-1)}_{L^s}\|\boldsymbol{u}\|^{(7s-9)/3(s-1)}_{L^3}, \tag{3.6}$$

or equivalently,

$$\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|^{2s/3(s-1)}_{L^3} \leqslant C\|\nabla\pi\|^{2s/3(s-1)}_{L^s} = C\frac{\|\nabla\pi\|^{2s/3(s-1)}_{L^s}}{1+\|\boldsymbol{u}\|^{2s/(3s-1)}_{L^3}}(1+\|\boldsymbol{u}\|^{2s/3(s-1)}_{L^3}).$$

Applying Gronwall inequality, we obtain

$$\sup_{\delta\leqslant t<\Gamma^*}(1+\|\boldsymbol{u}(t)\|^{2s/3(s-1)}_{L^3})$$
$$\leqslant (1+\|\boldsymbol{u}(\delta)\|^{2s/3(s-1)}_{L^3})\exp\left[C\int_0^T \frac{\|\nabla\pi(\tau)\|^{2s/3(s-1)}_{L^s}}{1+\|\boldsymbol{u}(\tau)\|^{2s/3(s-1)}_{L^3}}\,\mathrm{d}\tau\right] < \infty,$$

as desired. We complete the proof of Theorem 1.2 for $9/7 < s < 3$.

4. Proof of Theorem 1.3

For clarity, we split the proof into two cases.

Case 1: $3/(2-\gamma) < \beta < 3$. By the Sobolev inequality and Hölder's inequality, we obtain

$$\|\boldsymbol{u}\|_{L^q} \leqslant C\|\nabla\boldsymbol{u}\|_{L^s}, \quad \frac{3}{q} = -1 + \frac{3}{s},\ 1 \leqslant s < 3,\ 1 \leqslant q < \infty$$
$$\leqslant C\|\boldsymbol{\omega}\|_{L^s} = C\left\|\frac{\boldsymbol{\omega}}{|\boldsymbol{u}|^\gamma}\cdot|\boldsymbol{u}|^\gamma\right\|_{L^s} \leqslant C\left\|\frac{\boldsymbol{\omega}}{|\boldsymbol{u}|^\gamma}\right\|_{L^\beta}\|\boldsymbol{u}\|^\gamma_{L^q}, \quad \frac{1}{s} = \frac{1}{\beta} + \frac{\gamma}{q}.$$

Consequently,

$$\|\boldsymbol{u}\|^{1-\gamma}_{L^q} \leqslant C\left\|\frac{\boldsymbol{\omega}}{|\boldsymbol{u}|^\gamma}\right\|_{L^\beta} \Rightarrow \int_0^T \|\boldsymbol{u}(t)\|^{(1-\gamma)\alpha}_{L^q}\,\mathrm{d}t \leqslant C\int_0^T \left\|\frac{\boldsymbol{\omega}}{|\boldsymbol{u}|^\gamma}(t)\right\|^\alpha_{L^\beta}\,\mathrm{d}t. \tag{4.1}$$

It follows from $3/q = -1 + 3/s$ and $1/s = 1/\beta + \gamma/q$ that

$$\frac{3}{q} + 1 = \frac{3}{s} = \frac{3}{\beta} + \frac{3\gamma}{q} \Rightarrow \frac{3(1-\gamma)}{q} = \frac{3}{\beta} - 1 \Rightarrow \frac{3}{q} - \frac{3}{(1-\gamma)\beta} = -\frac{1}{1-\gamma}. \tag{4.2}$$

This together with the assumption $2/\alpha + 3/\beta = 2 - \gamma$ implies that

$$\frac{2}{(1-\gamma)\alpha} + \frac{3}{q} = \frac{1}{1-\gamma}\Big(\frac{2}{\alpha} + \frac{3}{\beta}\Big) - \frac{3}{(1-\gamma)\beta} + \frac{3}{q} = \frac{2-\gamma}{1-\gamma} - \frac{1}{1-\gamma} = 1.$$

Moreover, (4.2) and the assumption $3/(2-\gamma) < \beta < 3$ yield

$$q = \frac{3(1-\gamma)\beta}{3-\beta} \in (3, \infty).$$

This together with (4.1) verifies regularity criterion (1.3), which concludes the proof of Theorem in this case.

Case 2: $\beta = 3/(2-\gamma)$. In this case, $q = 3$, and we should invoke the margin case of (1.3): $\boldsymbol{u} \in L^\infty(0, T; L^3(\mathbb{R}^3))$ to finish the proof.

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