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INVERSE EIGENVALUE PROBLEM OF CELL MATRICES

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Abstract. We consider the problem of reconstructing an $n\times n$ cell matrix $D(\vec{x})$ constructed from a vector $\vec{x}=(x_1,x_2,\dots,x_n)$ of positive real numbers, from a given set of spectral data. In addition, we show that the spectra of cell matrices $D(\vec{x})$ and $D(\pi(\vec{x}))$ are the same for every permutation $\pi\in S_n$.

Keywords: cell matrix; inverse eigenvalue problem; Euclidean distance matrix

MSC 2010: 15B10, 15B05, 15B48, 35P30, 35P20

1. Introduction

An inverse eigenvalue problem (IEP) concerns the reconstruction of a matrix from given spectral data (see [2]). Usually, there are some specific applications of IEP such as system and control theory, system identification, seismic tomography, principal component analysis, exploration and remote sensing, antenna array processing, geophysics, molecular spectroscopy, particle physics, structure analysis, circuit theory, mechanical system simulation, etc. (see [1]). The objective of an inverse eigenvalue problem is to construct matrices that maintain the specific structure as well as the given spectral property (see [2]). There are many investigations about the inverse eigenvalue problem of matrices. For instance, an inverse eigenvalue problem for symmetric and normal matrices was studied by Radwan in [7]. An inverse eigenvalue problem for Jacobi matrices was studied by Wang and Zhong in [10]. Also a solution of the inverse eigenvalue problem of certain singular Hermitian matrices was studied by Gyamfi in [3]. Recently, in 2014 Nazari and Mahdinasab worked on the inverse eigenvalue problem of distance matrices by using the orthogonal matrices technique and they constructed Euclidean distance matrices with the eigenvalue list (see [6]). They provided a new method for construction of distance matrices and also added some conditions so that they could get regular spherical matrices

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having the given eigenvalues. They even considered providing a new algorithm for reconstructing distance matrices which are regular spherical matrices, however, they did not consider providing an algorithm for reconstructing cell matrices yet. Since the structure of cell matrices is much simpler than that of Euclidean distance matrices, it is theoretically interesting to find the inverse eigenvalue problem on cell matrices.

Cell matrices were first introduced by Jacklic and Modic in 2010 (see [4]), they are a special case of Euclidean distance matrices. In 2014, Tarazaga and Kurata studied the set of cell matrices and its relationship with the cone of positive semidefinite diagonal matrices (see [9]). Recently in 2015, Kurata and Tarazaga considered the problem of finding cell matrices that are closest to given Euclidean distance matrices with respect to the Frobenius norm (see [5]). They also discussed the majorization ordering of the eigenvalues of cell matrices.

In this study, we deal with the inverse eigenvalue problem on cell matrices for some lists of spectra in which we use elementary matrices to find the spectra of matrices. We consider the reconstruction of $n \times n$ cell matrices with the given set of at most $2k$ distinct eigenvalues in which we are free to choose k distinct eigenvalues. We also show that the spectra of cell matrices $D(\pi(\vec{x}))$ have the same spectra as cell matrices $D(\vec{x})$ for any $\pi \in S_n$.

2. Preliminary

An $n \times n$ matrix $D = (d_{ij})$ is said to be a *Euclidean distance matrix* (EDM) if there exist $x_1, x_2, \dots, x_n$ in some Euclidean space $\mathbb{R}^r$ $(r \leqslant n)$, such that $d_{ij} = \|x_i - x_j\|_2^2$ for all $i, j = 1, 2, \dots, n$, where $\|\cdot\|$ is the Euclidean norm. The following properties immediately hold according to the definition of EDM:

(1) D is a nonnegative matrix: $d_{ij} \geqslant 0$ for all $i, j = 1, 2, \dots, n$,

(2) D is a symmetric matrix: $d_{ij} = d_{ji}$ for all $i, j = 1, \dots, n$,

(3) all diagonal elements are zero: $d_{ii} = 0$ for all $i = 1, \dots, n$.

These matrices were introduced by Menger in 1928, later they were studied by Schoenberg (see [8]), when studying positive definite functions, and have received considerable attention. They are used in applications in geodesy, economics, genetics, psychology, biochemistry, engineering etc. (see [6]). A Euclidean distance matrix D is said to be *spherical* if the construction points of D lie on a hypersphere, otherwise, it is said to be non-spherical. Moreover, a spherical Euclidean distance matrix D is *regular* if the constructive points for D lie on a hypersphere whose center coincides with the centroid of those points.

Let $\vec{x} = (x_i)$, $i = 1, 2, \ldots, n$ be a vector of real numbers and $\vec{x} > 0$. A *cell matrix* $D \in \mathbb{R}^{n\times n}$ associated with $\vec{x}$, denoted by $D(\vec{x})$, is defined as

$$(D(\vec{x}))_{ij} = \begin{cases} 0 & \text{if } i = j, \\ x_i + x_j & \text{if } i \neq j. \end{cases}$$

It is well known that cell matrices are Euclidean distance matrices (see [9]). Furthermore, they are spherical EDM but need not be regular spherical EDM. According to [4] the determinants of principal sub-matrices of cell matrices $D(\vec{x})$ have the following form:

$$\det D^{(i)} = (-1)^{i-1} 2^{i-2} \left(4(i-1) + \sum_{j=1}^{i} \sum_{l=1}^{j-1} \frac{(x_j - x_l)^2}{x_j x_l} \right) \prod_{k=1}^{i} x_k,$$

where $D^{(i)} := D(1:i, 1:i)$, $i = 1, 2, \ldots, n$, are principal sub-matrices of a cell matrix. Also, each cell matrix has exactly one positive eigenvalue, the rest of the eigenvalues are negative (see [4]).

We denote the spectrum of a square matrix A by $\sigma(A)$ which is the set of all eigenvalues of A. To determine the spectra of matrices, we can use elementary matrices to simplify it. Throughout this investigation, we use two types of elementary matrices E:

(1) $E = W_{ij}$, the row swapping ($R_i \leftrightarrow R_j$) matrix (so $W_{ij}^{-1} = W_{ij}$),

(2) $E = S_{ij}(\lambda)$, the row sum ($R_i \to R_i + \lambda R_j$) matrix (so $S_{ij}^{-1}(\lambda) = S_{ij}(-\lambda)$).

It is well known that

$$\sigma(D(\vec{x})) = \sigma(ED(\vec{x})E^{-1})$$

for any elementary matrix E. Also, for the case of block matrices having the form

$$A = \begin{pmatrix} X & Y \\ O & Z \end{pmatrix},$$

where X, Z are square matrices and O is the zero matrix, we have $\sigma(A) = \sigma(X) \cup \sigma(Z)$. In particular, if we can express the $ED(\vec{x})E^{-1}$ as the upper or lower triangular matrix then the entries on the main diagonal are the eigenvalues of the matrix.

3. A construction of cell matrices with k distinct variables

We first consider the inverse eigenvalue problem for 3×3 cell matrices. To do this, it is necessary to concentrate only on the list of spectra with zero sum since their traces are all zero.

Proposition 3.1. *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3\} \subseteq \mathbb{R}$ with $\lambda_1 \geqslant 0 > \lambda_3 \geqslant \lambda_2$ and $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Then σ is the spectrum of the cell matrix, $D(\vec{a})$, constructed by $\vec{a} = \left(\sqrt{\frac{1}{2}|\lambda_1\lambda_2|} - \frac{1}{2}|\lambda_3|, \frac{1}{2}|\lambda_3|, \frac{1}{2}|\lambda_3|\right)$.*

Proof. Let $\vec{a} = (a_1, a_2, a_3)$, where a_i are positive real numbers. The explicit form of the cell matrix $D(\vec{a})$ is given by

$$D(\vec{a}) = \begin{bmatrix} 0 & a_1+a_2 & a_1+a_3 \\ a_1+a_2 & 0 & a_2+a_3 \\ a_1+a_3 & a_2+a_3 & 0 \end{bmatrix}.$$

So, its characteristic polynomial is

$$\Delta_x(D(\vec{a})) = x^3 - (\alpha^2+\beta^2+\gamma^2)x - 2\alpha\beta\gamma,$$

where $\alpha = a_1 + a_2$, $\beta = a_1 + a_3$ and $\gamma = a_2 + a_3$. If we require σ to be the spectrum of $D(\vec{a})$, then

$$\Delta_x(D(\vec{a})) = x^3 - (\lambda_1+\lambda_2+\lambda_3)x^2 + (\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)x - \lambda_1\lambda_2\lambda_3.$$

By comparing the coefficients of the polynomial $\Delta_x(D(\vec{a}))$ and by using the assumption that $\lambda_1 + \lambda_2 + \lambda_3 = 0$, we conclude that

$$\alpha^2+\beta^2+\gamma^2 = \lambda_1^2 - \lambda_2\lambda_3 \quad \text{and} \quad 2\alpha\beta\gamma = \lambda_1\lambda_2\lambda_3.$$

Now, we choose $\gamma = |\lambda_3|$, so $\gamma > 0$. Then

$$2\alpha\beta = \frac{\lambda_1\lambda_2\lambda_3}{\gamma} = \frac{\lambda_1\lambda_2\lambda_3}{|\lambda_3|} = -\lambda_1\lambda_2,$$

and thus

$$\begin{aligned}\alpha^2+\beta^2 &= \lambda_1^2 - \lambda_2\lambda_3 - \gamma^2 = \lambda_1^2 - \lambda_2\lambda_3 - \lambda_3^2 = (\lambda_1-\lambda_3)(\lambda_1+\lambda_3) - \lambda_2\lambda_3 \\ &= (\lambda_1-\lambda_3)(-\lambda_2) - \lambda_2\lambda_3 = -\lambda_1\lambda_2 = 2\alpha\beta.\end{aligned}$$

This implies that $\alpha = \beta$, which means $a_1 + a_2 = a_1 + a_3$ and hence $a_2 = a_3$. This also implies that $2\alpha^2 = -\lambda_1\lambda_2$ and hence $\alpha = \sqrt{\frac{1}{2}|\lambda_1\lambda_2|} = \beta$. Then

$$a_1 = \sqrt{\frac{|\lambda_1\lambda_2|}{2}} - a_2 \quad \text{and} \quad a_2 = \frac{2a_2}{2} = \frac{a_2 + a_3}{2} = \frac{\gamma}{2} = \frac{|\lambda_3|}{2} > 0.$$

Since $|\lambda_3| = \min\{|\lambda_1|, |\lambda_2|, |\lambda_3|\}$,

$$a_1 = \sqrt{\frac{|\lambda_1\lambda_2|}{2}} - \frac{|\lambda_3|}{2} > 0.$$

Therefore the cell matrix $D(\vec{a})$ constructed from $\vec{a} = (a_1, a_2, a_2)$ has the spectrum σ. □

For example, if $\sigma = \{-1, -2, 3\}$, then

$$D(\vec{a}) = \begin{bmatrix} 0 & \sqrt{3} & \sqrt{3} \\ \sqrt{3} & 0 & 1 \\ \sqrt{3} & 1 & 0 \end{bmatrix}$$

is a cell matrix with the given spectrum σ. It seems complicated to investigate the inverse eigenvalue problem in this way, even for 4×4 matrices. However, we have some observations on the reduction of the problem on cell matrices having a block form.

Here, we present a sufficient condition for the reconstruction of $n \times n$ cell matrices with the given set of at most $2k$ distinct eigenvalues in which we are free to choose k distinct eigenvalues.

Theorem 3.2. *Let $S = \{\lambda_1, \lambda_2, \ldots, \lambda_k, \lambda_{k+1}, \ldots, \lambda_{k+1}, \ldots, \lambda_{2k}, \ldots, \lambda_{2k}\}$ be a subset of real numbers in which $\lambda_{k+1}, \ldots, \lambda_{2k}$ have multiplicities $l_1 - 1, \ldots, l_k - 1$, respectively, where $l_1 + \ldots + l_k = n$ and $l_i > 1$ for $i = 1, 2, \ldots, k$. If elements of S satisfy*

(1) $\lambda_1 > 0$ *and* $\lambda_2, \lambda_3, \ldots, \lambda_k, \ldots, \lambda_{2k} < 0$,

(2) $\lambda_1, \lambda_2, \ldots, \lambda_k$ *are the roots of the characteristic polynomial of the $k \times k$ matrix $D^{(k)}(\vec{x})$ with*

$$(D^{(k)}(\vec{x}))_{ij} = \begin{cases} (l_{k-j+1} - 1)(2x_{k-j+1}) & \text{if } i = j, \\ l_{k-j+1}(x_{k-j+1} + x_{k-i+1}) & \text{if } i \neq j, \end{cases}$$

where $x_i := -\frac{1}{2}\lambda_{k+i}$ for $i = 1, 2, \ldots, k$ then there is a cell matrix $D(\vec{x})$ such that $\sigma(D(\vec{x})) = S$. Explicitly, $D(\vec{x})$ can be reconstructed from the vector

$$\vec{x} = \Big(\underbrace{-\frac{\lambda_{k+1}}{2}, \ldots, -\frac{\lambda_{k+1}}{2}}_{l_1}, \underbrace{-\frac{\lambda_{k+2}}{2}, \ldots, -\frac{\lambda_{k+2}}{2}}_{l_2}, \ldots, \underbrace{-\frac{\lambda_{2k}}{2}, \ldots, -\frac{\lambda_{2k}}{2}}_{l_k}\Big).$$

P r o o f. We construct $D(\vec{x})$ from the vector $\vec{x}$ given in the theorem. Denote by $1_{m\times n}$ the $m \times n$ matrix whose entries are all 1 and $x_i := -\frac{1}{2}\lambda_{k+i}$ for each $i = 1, 2, \ldots, k$. Then, $D(\vec{x})$ is a block matrix whose (s,t) block is given by $l_s \times l_t$ matrix

$$D^{st} = \begin{cases} (x_s + x_t)1_{l_s\times l_t} & \text{if } s \neq t, \\ 2x_s(1_{l_s\times l_t} - I_{l_s}) & \text{if } s = t \end{cases}$$

for $1 \leqslant s,\ t \leqslant k$.

For the first step, we use row sums and column sums on the rows $n, n-1, \ldots, n - l_k + 2$ by the row $n - l_k + 1 := j_k$; namely, we calculate

$$S_{n-l_k+2,j_k}(-1)\ldots S_{n-1,j_k}(-1)S_{n,j_k}(-1)D(\vec{x}) \\ S^{-1}_{n,j_k}(-1)S^{-1}_{n-1,j_k}(-1)\ldots S^{-1}_{n-l_k+2,j_k}(-1).$$

Then

$$D(\vec{x}) \sim \begin{bmatrix} D^1(\vec{x}) & * \\ 0 & D(\lambda_{2k}) \end{bmatrix},$$

where $D(\lambda_{2k}) = \operatorname{diag}(\underbrace{\lambda_{2k}, \ldots, \lambda_{2k}}_{l_k-1})$ and

$$D^1(\vec{x}) = \begin{bmatrix} D^{11} & D^{12} & \ldots & D^{1(k-1)} & \vec{v}_{1k} \\ D^{12} & D^{22} & \ldots & D^{2(k-1)} & \vec{v}_{2k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ D^{1(k-1)} & D^{2(k-1)} & \ldots & D^{(k-1)(k-1)} & \vec{v}_{(k-1)k} \\ \vec{w}^{\mathrm{T}}_{1k} & \vec{w}^{\mathrm{T}}_{2k} & \ldots & \vec{w}^{\mathrm{T}}_{(k-1)(k)} & (l_k-1)(2x_k) \end{bmatrix}$$

in which $\vec{v}_{ik} = (\underbrace{l_k(x_i + x_k), \ldots, l_k(x_i + x_k)}_{l_i})^{\mathrm{T}}$ and $\vec{w}_{ik} = (\underbrace{x_i + x_k, \ldots, x_i + x_k}_{l_i})^{\mathrm{T}}$.
Next, we swap the row j_k with the row $n - l_k - l_{k-1} + 1 := j_{k-1}$ of $D^1(\vec{x})$; namely, we calculate

$$W_{j_k,j_{k-1}}D^1(\vec{x})W_{j_k,j_{k-1}}.$$

Then $D^1(\vec{x}) \sim D^{11}(\vec{x})$, where

$$D^{11}(\vec{x}) = \begin{bmatrix} D^{11} & D^{12} & \ldots & D^{1(k-2)} & \vec{v}_{1k} & D^{1(k-1)} \\ D^{21} & D^{22} & \ldots & D^{2(k-2)} & \vec{v}_{2k} & D^{2(k-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ D^{1(k-2)} & D^{2(k-2)} & \ldots & D^{(k-2)(k-2)} & \vec{v}_{(k-2)k} & D^{(k-2)(k-1)} \\ \vec{w}^{\mathrm{T}}_{1k} & \vec{w}^{\mathrm{T}}_{2k} & \ldots & \vec{w}^{\mathrm{T}}_{(k-2)k} & (l_k-1)(2x_k) & \vec{w}^{\mathrm{T}}_{(k-1)k} \\ D^{1(k-1)} & D^{2(k-1)} & \ldots & D^{(k-2)(k-1)} & \vec{v}_{(k-1)k} & D^{(k-1)(k-1)} \end{bmatrix}.$$

For the second step, we use row sums and column sums on the row $j_k, j_k - 1, \ldots, j_{k-1}+3, j_{k-1}+2$ by the row $j_{k-1}+1$ on the matrix $D^{11}(\vec{x})$, namely, we calculate

$$S_{j_{k-1}+2,j_{k-1}+1}(-1)S_{j_{k-1}+3,j_{k-1}+1}(-1)\ldots S_{j_k-1,j_{k-1}+1}(-1)S_{j_k,j_{k-1}+1}(-1)D^{11}(\vec{x})$$
$$S^{-1}_{j_k,j_{k-1}+1}(-1)S^{-1}_{j_k-1,j_{k-1}+1}(-1)\ldots S^{-1}_{j_{k-1}+3,j_{k-1}+1}(-1)S^{-1}_{j_{k-1}+2,j_{k-1}+1}(-1).$$

Then

$$D^{11}(\vec{x}) \sim \begin{bmatrix} D^2(\vec{x}) & * \\ 0 & D(\lambda_{2k-1}) \end{bmatrix},$$

where $D(\lambda_{2k-1}) = \mathrm{diag}(\underbrace{\lambda_{2k-1}, \ldots, \lambda_{2k-1}}_{l_{k-1}-1})$ and

$$D^2(\vec{x}) = \begin{bmatrix} D^{11} & D^{12} & \ldots & D^{1(k-2)} & \vec{v}_{1k} & \vec{v}_{1(k-1)} \\ D^{12} & D^{22} & \ldots & D^{2(k-2)} & \vec{v}_{2k} & \vec{v}_{2(k-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ D^{1(k-2)} & D^{2(k-2)} & \ldots & D^{(k-2)(k-2)} & \vec{v}_{(k-2)k} & \vec{v}_{(k-2)(k-1)} \\ \vec{w}^{\mathrm{T}}_{1k} & \vec{w}^{\mathrm{T}}_{2k} & \ldots & \vec{w}^{\mathrm{T}}_{(k-2)k} & (l_k-1)(2x_k) & l_{k-1}(x_{k-1}+x_k) \\ \vec{w}^{\mathrm{T}}_{1(k-1)} & \vec{w}^{\mathrm{T}}_{2(k-1)} & \ldots & \vec{w}^{\mathrm{T}}_{(k-2)(k-1)} & l_k(x_{k-1}+x_k) & (l_{k-1}-1)(2x_{k-1}) \end{bmatrix}$$

in which

$$\vec{v}_{i(k-1)} = (\underbrace{l_{k-1}(x_i+x_{k-1}), \ldots, l_{k-1}(x_i+x_{k-1})}_{l_i})^{\mathrm{T}}$$

and

$$\vec{w}_{i(k-1)} = (\underbrace{x_i+x_{k-1}, \ldots, x_i+x_{k-1}}_{l_i})^{\mathrm{T}}.$$

Next, we swap the row $l_1 + \ldots + l_{k-3} + 1 = j_{k-2}$ with the row j_{k-1} and swap the row $j_{k-2}+1$ with the row $j_{k-1}+1$; namely, we calculate

$$W_{j_{k-2}+1,j_{k-1}+1}W_{j_{k-2},j_{k-1}}D^2(\vec{x})W_{j_{k-2},j_{k-1}}W_{j_{k-2}+1,j_{k-1}+1}.$$

Then $D^2(\vec{x}) \sim D^{22}(\vec{x})$, where $D^{22}(\vec{x})$ is the matrix

$$\begin{bmatrix} D^{11} & D^{12} & \ldots & D^{1(k-3)} & \vec{v}_{1k} & \vec{v}_{1(k-1)} & D^{1(k-2)} \\ D^{12} & D^{22} & \ldots & D^{2(k-3)} & \vec{v}_{2k} & \vec{v}_{2(k-1)} & D^{2(k-2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ D^{1(k-3)} & D^{2(k-3)} & \ldots & D^{(k-3)(k-3)} & \vec{v}_{(k-3)k} & \vec{v}_{(k-3)(k-1)} & D^{(k-3)(k-3)} \\ \vec{w}^{\mathrm{T}}_{1k} & \vec{w}^{\mathrm{T}}_{2k} & \ldots & \vec{w}^{\mathrm{T}}_{(k-3)k} & (l_k-1)(2x_k) & l_{k-1}(x_{k-1}+x_k) & \vec{w}^{\mathrm{T}}_{(k-2)k} \\ \vec{w}^{\mathrm{T}}_{1(k-1)} & \vec{w}^{\mathrm{T}}_{2(k-1)} & \ldots & \vec{w}^{\mathrm{T}}_{(k-3)(k-1)} & l_k(x_{k-1}+x_k) & (l_{k-1}-1)(2x_{k-1}) & \vec{w}^{\mathrm{T}}_{(k-2)(k-1)} \\ D^{1(k-2)} & D^{2(k-2)} & \ldots & D^{(k-3)(k-3)} & \vec{v}_{(k-2)k} & \vec{v}_{(k-2)(k-1)} & D^{(k-2)(k-2)} \end{bmatrix}$$

We continue the same process until the kth step, where we use row sums and column sums on the row $l_1, l_1-1, \ldots, 3, 2$ by the row 1 on the matrix $D_{k-1} := D^{((k-1)(k-1))}(\vec{x})$, namely, we calculate

$$S_{2,1}(-1)S_{3,1}(-1)\ldots S_{l_1-1,1}(-1)S_{l_1,1}(-1)D_{(k-1)} \\ S_{l_1,1}^{-1}(-1)S_{l_1-1,1}^{-1}(-1)\ldots S_{3,1}^{-1}(-1)S_{2,1}^{-1}(-1).$$

Then

$$D_{k-1} \sim \begin{bmatrix} D^k(\vec{x}) & * \\ 0 & D(\lambda_{k+1}) \end{bmatrix},$$

where $D(\lambda_{k+1}) = \mathrm{diag}(\underbrace{\lambda_{k+1}, \ldots, \lambda_{k+1}}_{l_1-1})$ and

$$D^k(\vec{x}) = \begin{bmatrix} (l_k-1)(2x_k) & l_{k-1}(x_{k-1}+x_k) & \ldots & l_2(x_2+x_k) & l_1(x_1+x_k) \\ l_k(x_{k-1}+x_k) & (l_{k-1}-1)(2x_{k-1}) & \ldots & l_2(x_2+x_{k-1}) & l_1(x_1+x_{k-1}) \\ l_k(x_{k-2}+x_k) & l_{k-1}(x_{k-2}+x_{k-1}) & \ldots & l_2(x_2+x_{k-2}) & l_1(x_1+x_{k-2}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ l_k(x_2+x_k) & l_{k-1}(x_2+x_{k-1}) & \ldots & (l_2-1)(2x_2) & l_1(x_1+x_2) \\ l_k(x_1+x_k) & l_{k-1}(x_1+x_{k-1}) & \ldots & l_2(x_1+x_2) & (l_1-1)(2x_1) \end{bmatrix}_{k\times k}.$$

If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the roots of the characteristic polynomial of the matrix $D^{(k)}(\vec{x})$, then $D(\vec{x})$ is a cell matrix with the given spectrum S. □

By this theorem, we can select a set of eigenvalues which is the spectrum of a cell matrix as follows. First, we are free to pick any distinct k negative real numbers, say, $\lambda_{k+1}, \ldots, \lambda_{2k}$, with multiplicities $l_1-1, \ldots, l_k-1$, respectively, where $l_i \geqslant 2$ for each $i = 1, \ldots, k$. Next, we calculate the eigenvalues (say $\lambda_1, \ldots, \lambda_k$) of the $k \times k$ matrix $D^{(k)}(\vec{x})$ defined in the theorem. Then the set

$$S = \{\lambda_1, \ldots, \lambda_k, \underbrace{\lambda_{k+1}, \ldots, \lambda_{k+1}}_{l_1-1}, \ldots, \underbrace{\lambda_{2k}, \ldots, \lambda_{2k}}_{l_k-1}\}$$

will be the spectrum of the cell matrix $D(\vec{x})$ of size $n = l_1 + \ldots + l_k$ which is bigger than or equal to $2k$.

In particular, when $k = 1$, we have:

Corollary 3.3. *Let n be a positive integer and λ a positive real number. If*

$$S = \{(n-1)\lambda, \underbrace{-\lambda, -\lambda, \ldots, -\lambda}_{n-1}\},$$

then there is an $n \times n$ *cell matrix* $D(\vec{x})$ *with* $\sigma(D(\vec{x})) = S$. *Precisely,* $D(\vec{x})$ *is constructed from the vector* $\vec{x} = (\underbrace{\frac{1}{2}\lambda, \frac{1}{2}\lambda, \dots, \frac{1}{2}\lambda}_{n})$, *which is the distance matrix with constant values* λ *for every off diagonal entry.*

Also, when $k = 2$ we have:

Corollary 3.4. *If* $S = \{\lambda_1, \lambda_2, \underbrace{\lambda_3, \lambda_3, \dots, \lambda_3}_{l_1-1}, \underbrace{\lambda_4, \lambda_4, \dots, \lambda_4}_{l_2-1}\}$ *with* $l_1 + l_2 = n$ *and* $l_1, l_2 > 0$ *satisfies:*
(1) $\lambda_1 > 0$ *and* $\lambda_2, \lambda_3, \lambda_4 < 0$,
(2)

$$\lambda_1 = [(l_1 - 1)(-\tfrac{1}{2}\lambda_3) + (l_2 - 1)(-\tfrac{1}{2}\lambda_4)] + \sqrt{(l_1(n-2)+1)(-\tfrac{1}{2}\lambda_3)^2 + \tfrac{1}{2}(n-1)\lambda_3\lambda_4 + (n^2 - n(l_1+2) + 2l_1 + 1)(-\tfrac{1}{2}\lambda_4)^2},$$

(3)

$$\lambda_2 = [(l_1 - 1)(-\tfrac{1}{2}\lambda_3) + (l_2 - 1)(-\tfrac{1}{2}\lambda_4)] - \sqrt{(l_1(n-2)+1)(-\tfrac{1}{2}\lambda_3)^2 + \tfrac{1}{2}(n-1)\lambda_3\lambda_4 + (n^2 - n(l_1+2) + 2l_1 + 1)(-\tfrac{1}{2}\lambda_4)^2},$$

then there is a cell matrix $D(\vec{x})$ *such that* $\sigma(D(\vec{x})) = S$. *Explicitly,* $D(\vec{x})$ *can be reconstructed from the vector*

$$\vec{x} = \Big(\underbrace{-\frac{\lambda_3}{2}, -\frac{\lambda_3}{2}, \dots, -\frac{\lambda_3}{2}}_{l_1}, \underbrace{-\frac{\lambda_4}{2}, -\frac{\lambda_4}{2}, \dots, -\frac{\lambda_4}{2}}_{l_2}\Big).$$

P r o o f. By Theorem 3.2, for the case $k = 2$, we have

$$D^2(\vec{x}) = \begin{bmatrix} -(l_2-1)\lambda_4 & -\frac{1}{2}l_1(\lambda_3+\lambda_4) \\ -\frac{1}{2}l_2(\lambda_3+\lambda_4) & -(l_1-1)\lambda_3 \end{bmatrix}.$$

So, $\sigma(D^2(\vec{x})) = \{\lambda_1, \lambda_2\}$. □

For example, if we choose $\lambda_3 = -2$, $\lambda_4 = -4$ and $l_1 = 5$, $l_2 = 6$, then we compute by Corollary 3.4 that $\lambda_1 = 14 + \sqrt{306}$ and $\lambda_2 = 14 - \sqrt{306}$. Then,

$$\sigma = \big\{14 + \sqrt{306}, 14 - \sqrt{306}, -2, -2, -2, -2, -4, -4, -4, -4, -4\big\}$$

is the spectrum of the 11×11 cell matrix constructed from the vector

$$\vec{x} = (1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2).$$

Precisely,

$$D(\vec{x}) = \begin{bmatrix} 0 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\ 2 & 0 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 0 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 0 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 0 & 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 & 0 & 4 & 4 & 4 & 4 & 4 \\ 3 & 3 & 3 & 3 & 3 & 4 & 0 & 4 & 4 & 4 & 4 \\ 3 & 3 & 3 & 3 & 3 & 4 & 4 & 0 & 4 & 4 & 4 \\ 3 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 0 & 4 & 4 \\ 3 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 0 & 4 \\ 3 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 0 \end{bmatrix}_{(11\times 11)}$$

is a cell matrix with the spectrum σ.

Example 3.5. We construct a cell matrix of order 13 by using Theorem 3.2. Here, we choose $\lambda_4 = -2$, $\lambda_5 = -3$, $\lambda_6 = -5$ and choose $l_1 = 4$, $l_2 = 4$, $l_3 = 5$. Then the eigenvalues of $D^{(3)}(\vec{x})$ are $20 + \sqrt{511}$, $20 - \sqrt{511}$ and -5. So

$$\sigma = \left\{20 + \sqrt{511}, 20 - \sqrt{511}, -5, -2, -2, -2, -3, -3, -3, -5, -5, -5, -5\right\}$$

satisfies the conditions of Theorem 3.2. Thus we may construct the cell matrix from the vector

$$\vec{x} = \left(1, 1, 1, 1, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \tfrac{5}{2}, \tfrac{5}{2}, \tfrac{5}{2}, \tfrac{5}{2}\right).$$

In fact,

$$D(\vec{x}) = \begin{bmatrix} 0 & 2 & 2 & 2 & \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & \frac{7}{2} & \frac{7}{2} & \frac{7}{2} & \frac{7}{2} & \frac{7}{2} \\ 2 & 0 & 2 & 2 & \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & \frac{7}{2} & \frac{7}{2} & \frac{7}{2} & \frac{7}{2} & \frac{7}{2} \\ 2 & 2 & 0 & 2 & \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & \frac{7}{2} & \frac{7}{2} & \frac{7}{2} & \frac{7}{2} & \frac{7}{2} \\ 2 & 2 & 2 & 0 & \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & \frac{7}{2} & \frac{7}{2} & \frac{7}{2} & \frac{7}{2} & \frac{7}{2} \\ \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & 0 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 4 \\ \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & 3 & 0 & 3 & 3 & 4 & 4 & 4 & 4 & 4 \\ \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & 3 & 3 & 0 & 3 & 4 & 4 & 4 & 4 & 4 \\ \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & 3 & 3 & 3 & 0 & 4 & 4 & 4 & 4 & 4 \\ \frac{7}{2} & \frac{7}{2} & \frac{7}{2} & \frac{7}{2} & 4 & 4 & 4 & 4 & 0 & 5 & 5 & 5 & 5 \\ \frac{7}{2} & \frac{7}{2} & \frac{7}{2} & \frac{7}{2} & 4 & 4 & 4 & 4 & 5 & 0 & 5 & 5 & 5 \\ \frac{7}{2} & \frac{7}{2} & \frac{7}{2} & \frac{7}{2} & 4 & 4 & 4 & 4 & 5 & 5 & 0 & 5 & 5 \\ \frac{7}{2} & \frac{7}{2} & \frac{7}{2} & \frac{7}{2} & 4 & 4 & 4 & 4 & 5 & 5 & 5 & 0 & 5 \\ \frac{7}{2} & \frac{7}{2} & \frac{7}{2} & \frac{7}{2} & 4 & 4 & 4 & 4 & 5 & 5 & 5 & 5 & 0 \end{bmatrix}_{(13\times 13)}$$

is a cell matrix with the spectrum σ.

Note that the $k \times k$ matrix $D^{(k)}(\vec{x})$ in Theorem 3.2 is a positive real matrix which may not be symmetric matrix depending on l_i's and λ_{k+i}'s. However, since its spectrum is $\{\lambda_1, \ldots, \lambda_k\}$ which is a subset of the spectrum of the cell matrix $D(\vec{x})$ with $\lambda_1 > 0$, it always holds that the spectrum of $D^{(k)}(\vec{x})$ must be a subset of real numbers with only one positive eigenvalue λ_1 satisfying $\lambda_1 > |\lambda_2| + \ldots + |\lambda_k|$.

4. Invariance of the spectrum

For a vector $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and a permutation $\pi \in S_n$, we denote by $\pi(\vec{x})$ the vector $(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$.

Lemma 4.1. *Let $D(\vec{x})$ be the cell matrix constructed from $\vec{x} = (x_1, x_2, \ldots, x_n)$. Let $\pi_1 = (l, k)$ for some distinct $l, k \in \{1, 2, \ldots, n\}$, be a transposition in S_n. If P is the permutation matrix corresponding to π_1, then $PD(\pi_1(\vec{x}))P = D(\vec{x})$.*

Proof. Since π_1 is a transposition (l, k),

$$(\pi_1(\vec{x}))_i = \begin{cases} x_i & \text{if } i \neq l, k, \\ x_l & \text{if } i = k, \\ x_k & \text{if } i = l. \end{cases}$$

Note also that $PD(\pi_1(\vec{x}))P$ is the matrix obtained from $D(\pi_1(\vec{x}))$ by swapping the lth row with the kth row and the lth column with the kth column. Then, we can list the entries of $D(\vec{x})$, $D(\pi_1(\vec{x}))$ and $P(D(\pi_1(\vec{x})))P$ as in the table below:

(i,j)	$D_{ij} = (D(\vec{x}))_{ij}$	$\widetilde{D}_{ij} = (D(\pi_1(\vec{x})))_{ij}$	$(P\widetilde{D}P)_{ij}$
$i = j$	0	0	0
$i = l$, $j = k$	$x_l + x_k$	$x_k + x_l$	$\widetilde{D}_{kl} = x_l + x_k$
$i = k$, $j = l$	$x_k + x_l$	$x_l + x_k$	$\widetilde{D}_{lk} = x_k + x_l$
$i \notin \{l, k\}$, $j = k$	$x_i + x_k$	$x_i + x_l$	$\widetilde{D}_{il} = x_i + x_k$
$i \notin \{l, k\}$, $j = l$	$x_i + x_l$	$x_i + x_k$	$\widetilde{D}_{ik} = x_i + x_l$
$i = l$, $j \notin \{l, k\}$	$x_l + x_j$	$x_k + x_j$	$\widetilde{D}_{kj} = x_l + x_j$
$i = k$, $j \notin \{l, k\}$	$x_k + x_j$	$x_l + x_j$	$\widetilde{D}_{lj} = x_k + x_j$
$i, j \notin \{l, k\}$	$x_i + x_j$	$x_i + x_j$	$\widetilde{D}_{ij} = x_i + x_j$

Hence $PD(\pi_1(\vec{x}))P = D(\vec{x})$. □

Theorem 4.2. *Let $D(\vec{x})$ be an $n \times n$ cell matrix with $\vec{x} = (x_1, x_2, \ldots, x_n)$. If $\pi \in S_n$ then $D(\vec{x})$ and $D(\pi(\vec{x}))$ have the same spectrum.*

P r o o f. Let $\pi \in S_n$. Note that π can be written as a composition of transpositions in S_n, say,

$$\pi = \pi_m \circ \pi_{m-1} \circ \ldots \circ \pi_1,$$

where π_i is a transposition in S_n for each $i = 1, 2, \ldots, m$. We denote $\vec{x}_i = \pi_i(\vec{x}_{i-1})$ for $i = 1, 2, \ldots, m$, and $\vec{x}_0 = \vec{x}$. So, $\vec{x}_m = \pi(\vec{x})$. Let P_i be the permutation matrix corresponding to the transposition π_i. By Lemma 4.1, we have

$$P_i(D(\pi_i(\vec{x}_{i-1})))P_i = D(\vec{x}_{i-1}).$$

Since $P_i^{-1} = P_i$, we have that $D(\pi_i(\vec{x}_{i-1}))$ is similar to $D(\vec{x}_{i-1})$. Hence

$$\sigma(D(\pi_i(\vec{x}_{i-1}))) = \sigma(D(\vec{x}_{i-1}))$$

for each $i = 1, 2, \ldots, m$. Therefore

$$\sigma(D(\pi(\vec{x}))) = \sigma(D(\vec{x}_m)) = \sigma(D(\vec{x}_{m-1})) = \ldots = \sigma(D(\vec{x}_0)) = \sigma(D(\vec{x})).$$

Hence $\sigma(D(\pi(\vec{x}))) = \sigma(D(\vec{x}))$. □

It is well known that a permutation of rows or columns of a matrix does effect the spectrum of the matrix, e.g., $\sigma(A) \cap \sigma(B) = \emptyset$, where

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}.$$

However, by Theorem 4.2, some permutations of elements in the cell matrix $D(\vec{x})$ based on any swapping of elements in $\vec{x}$ do not effect the spectrum of $D(\vec{x})$. For example, if $\vec{x} = (1, 2, 3, 4, 5, 6, 7)$ and $\pi = (1\,4)(2\,5)(3\,7\,6) \in S_7$, then $\pi(\vec{x}) = (4, 5, 7, 1, 2, 3, 6)$. Hence, the cell matrices $D(\vec{x})$ and $D(\pi(\vec{x}))$ have the same spectrum

$$D(\vec{x}) = \begin{pmatrix} 0 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 0 & 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 0 & 7 & 8 & 9 & 10 \\ 5 & 6 & 7 & 0 & 9 & 10 & 11 \\ 6 & 7 & 8 & 9 & 0 & 11 & 12 \\ 7 & 8 & 9 & 10 & 11 & 0 & 13 \\ 8 & 9 & 10 & 11 & 12 & 13 & 0 \end{pmatrix},$$

$$D(\pi(\vec{x})) = \begin{pmatrix} 0 & 9 & 11 & 5 & 6 & 7 & 10 \\ 9 & 0 & 12 & 6 & 7 & 8 & 11 \\ 11 & 12 & 0 & 8 & 9 & 10 & 13 \\ 5 & 6 & 8 & 0 & 3 & 4 & 7 \\ 6 & 7 & 9 & 3 & 0 & 5 & 8 \\ 7 & 8 & 10 & 4 & 5 & 0 & 9 \\ 10 & 11 & 13 & 7 & 8 & 9 & 0 \end{pmatrix}.$$

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