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ON THE GENERALIZED VANISHING CONJECTURE

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Abstract. We show that the GVC (generalized vanishing conjecture) holds for the differential operator $\Lambda = (\partial_x - \Phi(\partial_y))\partial_y$ and all polynomials $P(x,y)$, where $\Phi(t)$ is any polynomial over the base field. The GVC arose from the study of the Jacobian conjecture.

Keywords: Jacobian conjecture; generalized vanishing conjecture; differential operator

MSC 2010: 14R15, 13N15

1. Introduction

The well-known Jacobian conjecture (JC for short) was first proposed by Keller in 1939 (see [2] and [13]). It asserts that any polynomial map F from the complex affine n-space $\mathbb{C}^n$ to itself with $\det JF = 1$ must be an automorphism of $\mathbb{C}^n$. Various special cases of this still mysterious conjecture have been investigated, and connections with some other notable problems have been established. For example, the JC is related to some problems in combinatorics (cf. [18]), it is equivalent to the Dixmer conjecture (cf. [1], [3], [12]) and also to the Mathieu conjecture proposed by Mathieu in 1995 (see [9]).

It was shown independently by de Bondt and van den Essen (see [6]) and Meng (see [10]) that for the JC one only needs to consider all polynomial maps of the form $X + H\colon \mathbb{C}^n \to \mathbb{C}^n$ for all dimensions n, where H is cubic homogeneous and JH is symmetric and nilpotent. Based on this result, Zhao proposed in 2007 the vanishing conjecture (see [17], [20]) and generalized it later in [19] to the following form.

Generalized Vanishing Conjecture (GVC). Let Λ be any differential operator on $\mathbb{C}[z] := \mathbb{C}[z_1, z_2, \ldots, z_n]$ with constant coefficients. If $P \in \mathbb{C}[z]$ is such that

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$\Lambda^m(P^m) = 0$ for all $m \geqslant 1$, then for any polynomial $Q \in \mathbb{C}[z]$ we have $\Lambda^m(P^m Q) = 0$ for all sufficiently large m.

In fact, Zhao showed in [17] and [20] that the JC holds for all dimensions n if and only if the GVC holds for all dimensions n for the case where Λ is the Laplace operator $\sum_{i=1}^{n} \partial_{z_i}^2$ and P is homogeneous.

Up to now, the GVC was verified in the following special cases:

(1) $n = 1$;

(2) $n = 2$ and $\Lambda = \partial_{z_1} - \Phi(\partial_{z_2})$;

(3) $\Lambda(t)$ (or $P(z)$) is a linear combination of two monomials with different degrees (see [15]);

(4) $n = 2$, Λ is homogeneous (see [4]);

(5) $n \leqslant 4$, Λ is the Laplace operator;

(6) $n = 5$, Λ is the Laplace operator and P is homogeneous (due to [5], [7] and [19]).

In this paper, we will show that the GVC holds for the differential operator $\Lambda = (\partial_x - \Phi(\partial_y))\partial_y$ on $\mathbb{C}[x, y]$ and all polynomials $P(x, y) \in \mathbb{C}[x, y]$, where $\Phi(t)$ is any polynomial over $\mathbb{C}$. The conclusion is in fact valid for any field of characteristic zero.

A more general conjecture concerning the image of differential operators, the Image Conjecture, which implies the GVC, was proposed by Zhao in [21], and for the study of the Image Conjecture we refer the reader to [8], [11], [14], [16], [17] etc.

2. The proof of GVC for $\Lambda = (\partial_x - \Phi(\partial_y))\partial_y$

Throughout this section, K stands for a field of characteristic zero. For simplicity, we write $K[x, y]$ instead of $K[z_1, z_2]$. We consider the GVC for the differential operator $\Lambda = (\partial_x - \Phi(\partial_y))\partial_y$, where $\Phi(t)$ is an arbitrary polynomial over K. We write $\Phi(t)$ as

$$\Phi(t) = q_0 + q_1 t + \ldots + q_s t^s$$

where $q_i \in K$, $0 \leqslant i \leqslant s$. And we denote by $o(\Phi(t))$ or $o(\Phi)$ the order of the polynomial $\Phi(t)$, i.e., the least integer $m \geqslant 0$ such that $q_m \neq 0$.

We will show the following theorem.

Theorem 2.1. *The GVC holds for the differential operator* $\Lambda = (\partial_x - \Phi(\partial_y))\partial_y$ *and all polynomials* $P(x, y) \in K[x, y]$.

We start with some lemmas.

Lemma 2.2. *Let* $\Lambda = (\partial_x - \Phi(\partial_y))\partial_y$ *and let* $0 \neq P(x, y) \in K[x, y]$ *be such that* $\Lambda P = 0$. *Then*

(1) $q_0 = 0$ *(i.e.* $\Phi(t) = 0$ *or* $o(\Phi) > 1$*) or* $P(x,y) \in K[x]$*;*
(2) $P(x,y) = \mathrm{e}^{x\Phi(\partial_y)}(f(x)+g(y))$ *for some* $f(x) \in K[x]$ *and* $g(y) \in K[y]$*.*

Proof. (1) Suppose that $P(x,y) \notin K[x]$ and let $p_m(x)y^m$ be the leading term of $P(x,y)$ with respect to y. Note that

$$\Lambda = (\partial_x - \Phi(\partial_y))\partial_y = (\partial_x - (q_0 + q_1\partial_y + \ldots + q_s\partial_y^s))\partial_y.$$

Let cx^l be the leading term of $p_m(x)$ with respect to x. Since $\Lambda P = 0$, the leading term of ΛP with respect to y is zero, i.e., $(\partial_x - q_0)\partial_y(p_m(x)y^m) = 0$, which implies that $q_0\partial_y(cx^l y^m) = 0$ and thus $q_0 = 0$.

(2) One may verify that $\partial_x \mathrm{e}^{-x\Phi(\partial_y)} = \mathrm{e}^{-x\Phi(\partial_y)}(\partial_x - \Phi(\partial_y))$. So

$$\begin{aligned}\partial_x\partial_y(\mathrm{e}^{-x\Phi(\partial_y)}P) &= \partial_x \mathrm{e}^{-x\Phi(\partial_y)}(\partial_y P) = \mathrm{e}^{-x\Phi(\partial_y)}(\partial_x - \Phi(\partial_y))\partial_y P\\ &= \mathrm{e}^{-x\Phi(\partial_y)}(\Lambda P) = 0.\end{aligned}$$

So there are no terms $x^a y^b$ with $a \geqslant 1$ and $b \geqslant 1$ in $\mathrm{e}^{-x\Phi(\partial_y)}P$, namely

$$\mathrm{e}^{-x\Phi(\partial_y)}P = f(x) + g(y)$$

for some $f(x) \in K[x]$ and $g(y) \in K[y]$. Applying $\mathrm{e}^{x\Phi(\partial_y)}$ to both sides of the last equation, we obtain that $P(x,y) = \mathrm{e}^{x\Phi(\partial_y)}(f(x)+g(y))$. $\square$

Now we write $f(x)$ and $g(x)$ above as

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_kx^k, \quad g(y) = b_0 + b_1y + b_2y^2 + \ldots + b_dy^d,$$

where a_j, $b_t \in K$, $0 \leqslant j \leqslant s$, $0 \leqslant t \leqslant d$. We may assume that $a_0 = 0$.

Lemma 2.3. *Let* $\Lambda = (\partial_x - \Phi(\partial_y))\partial_y$ *and* $0 \neq P(x,y) \in K[x,y]$*. If* $o(\Phi) \geqslant 2$ *and* $\Lambda P = \Lambda^2(P^2) = 0$*, then* $o(\Phi) \geqslant \deg g$ *and* $P(x,y) = f(x) + g(y)$*. Furthermore,* $\deg f(x) < 2$ *or* $\deg g(y) < 2$*.*

Proof. Since $\Lambda P = 0$, by Lemma 2.2, $P(x,y) = \mathrm{e}^{x\Phi(\partial_y)}(f(x)+g(y))$, where $f(x)$, $g(y)$ are as above and $a_0 = f(0) = 0$. So

$$\begin{aligned}0 &= \Lambda^2(P^2) = \Lambda^2[\mathrm{e}^{x\Phi(\partial_y)}(f(x)+g(y))]^2\\ &= (\partial_x^2 - 2\partial_x\Phi + \Phi^2)\partial_y^2\Big[f + g + x\Phi(g) + \frac{x^2}{2!}\Phi^2(g) + \frac{x^3}{3!}\Phi^3(g) + \ldots\Big]^2\\ &= (\partial_x^2 - 2\partial_x\Phi + \Phi^2)\partial_y^2\Big[(f^2 + g^2 + 2fg) + (2f\Phi(g) + 2g\Phi(g))x\\ &\quad + (\Phi(g)^2 + f\Phi^2(g) + g\Phi^2(g))x^2\\ &\quad + \Big(\frac{1}{3}f\Phi^3(g) + \frac{1}{3}g\Phi^3(g) + \Phi(g)\Phi^2(g)\Big)x^3 + \ldots\Big].\end{aligned}$$

Viewing $\Lambda^2(P^2)$ as a polynomial in $K[y][x]$ and looking at its constant term, we obtain that

$$\begin{aligned}0 = \Lambda^2 P^2|_{x=0} &= \partial_y^2(4a_2 g + 4a_1\Phi(g) + 2(\Phi(g))^2 + 2g\Phi^2(g) \\ &\quad - 4a_1\Phi(g) - 4\Phi(g\Phi(g)) + \Phi^2(g^2)) \\ &= \partial_y^2(4a_2 g + 2(\Phi(g))^2 + 2g\Phi^2(g) - 4\Phi(g\Phi(g)) + \Phi^2(g^2)).\end{aligned}$$

It follows that

$$u(y) := 4a_2 g + 2(\Phi(g))^2 + 2g\Phi^2(g) - 4\Phi(g\Phi(g)) + \Phi^2(g^2) \in Ky + K.$$

By the hypothesis of the lemma, $r := o(\Phi(t)) \geqslant 2$. Note that $\Phi(\partial_y) = q_r\partial_y^r +$ higher-order terms, and that $g = b_d y^d +$ lower terms.

Claim: $r \geqslant d$.

Suppose conversely that $d > r$. Observe that the first polynomial in the expression of $u(y)$ is of degree d if $a_2 \neq 0$, and the others are all of degree $2d - 2r$.

(1) If $d > 2r$, then $2d - 2r > d$, whence the coefficient of the term in $u(y)$ with degree $2d - 2r$ is

$$q_r{}^2 b_d{}^2\Big(2\frac{d!\,d!}{(d-r)!(d-r)!} + 2\frac{d!}{(d-2r)!} - 4\frac{d!(2d-r)!}{(d-r)!(2d-2r)!} + \frac{(2d)!}{(2d-2r)!}\Big),$$

which must be zero since $u(y) \in Ky + K$. But in the last formula, the last number is greater than the third, and thus the coefficient is nonzero, a contradiction.

(2) If $2r > d > r$, then $d > 2d - 2r$, whence the coefficient of the term in $u(y)$ with degree d is $4a_2 b_d$, which must be zero since $u(y) \in Ky + K$, and thus $a_2 = 0$. Then one may observe the coefficient of the term in $u(y)$ with degree $2d - 2r$ and arrive at a contradiction as in the case $d > 2r$.

(3) If $d = 2r$, then all polynomials in the expression of $u(y)$ are of degree $2r$, whence the coefficient of the term in $u(y)$ with degree $2r$ is

$$\begin{aligned}0 &= 4b_d a_2 + b_d^2 q_r^2\frac{2(2r)!(2r)!}{r!r!} + b_d^2 q_r^2 2(2r)! - b_d^2 q_r^2\frac{4(3r)!}{r!} + b_d^2 q_r^2\frac{(4r)!}{(2r)!} \\ &= b_d\Big(4a_2 + b_d q_r^2\Big(\frac{2(2r)!(2r)!}{r!r!} + 2(2r)! - \frac{4(3r)!}{r!} + \frac{(4r)!}{(2r)!}\Big)\Big).\end{aligned} \tag{2.1}$$

Now observe the term x of $\Lambda^2(P^2) \in K[y][x]$, which is

$$\begin{aligned}0 = \partial_y^2(&12a_3 g + 12a_2\Phi(g) + 6\Phi(g)\Phi^2(g) - 8a_2\Phi(g) \\ &- 4\Phi(\Phi(g)^2) - 4\Phi(g\Phi^2(g)) + 2\Phi^2(g\Phi(g))).\end{aligned}$$

It follows that

$$\begin{aligned} v(y) := 12a_3 g &+ 12a_2\Phi(g) + 6\Phi(g)\Phi^2(g) - 8a_2\Phi(g) \\ &- 4\Phi(\Phi(g)^2) - 4\Phi(g\Phi^2(g)) + 2\Phi^2(g\Phi(g)) \in Ky + K. \end{aligned}$$

The first polynomial $12a_3 g$ in the expression of $v(y)$ is of degree $2r$ if $a_3 \neq 0$, and all the others are of degree r. So the coefficient of the $2r$-degree term of $v(y)$ is $12a_3 b_d = 0$ and thus $a_3 = 0$, and then the coefficient of the r-degree term of $v(y)$ is

$$\begin{aligned} 0 &= 12a_2 b_d q_r \frac{(2r)!}{r!} + 6b_d^2 q_r^3 \frac{(2r)!(2r)!}{r!} - 8a_2 b_d q_r \frac{(2r)!}{r!} - 4b_d^2 q_r^3 \frac{(2r)!(2r)!(2r)!}{r!r!r!} \\ &\quad - 4b_d^2 q_r^3 \frac{(2r)!(2r)!}{r!} + 2b_d^2 q_r^3 \frac{(2r)!(3r)!}{r!r!} \\ &= b_d q_r \frac{(2r)!}{r!}\Big(4a_2 + b_d q_r^2 \Big(2(2r)! - 4\frac{(2r)!(2r)!}{r!r!} + 2\frac{(3r)!}{r!}\Big)\Big). \end{aligned} \tag{2.2}$$

From the equalities (2.1) and (2.2), we obtain that

$$2(2r)! - \frac{4(2r)!(2r)!}{r!r!} + \frac{2(3r)!}{r!} = \frac{2(2r)!(2r)!}{r!r!} + 2(2r)! - \frac{4(3r)!}{r!} + \frac{(4r)!}{(2r)!},$$

i.e.,

$$\frac{6(2r)!(2r)!}{r!r!} - \frac{6(3r)!}{r!} + \frac{(4r)!}{(2r)!} = 0. \tag{2.3}$$

Then

$$\frac{(4r)!}{(2r)!} < \frac{6(3r)!}{r!},$$

which is only possible when $r = 2$. But when $r = 2$, the left hand of the equality (2.3) is

$$\frac{6(4)!(4)!}{2!2!} - \frac{6(6)!}{2!} + \frac{8!}{4!} = 6 \times 64,$$

a contradiction.

Thus we have proved the claim that $r \geqslant d$.

In the case $r > d$, we have

$$P(x, y) = \mathrm{e}^{x\Phi(\partial_y)}(f(x) + g(y)) = f(x) + g(y).$$

In the case $r = d$, we have

$$P(x, y) = \mathrm{e}^{x\Phi(\partial_y)}(f(x) + g(y)) = f(x) + g(y) + x\Phi(g),$$

where $\Phi(g)$ is a constant, and thus in this case $P(x, y)$ is also of the form $f(x) + g(y)$.

Finally, observing that

$$
\begin{aligned}
0 = \Lambda^2(P^2) &= (\partial_x - \Phi)^2\partial_y^2(f^2 + 2fg + g^2) \\
&= (\partial_x^2 - 2\partial_x\Phi + \Phi^2)(2f\partial_y^2 g + \partial_y^2 g^2) = 2(\partial_x^2 f)(\partial_y^2 g),
\end{aligned}
$$

we have $\partial_x^2 f$ or $\partial_y^2 g = 0$, i.e., $\deg f < 2$ or $\deg g < 2$. □

Now we are in the position to prove Theorem 2.1.

Proof of Theorem 2.1. Suppose that $\Lambda^m(P^m) = 0$ for all $m \geqslant 1$. We need to show that for any $h \in K[x,y]$ we have $\Lambda^m(P^m h) = 0$ for all sufficiently large m. It suffices to take $h = x^a y^b$, $a \geqslant 0$, $b \geqslant 0$.

The case $P = 0$ is obvious and thus suppose that $P \neq 0$. By Lemma 2.2, $q_0 = 0$ (i.e., $o(\varphi) \geqslant 1$) or $P(x,y) \in K[x]$. If $P(x,y) \in K[x]$, then

$$
\Lambda^m(P^m x^a y^b) = (\partial_x - \Phi(\partial_y))^m \partial_y^m (P^m x^a y^b) = 0 \quad \forall\, m > b.
$$

So we may assume that $q_0 = 0$, and then $\Lambda = (\partial_x - \Phi(\partial_y))\partial_y = (\partial_x - q_1\partial_y - \ldots - q_s\partial_y^s)\partial_y$. Using a linear coordinate change, we may assume that $q_1 = 0$ i.e., $o(\varphi) \geqslant 2$. By Lemma 2.3, $r = o(\Phi) \geqslant \deg g = d$ and $P(x,y) = f(x) + g(y)$, where $\deg f(x) < 2$ or $\deg g(y) < 2$.

In the case $\deg f(x) < 2$, observe that

$$
\begin{aligned}
\Lambda^m(P^m x^a y^b) &= (\partial_x - \Phi(\partial_y))^m \partial_y^m ((f+g)^m x^a y^b) \\
&= \left(\sum_{i=0}^{m} (-1)^i C_m^i \partial_x^{m-i} \Phi^i(\partial_y)\right) \partial_y^m \left(\sum_{j=0}^{m} C_m^j f^{m-j} g^j x^a y^b\right) \\
&= \sum_{i=0}^{m}\sum_{j=0}^{m} (-1)^i C_m^i C_m^j \cdot \partial_x^{m-i}(f^{m-j} x^a) \cdot \Phi^i(\partial_y)\partial_y^m (g^j y^b).
\end{aligned}
$$

When $m - i > m - j + a$, we have $\partial_x^{m-i}(f^{m-j}x^a) = 0$.

When $m - i \leqslant m - j + a$, i.e., $a + i \geqslant j$, we have

$$
o(\Phi^i(\partial_y)\partial_y^m) = ir + m \quad \text{and} \quad \deg_y(g^j y^b) = dj + b,
$$

If $m > b + ar$, then noticing that $a + i \geqslant j$ and $r \geqslant d$, we have

$$
m + ir > b + ar + ir \geqslant b + jr \geqslant b + dj,
$$

whence $\Phi^i(\partial_y)\partial_y^m(g^j y^b) = 0$. Therefore,

$$
\Lambda^m(P^m x^a y^b) = 0 \quad \forall\, m > b + ar.
$$

In the case $\deg g < 2$, we have

$$\partial_y^m((f+g)^m x^a y^b) = \sum_{i=0}^{m} C_m^i f^i x^a \partial_y^m(g^{m-i} y^b) = \sum_{i=0}^{b} C_m^i f^i x^a \partial_y^m(g^{m-i} y^b),$$

the degree of which is no more than $b(\deg f) + a$. So for all $m > b(\deg f) + a$, we have

$$\Lambda^m(P^m x^a y^b) = (\partial_x - \Phi(\partial_y))^m \partial_y^m((f+g)^m x^a y^b) = 0,$$

which completes the proof. □

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