Czechoslovak Mathematical Journal

Hongfen Yuan

On representations of real analytic functions by monogenic functions

Czechoslovak Mathematical Journal, Vol. 69 (2019), No. 4, 997–1013

Persistent URL: `http://dml.cz/dmlcz/147909`

Czechoslovak Mathematical Journal, 69 (144) (2019), 997–1013

ON REPRESENTATIONS OF REAL ANALYTIC FUNCTIONS BY MONOGENIC FUNCTIONS

Hongfen Yuan, Handan

Received December 15, 2017. Published online April 15, 2019.

Abstract. Using the method of normalized systems of functions, we study one representation of real analytic functions by monogenic functions (i.e., solutions of Dirac equations), which is an Almansi's formula of infinite order. As applications of the representation, we construct solutions of the inhomogeneous Dirac and poly-Dirac equations in Clifford analysis.

Keywords: monogenic function; inhomogeneous Dirac equation; inhomogeneous poly-Dirac equation; Almansi's formula of infinite order; Clifford analysis

MSC 2010: 30G35, 35J05, 35C10

1. Introduction

Normalized systems of functions were introduced by Bondarenko in [3]. The method of f-normalized system of functions with respect to a partial differential operator was considered earlier by Karachik in [10] for construction and investigation of polynomial solutions to a linear PDE with constant coefficients, such as the polyharmonic equation, the Helmholtz equation, the Possion equation and so on, see [11], [12]. The proposed method was also used in the study of polynomial solutions of boundary value problems for polyharmonic equation and the Helmholtz equation, more specifically the Dirichlet problems, Neumann problems and so on, see [13], [14]. In this paper, by the normalized system of functions, we consider the classical solutions of generalized Dirac equations in Clifford analysis. This is a starting point for further research, in particular on generalized Dirac equations with help

---

This work was supported by the NNSF of China (No. 11426082), the Natural Science Foundation of Hebei Province (No. A2016402034), Project of Handan Municipal Science and Technology Bureau (No. 1534201097-10), and Foundation of Hebei University of Engineering under Grant Nos. 16121002014, 17129033049, 86210022, 00070348.

DOI: 10.21136/CMJ.2019.0573-17

of the normalized system of functions, for which the corresponding boundary value problems will be published in a forthcoming paper.

The Dirac equation in Euclidean Clifford analysis defined by Delanghe et al., has the form $\sum\limits_{i=1}^{m} e_i\partial_{x_i} f = 0$, where e_i are the generators of a real Clifford algebra $R_{0,m}$, see [4], [7]. Euclidean Clifford analysis is a higher dimensional function theory centered around monogenic functions, i.e., null solutions of the Dirac equation. It is well known that the fundamental solution of the Dirac equation is the function $G(x) = \omega_n^{-1} x/\|x\|^n$, which is a generalization of the Cauchy kernel from one-variable complex analysis. Applying the fundamental solution, scholars constructed solutions of generalized Dirac equations in Clifford analysis, such as the inhomogeneous Dirac equations, polynomial Dirac equations, etc., see [1], [6], [9], [15], [16], [17], [19], [20]. In this paper, we consider solutions of generalized Dirac equations in Clifford analysis by normalized systems of functions, which is another method with no fundamental solution.

The purpose of the present article is to generalize the method of normalized systems of functions to the setting of Clifford analysis. A great challenge arising from extensions to the setting of Clifford analysis is the lack of commutativity. To overcome the noncommutativity, we introduce the intertwine relations between the operators in Clifford analysis. Based on the intertwine relations, we construct the 0-normalized system with respect to the Dirac operator. Furthermore, with the system, we consider the representation of real analytic functions by monogenic functions, which is closely related to the Fischer decomposition for monogenic functions, see [5], [7], [8]. Applying the representation, we obtain solutions of the modified Dirac equation $(D - \lambda)g = 0$, the inhomogeneous Dirac equation $Dg = f$, and the inhomogeneous poly-Dirac equation $D^k g = f$ in Clifford analysis. To obtain the classical solutions of these equations, we prove some infinite series converge absolutely and uniformly in some starlike domain with center 0.

2. PRELIMINARIES

Clifford analysis is a hypercomplex function theory with functions defined in the Euclidean space $\mathbb{R}^m$ and taking values in Clifford algebra (see [4], [7]).

Let $\{e_1, e_2, \ldots, e_m\}$ be the standard orthonormal basis of the Euclidean space $\mathbb{R}^m$. We introduce a product subject to the rules $e_ie_j + e_je_i = -2\delta_{i,j}$, $i, j = 1, \ldots, m$, where $\delta_{i,j}$ is the Kronecker symbol. This non-commutative product generates the real Clifford algebra denoted by $R_{0,m}$.

Each of the elements in $R_{0,m}$ may be written as

$$a = \sum_A a_A e_A,$$

where a_A are real numbers and $e_A = e_{\alpha_1}e_{\alpha_2}\dots e_{\alpha_h}$ with $A = \{\alpha_1, \dots, \alpha_h\} \subset \{1, \dots, m\}$ and $1 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \dots \leqslant \alpha_h \leqslant m$. We define the norm of a as

$$|a| = \left(\sum_A |a_A|^2\right)^{1/2}.$$

A vector of $R_{0,m}$ is denoted by

$$x = x_1e_1 + x_2e_2 + \dots + x_me_m$$

with $x_i \in \mathbb{R}$. One can calculate that $x^2 = -|x|^2$.

Let Ω be an open subset of $\mathbb{R}^m$ with piecewise smooth boundary. A Clifford-valued function on Ω is a mapping $f\colon\ \Omega \to R_{0,m}$ with

$$f(x) = \sum_A f_A(x)e_A,$$

where the functions $f_A(x)$ are real valued functions.

The set of C^k-functions in Ω with values in $R_{0,m}$ is denoted by

$$C^k(\Omega, R_{0,m}) = \left\{f \mid f\colon\ \Omega \to R_{0,m}, f(x) = \sum_A f_A(x)e_A,\ f_A(x) \in C^k(\Omega)\right\},$$

where $C^k(\Omega)$ denotes the space of the k-times continuously differentiable real-valued functions defined in a domain Ω of $\mathbb{R}^m$.

The Dirac operator in $\mathbb{R}^m$ is the first order differential operator

$$D = \sum_{i=1}^m e_i\partial_{x_i},$$

acting on C^1 functions. A function $f\colon\ \Omega \to R_{0,m}$ of class C^1 is said to be monogenic in Ω if it verifies $Df = 0$ in Ω.

The Euler operator in Clifford analysis is defined by

$$E = \sum_{i=1}^m x_i\partial_{x_i}.$$

Let $\mathcal{P}$ be the set of all homogeneous polynomials of degree k. If we consider a monomial $\varphi = x_1^{\alpha_1}\dots x_m^{\alpha_m} \in \mathcal{P}$ with $\alpha_i \in \mathbb{N}$, we have

$$E\varphi = k\varphi,$$

where $k = \sum\limits_{i=1}^m \alpha_i$.

3. REPRESENTATIONS OF REAL ANALYTIC FUNCTIONS BY MONOGENIC FUNCTIONS

In this section, we introduce the generalized Euler operator E_s and the integral operator J_s. Then, based on the operators E_s, J_s and the intertwine relations between the operators x, ∂_x, E, we construct the 0-normalized system with respect to the Dirac operator in Clifford analysis. Finally, with the system, we obtain the representation of real analytic functions by monogenic functions. We first give the following definitions:

Definition 3.1 ([2]). Suppose that Ω_0 is a domain in $\mathbb{R}^m$ with $0 \in \Omega_0$. The domain Ω_0 is said to be a starlike domain with center 0 if $x \in \Omega_0$ implies $tx \in \Omega_0$ holds for each $0 \leqslant t \leqslant 1$.

Definition 3.2. A sequence of functions $\{F_k(x; f)\}_{k=0}^{\infty}$ in Ω_0 is called 0-normalized with respect to D if $DF_0(x; f) = 0$ and $DF_k(x; f) = F_{k-1}(x; f)$.

Definition 3.3. The operator $J_s \colon C(\Omega_0, R_{0,m}) \to C(\Omega_0, R_{0,m})$ is defined by

$$J_s f(x) = \int_0^1 (1-\alpha)^{s-1} \alpha^{m/2-1} f(\alpha x)\, \mathrm{d}\alpha,$$

where m is the dimension and $s > 0$.

Definition 3.4. For any $s > 0$, the generalized Euler operator on a domain Ω_0 is defined by

$$E_s = sI + E = sI + \sum_{i=1}^{m} x_i \partial_{x_i},$$

where I is the identical operator and E is the Euler operator.

In the sequel, we will need the following lemmas (i.e., Lemmas 3.1, 3.2), which are well known in Clifford analysis.

Lemma 3.1 ([7]). *The operators x, D, and E have the following properties:*

$$xD + Dx = -(2E + m), \tag{3.1}$$

$$Ex - xE = x. \tag{3.2}$$

Lemma 3.1 states the most important intertwine relations between the operators in Clifford analysis. Applying Lemma 3.1, we have Lemma 3.2.

Lemma 3.2 ([7]). *If $f(x) \in C^1(\Omega_0, R_{0,m})$, then for any $l \in \mathbb{N}$,*

$$\begin{cases} D(x^{2l} f(x)) = -2l x^{2l-1} f(x) + x^{2l} Df(x), \\ D(x^{2l-1} f(x)) = -2x^{2(l-1)} E_{m/2+l-1} f(x) - x^{2l-1} Df(x). \end{cases} \tag{3.3}$$

By direct calculation, we have Lemma 3.3.

Lemma 3.3 ([20]). *If $f(x) \in C^1(\Omega_0, R_{0,m})$, then for $s > 1$,*

$$E_{m/2+s-1} J_s f(x) = (s-1) J_{s-1} f(x). \tag{3.4}$$

Now we suppose that $f \in C^1(\Omega_0, R_{0,m})$ is monogenic. Then we define the sequence of functions $\{F_k(x; f) \in C^1(\Omega_0, R_{0,m})\colon\ k = 0, 1, 2, \dots\}$ by

$$\begin{cases} F_0(x;f) = f(x), \quad k = 0, \\ F_{2s}(x;f) = \dfrac{x^{2s}}{4^s s!\,(s-1)!} \displaystyle\int_0^1 (1-\alpha)^{s-1} \alpha^{m/2-1} f(\alpha x)\,\mathrm{d}\alpha, \quad k = 2s, \\ F_{2s-1}(x;f) \\ \quad = \dfrac{-x^{2s-1}}{2 \cdot 4^{s-1} (s-1)!\,(s-1)!} \displaystyle\int_0^1 (1-\alpha)^{s-1} \alpha^{m/2-1} f(\alpha x)\,\mathrm{d}\alpha, \quad k = 2s-1, \end{cases} \tag{3.5}$$

where $s = 1, 2, \dots$.

Theorem 3.1. *The sequence of functions $F_k(x; f)$ in Ω_0 is 0-normalized with respect to the operator D.*

Proof. Note that $DF_0(x; f) = Df(x) = 0$. We will prove that $DF_k(x; f) = F_{k-1}(x; f)$ for any $k \in \mathbb{N}$. For $k = 2s$, it is easy to obtain the result by Lemma 3.2. For $k = 2s-1$, using Lemmas 3.2 and 3.3, we have

$$\begin{aligned} DF_{2s-1}(x;f) &= D\biggl(-\frac{x^{2s-1}}{2 \cdot 4^{s-1}(s-1)!\,(s-1)!} \int_0^1 (1-\alpha)^{s-1} \alpha^{m/2-1} f(\alpha x)\,\mathrm{d}\alpha\biggr) \\ &= -\frac{1}{2 \cdot 4^{(s-1)}(s-1)!\,(s-1)!} D\biggl[x^{2s-1} \int_0^1 (1-\alpha)^{s-1} \alpha^{m/2-1} f(\alpha x)\,\mathrm{d}\alpha\biggr] \\ &= -\frac{1}{2 \cdot 4^{(s-1)}(s-1)!\,(s-1)!} \bigl[-2x^{2(s-1)} E_{m/2+s-1} J_s f(x) - x^{2s-1} D J_s f(x)\bigr] \\ &= \frac{2x^{2(s-1)}}{2 \cdot 4^{s-1}(s-1)!\,(s-1)!} E_{m/2+s-1} \int_0^1 (1-\alpha)^{s-1} \alpha^{m/2-1} f(\alpha x)\,\mathrm{d}\alpha \\ &= \frac{x^{2(s-1)}}{4^{s-1}(s-1)!\,(s-1)!} (s-1) \int_0^1 (1-\alpha)^{s-2} \alpha^{m/2-1} f(\alpha x)\,\mathrm{d}\alpha \\ &= \frac{x^{2(s-1)}}{4^{s-1}(s-1)!\,(s-2)!} \int_0^1 (1-\alpha)^{s-2} \alpha^{m/2-1} f(\alpha x)\,\mathrm{d}\alpha = F_{2s-2}(x;f). \end{aligned}$$

Thus, we complete the proof. □

Lemma 3.4 ([18]). *If* $g(x) \in C^0(\Omega_0, R_{0,m})$, *then*

$$(E+l+1)\int_0^1 \alpha^l g(\alpha x)\,\mathrm{d}\alpha = g(x), \tag{3.6}$$

and

$$(E+l+1)\int_0^1 \frac{(1-\alpha)^q}{q!}\alpha^l g(\alpha x)\,\mathrm{d}\alpha = \int_0^1 \frac{(1-\alpha)^{q-1}}{(q-1)!}\alpha^{l+1} g(\alpha x)\,\mathrm{d}\alpha, \tag{3.7}$$

where $q \in \mathbb{N}$ *and* $l \geqslant 0$.

Now we give the main theorem in this section.

Theorem 3.2. *If* $G(x) \in C^\infty(\Omega_0, R_{0,m})$ *is a real analytic function, then there exist monogenic functions* $f_j(x)$, $j = 0, 1, \ldots$, *such that*

$$\begin{aligned} G(x) &= F_0(x; f_0) + \sum_{i=1}^{\infty} F_{2i-1}(x; f_{2i-1}) + \sum_{i=1}^{\infty} F_{2i}(x; f_{2i}) \\ &= f_0(x) - \sum_{i=1}^{\infty} \frac{x^{2i-1}}{2\cdot 4^{i-1}} \int_0^1 \frac{(1-\alpha)^{i-1}\alpha^{m/2-1}}{(i-1)!\,(i-1)!} f_{2i-1}(\alpha x)\,\mathrm{d}\alpha \\ &\quad + \sum_{i=1}^{\infty} \frac{x^{2i}}{4^i} \int_0^1 \frac{(1-\alpha)^{i-1}\alpha^{m/2-1}}{i!\,(i-1)!} f_{2i}(\alpha x)\,\mathrm{d}\alpha, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} f_j(x) &= D^j G(x) - \sum_{s=1}^{\infty} \frac{(-1)^s x^{2s-1}}{2\cdot 4^{s-1}} \int_0^1 \frac{(1-\beta)^{s-1}\beta^{s+m/2-2}}{(s-1)!\,(s-1)!} D^{j+2s-1} G(\beta x)\,\mathrm{d}\beta \\ &\quad + \sum_{s=1}^{\infty} \frac{(-1)^s x^{2s}}{4^s s!\,(s-1)!} \int_0^1 (1-\beta)^{s-1}\beta^{s+m/2-1} D^{j+2s} G(\beta x)\,\mathrm{d}\beta. \end{aligned} \tag{3.9}$$

Before proving Theorem 3.2, we need the following lemmas:

Lemma 3.5. *If* $f(x)$ *is a real analytic function on* Ω_0, *then the series*

$$G_1(x) = \sum_{i=0}^{\infty} \frac{(-1)^{i+1} x^{2i+1}}{2\cdot 4^i \cdot i! \cdot i!} \int_0^1 (1-\alpha)^i \alpha^{m/2+i-1} D^{2i} f(\alpha x)\,\mathrm{d}\alpha \tag{3.10}$$

and

$$G_2(x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2(i+1)}}{4^{i+1}\cdot (i+1)! \cdot i!} \int_0^1 (1-\alpha)^i \alpha^{m/2+i} D^{2i+1} f(\alpha x)\,\mathrm{d}\alpha \tag{3.11}$$

converge absolutely and uniformly in x *on* Ω_0.

Proof. Suppose that the function $f(x)$ is real analytic at $\widetilde{x} \in \Omega_0$. Then

$$f(x) = \sum_{\gamma} f_\gamma (x - \widetilde{x})^\gamma \tag{3.12}$$

and the series converges absolutely in some neighborhood Ω_ε about $\widetilde{x}$. Therefore, there exists $0 < \varepsilon < 1$ such that $(\widetilde{x}_1 + \varepsilon, \ldots, \widetilde{x}_m + \varepsilon) \in \Omega_\varepsilon \subset \Omega_0$. For all k we have

$$\sum_{|\gamma|=k} |f_\gamma| \leqslant C\varepsilon^{-k}.$$

Note that $|x_i - \widetilde{x}_i| \leqslant |x - \widetilde{x}|$. It follows that

$$\begin{aligned} |f(x)| &\leqslant \sum_{k=0}^{\infty} \sum_{|\gamma|=k} |f_\gamma| |(x - \widetilde{x})^\gamma| \leqslant \sum_{k=0}^{\infty} \Big(\frac{|x - \widetilde{x}|}{\varepsilon}\Big)^k \varepsilon^k \sum_{|\gamma|=k} |f_\gamma| \\ &\leqslant C \sum_{k=0}^{\infty} \Big(\frac{|x - \widetilde{x}|}{\varepsilon}\Big)^k = \varphi(|x - \widetilde{x}|), \end{aligned}$$

which is valid for $|x - \widetilde{x}| < \varepsilon$.

Let $\gamma = (\gamma_1, \ldots, \gamma_m)$ and $\beta = (\beta_1, \ldots, \beta_m)$. Then we have

$$\frac{\gamma!}{(\gamma - \beta)!} \leqslant \frac{|\gamma|!}{(|\gamma| - |\beta|)!}$$

by induction.

We now estimate the derivative $D^\beta f(x)$. Grounding on (3.12) for $\varrho = |x - \widetilde{x}| < \varepsilon$, we have

$$\begin{aligned} |D^\beta f(x)| &\leqslant \sum_{\gamma \geqslant \beta} C_{\gamma,\beta} |f_\gamma| |(x - \widetilde{x})^{\gamma-\beta}| \leqslant C \sum_{\gamma \geqslant \beta} \frac{\gamma!}{(\gamma - \beta)!} \varepsilon^{-k} \varrho^{\gamma-\beta} \\ &\leqslant C \sum_{k=|\beta|}^{\infty} \frac{k!}{(k - |\beta|)!} \varepsilon^{-k} \varrho^{k-|\beta|} \leqslant C \sum_{k=|\beta|}^{\infty} \varepsilon^{-k} D_\varrho^{|\beta|} \varrho^k = D_\varrho^{|\beta|} \varphi(\varrho)|_{\varrho=|x-\widetilde{x}|}, \end{aligned}$$

where $\varphi(\varrho) = C\varepsilon/(\varepsilon - \varrho)$, $C > 0$ and $|\gamma| = k$.

For $\widetilde{x} = 0$, we have

$$|D^{2i} f(x)| \leqslant m^i D_\varrho^{2i} \varphi(\varrho)|_{\varrho=|x|}. \tag{3.13}$$

For $\alpha \in [0, 1]$, we see that $|D^{2i} f(\alpha x)| \leqslant m^i D_\varrho^{2i} \varphi(\varrho)|_{\varrho=\alpha|x|}$. Applying the above inequality, we have

$$\begin{aligned} |G_1(x)| \leqslant{}& \frac{|x|}{2} \Big| \int_0^1 \alpha^{m/2-1} f(\alpha x) \, \mathrm{d}\alpha \Big| \\ &+ \frac{|x|}{2} \sum_{i=1}^{\infty} \frac{(2i-1)!}{4^i \cdot i! \cdot i!} \int_0^1 (1 - \alpha)^i \alpha^{m/2+i-1} \frac{m^i |x|^{2i}}{(2i - 1)!} D_\varrho^{2i} \varphi(\alpha |x|) \, \mathrm{d}\alpha. \end{aligned} \tag{3.14}$$

The first integral term is estimated as follows:

$$\frac{|x|}{2}\left|\int_0^1 \alpha^{m/2-1} f(\alpha x)\,\mathrm{d}\alpha\right| \leqslant \frac{\varepsilon}{2}|f(\alpha x)|\int_0^1 \alpha^{m/2-1}\,\mathrm{d}\alpha \leqslant \varepsilon\varphi(|\alpha x|) \leqslant \varphi(|x|),$$

where $0<\varepsilon<1$ and $|x|<\varepsilon$.

For $|x|<\varepsilon$ and $m\geqslant 2$, the second integral term is estimated as follows:

$$\begin{aligned}\int_0^1 &(1-\alpha)^i \alpha^{m/2+i-1}\frac{m^i|x|^{2i}}{(2i-1)!}D_\varrho^{2i}\varphi(\alpha|x|)\,\mathrm{d}\alpha\\ &\leqslant \frac{m^i|x|^{2i-1}}{(2i-1)!}\int_0^{|x|} D_\varrho^{2i}\varphi(\varrho)\,\mathrm{d}\varrho \leqslant \frac{m^i|x|^{2i-1}}{(2i-1)!}\varphi^{(2i-1)}(|x|).\end{aligned}$$

For $m=1$ and $i=1$, we have

$$\begin{aligned}|x|^3\int_0^1 \frac{\varphi''(\alpha|x|)}{\sqrt{\alpha}}\,\mathrm{d}\alpha &\leqslant \frac{C|x|^3}{\varepsilon^2}\int_0^1 \sum_{k=2}^{\infty} k(k-1)\Big(\frac{|x|}{\varepsilon}\Big)^{k-2}\alpha^{k-5/2}\,\mathrm{d}\alpha\\ &= \frac{C|x|^3}{\varepsilon^2}\sum_{k=2}^{\infty}\frac{k(k-1)}{k-\frac{3}{2}}\Big(\frac{|x|}{\varepsilon}\Big)^{k-2}\\ &\leqslant \frac{2C|x|^2}{\varepsilon}\sum_{k=1}^{\infty}k\Big(\frac{|x|}{\varepsilon}\Big)^{k-1} \leqslant 2|x|\varphi'(|x|).\end{aligned}$$

Thus, we obtain

$$|G_1(x)| \leqslant \varphi(|x|) + \sum_{i=1}^{\infty}\frac{m^i|x|^{2i-1}}{(2i-1)!}\varphi^{(2i-1)}(|x|).$$

Note that

$$\sum_{i=1}^{\infty}\frac{\varrho^{2i-1}}{(2i-1)!}\varphi^{(2i-1)}(|x|) \leqslant \frac{\varphi(|x|+\varrho)-\varphi(|x|-\varrho)}{2}$$

for $\varrho<\varepsilon-|x|$. It follows that

$$\begin{aligned}|G_1(x)| &\leqslant \varphi(|x|)+\frac{\sqrt{m}}{4}(\varphi((1+\sqrt{m})|x|)-\varphi((1-\sqrt{m})|x|))\\ &\leqslant \varphi(|x|)+\frac{\sqrt{m}}{4}\varphi((1+\sqrt{m})|x|) \leqslant \Big(\frac{\sqrt{m}}{4}+1\Big)\varphi((1+\sqrt{m})|x|),\end{aligned} \tag{3.15}$$

where $|x|<\varepsilon/(1+\sqrt{m})$. Here we have used the inequalities

$$\varphi((1-\sqrt{m})|x|)>0 \quad\text{and}\quad \varphi((1+\sqrt{m})|x|)>\varphi(|x|).$$

Put $0 < \varepsilon' < \varepsilon$ and $\Omega_{\varepsilon'} = \{|x| < \varepsilon'/(1+\sqrt{m})\}$. Since the terms of the dominating series in (3.15) are uniformly bounded by their values at $|x| = \varepsilon'/(1+\sqrt{m})$ in Ω_0, the Weierstrass test implies that $G_1(x)$ converges uniformly on Ω_0. From the above estimates we see that the series $D^\gamma G_1(x)$ also converges uniformly on Ω_0, which implies that the series $G_1(x)$ admits termwise differentiation of D^γ. In a similar way, we can prove $G_2(x)$ converges absolutely and uniformly in x on Ω_0. Thus, we finish the proof. □

Similarly, we can prove Lemmas 3.6 and 3.7.

Lemma 3.6. *If $f(x)$ is a real analytic function on Ω_0, then for $j = 0, 1, \ldots,$ the series*

$$G_3(x) = \sum_{i=1}^{\infty} \frac{(-1)^i x^{2i}}{4^i} \int_0^1 \frac{(1-\alpha)^{i-1}}{i!\,(i-1)!} \alpha^{i+m/2-1} D^{j+2i} f(\alpha x)\,\mathrm{d}\alpha \tag{3.16}$$

and

$$G_4(x) = \sum_{i=1}^{\infty} \frac{(-1)^i x^{2i-1}}{2\cdot 4^{i-1}} \int_0^1 \frac{(1-\alpha)^{i-1}}{(i-1)!\,(i-1)!} \alpha^{i+m/2-2} D^{j+2i-1} f(\alpha x)\,\mathrm{d}\alpha \tag{3.17}$$

converge absolutely and uniformly in x on Ω_0.

Lemma 3.7. *Let the functions $f_{2i}(x)$ and $f_{2i-1}(x)$ be defined by the formula* (3.8) *on some star domain Ω_0. Then the series*

$$G_5(x) = f_0(x) + \sum_{i=1}^{\infty} \frac{x^{2i}}{4^i i!\,(i-1)!} \int_0^1 (1-\alpha)^{i-1} \alpha^{m/2-1} f_{2i}(\alpha x)\,\mathrm{d}\alpha \tag{3.18}$$

and

$$G_6(x) = \sum_{i=1}^{\infty} \frac{x^{2i-1}}{2\cdot 4^{i-1}(i-1)!\,(i-1)!} \int_0^1 (1-\alpha)^{i-1} \alpha^{m/2-1} f_{2i-1}(\alpha x)\,\mathrm{d}\alpha \tag{3.19}$$

converge absolutely and uniformly in x on Ω_0.

Lemmas 3.5–3.7 state that for every j, the series in (3.8) and (3.9) converge absolutely and uniformly in x on some star-shaped domain Ω_0, and they are termwise differentiable in Ω_0.

Now we come to the proof of Theorem 3.2.

P r o o f. First we will prove that $f_j(x)$ are monogenic functions. We apply the operator D to both sides of the equality (3.9):

(3.20)

$$\begin{aligned} Df_j(x) &= D^{j+1}G(x) \\ &\quad+\sum_{s=1}^{\infty}\frac{(-1)^s x^{2(s-1)}}{4^{s-1}}E_{m/2+s-1}\int_0^1\frac{(1-\beta)^{s-1}\beta^{s+m/2-2}}{(s-1)!\,(s-1)!}D^{j+2s-1}G(\beta x)\,\mathrm{d}\beta \\ &\quad+\sum_{s=1}^{\infty}\frac{(-1)^s x^{2s-1}}{2\cdot 4^{s-1}(s-1)!\,(s-1)!}\int_0^1(1-\beta)^{s-1}\beta^{s+m/2-1}D^{j+2s}G(\beta x)\,\mathrm{d}\beta \\ &\quad-\sum_{s=1}^{\infty}\frac{(-1)^s x^{2s-1}}{2\cdot 4^{s-1}(s-1)!\,(s-1)!}\int_0^1(1-\beta)^{s-1}\beta^{s+m/2-1}D^{j+2s}G(\beta x)\,\mathrm{d}\beta \\ &\quad+\sum_{s=1}^{\infty}\frac{(-1)^s x^{2s}}{4^s s!\,(s-1)!}\int_0^1(1-\beta)^{s-1}\beta^{s+m/2}D^{j+2s+1}G(\beta x)\,\mathrm{d}\beta. \end{aligned}$$

Using Lemma 3.4, we transform the first sum in the expression (3.20) on the right-hand side as follows:

$$\begin{aligned} &\sum_{s=1}^{\infty}\frac{(-1)^s x^{2(s-1)}}{4^{s-1}}E_{m/2+s-1}\int_0^1\frac{(1-\beta)^{s-1}\beta^{s-1}}{(s-1)!\,(s-1)!}\beta^{m/2-1}D^{j+2s-1}G(\beta x)\,\mathrm{d}\beta \\ &= -(E+m/2)\int_0^1\beta^{m/2-1}D^{j+1}G(\beta x)\,\mathrm{d}\beta+\sum_{s=2}^{\infty}\frac{(-1)^s x^{2(s-1)}}{4^{s-1}}(E+m/2+s-1) \\ &\quad\times\int_0^1\frac{(1-\beta)^{s-1}\beta^{s+m/2-2}}{(s-1)!\,(s-1)!}D^{j+2s-1}G(\beta x)\,\mathrm{d}\beta \\ &= -D^{j+1}G(x)+\sum_{s=2}^{\infty}\frac{(-1)^s x^{2(s-1)}}{4^{s-1}(s-1)!}\int_0^1\frac{(1-\beta)^{s-2}\beta^{s+m/2-1}}{(s-2)!}D^{j+2s-1}G(\beta x)\,\mathrm{d}\beta \\ &= -D^{j+1}G(x)+\sum_{s=1}^{\infty}\frac{(-1)^{s+1} x^{2s}}{4^s s!}\int_0^1\frac{(1-\beta)^{s-1}\beta^{s+m/2}}{(s-1)!}D^{j+2s+1}G(\beta x)\,\mathrm{d}\beta \\ &= -D^{j+1}G(x)-\sum_{s=1}^{\infty}\frac{(-1)^s x^{2s}}{4^s s!}\int_0^1\frac{(1-\beta)^{s-1}\beta^{s+m/2}}{(s-1)!}D^{j+2s+1}G(\beta x)\,\mathrm{d}\beta. \end{aligned}$$

Substituting the resulting expression into (3.20), it is easy to see that $Df_j(x)=0$, which implies that $f_j(x),\ j=0,\dots,k-1,\dots,$ are monogenic.

Next, we prove that formula (3.8) holds. Substituting $f_j(x)$ into the right-hand side of formula (3.8), we have

$$(3.21)\quad G(x)-\sum_{i=1}^{\infty}\frac{(-1)^i x^{2i-1}}{2\cdot 4^{i-1}(i-1)!\,(i-1)!}\int_0^1(1-\beta)^{i-1}\beta^{i+m/2-2}D^{2i-1}G(\beta x)\,\mathrm{d}\beta$$

$$
\begin{aligned}
&+\sum_{i=1}^{\infty}\frac{(-1)^i x^{2i}}{4^i i!\,(i-1)!}\int_0^1 (1-\beta)^{i-1}\beta^{i+m/2-1}D^{2i}G(\beta x)\,\mathrm{d}\beta\\
&-\sum_{i=1}^{\infty}\frac{x^{2i-1}}{2\cdot 4^{i-1}(i-1)!\,(i-1)!}\int_0^1 (1-\beta)^{i-1}\beta^{m/2-1}D^{2i-1}G(\beta x)\,\mathrm{d}\beta\\
&+\sum_{i=1}^{\infty}\frac{x^{2i-1}}{2\cdot 4^{i-1}(i-1)!\,(i-1)!}\int_0^1 (1-\alpha)^{i-1}\alpha^{m/2-1}\\
&\quad\times\sum_{s=1}^{\infty}\frac{(-1)^s(\alpha x)^{2s-1}}{2\cdot 4^{s-1}(s-1)!}\int_0^1\frac{(1-\beta)^{s-1}\beta^{s+m/2-2}}{(s-1)!}D^{2i+2s-2}G(\alpha\beta x)\,\mathrm{d}\beta\,\mathrm{d}\alpha\\
&-\sum_{i=1}^{\infty}\frac{x^{2i-1}}{2\cdot 4^{i-1}(i-1)!\,(i-1)!}\int_0^1 (1-\alpha)^{i-1}\alpha^{m/2-1}\\
&\quad\times\sum_{s=1}^{\infty}\frac{(-1)^s(\alpha x)^{2s}}{4^s s!}\int_0^1\frac{(1-\beta)^{s-1}\beta^{s+m/2-1}}{(s-1)!}D^{2i+2s-1}G(\alpha\beta x)\,\mathrm{d}\beta\,\mathrm{d}\alpha\\
&+\sum_{i=1}^{\infty}\frac{x^{2i}}{4^i i!\,(i-1)!}\int_0^1 (1-\beta)^{i-1}\beta^{m/2-1}D^{2i}G(\beta x)\,\mathrm{d}\beta\\
&-\sum_{i=1}^{\infty}\frac{x^{2i}}{4^i i!\,(i-1)!}\int_0^1 (1-\alpha)^{i-1}\alpha^{m/2-1}\\
&\quad\times\sum_{s=1}^{\infty}\frac{(-1)^s(\alpha x)^{2s-1}}{2\cdot 4^{s-1}(s-1)!}\int_0^1\frac{(1-\beta)^{s-1}\beta^{s+m/2-2}}{(s-1)!}D^{2i+2s-1}G(\alpha\beta x)\,\mathrm{d}\beta\,\mathrm{d}\alpha\\
&+\sum_{i=1}^{\infty}\frac{x^{2i}}{4^i i!\,(i-1)!}\int_0^1 (1-\alpha)^{i-1}\alpha^{m/2-1}\\
&\quad\times\sum_{s=1}^{\infty}\frac{(-1)^s(\alpha x)^{2s}}{4^s s!}\int_0^1\frac{(1-\beta)^{s-1}\beta^{s+m/2-1}}{(s-1)!}D^{2i+2s}G(\alpha\beta x)\,\mathrm{d}\beta\,\mathrm{d}\alpha.
\end{aligned}
$$

Denote by $T_1(x)$ the fourth sum of the expression (3.21). Then

(3.22)

$$
\begin{aligned}
T_1(x) = {}&\sum_{i=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^s x^{2i+2s-2}}{4^{i+s-1}(i-1)!\,(i-1)!\,(s-1)!\,(s-1)!}\\
&\times\int_0^1 (1-\alpha)^{i-1}\alpha^{m/2+2s-2}\int_0^1 (1-\beta)^{s-1}\beta^{s+m/2-2}D^{2i+2s-2}G(\alpha\beta x)\,\mathrm{d}\beta\,\mathrm{d}\alpha\\
={}&\sum_{i=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^s x^{2i+2s-2}}{4^{i+s-1}(i-1)!\,(i-1)!\,(s-1)!\,(s-1)!}\\
&\times\int_0^1 \alpha(1-\alpha)^{i-1}\int_0^1 (\alpha-\alpha\beta)^{s-1}(\alpha\beta)^{s+m/2-2}D^{2i+2s-2}G(\alpha\beta x)\,\mathrm{d}\beta\,\mathrm{d}\alpha.
\end{aligned}
$$

Put $t = \alpha\beta$. Then the repeated integral in (3.22) turns into

$$\begin{aligned}
&\int_0^1 (1-\alpha)^{i-1} \int_0^\alpha (\alpha - t)^{s-1} t^{m/2+s-2} D^{2i+2s-2} G(tx)\,\mathrm{d}t\,\mathrm{d}\alpha \\
&\qquad = \int_0^1 t^{m/2+s-2} D^{2i+2s-2} G(tx) \int_t^1 (1-\alpha)^{i-1} (\alpha - t)^{s-1}\,\mathrm{d}\alpha\,\mathrm{d}t \\
&\qquad = \frac{(i-1)!\,(s-1)!}{(s+i-1)!} \int_0^1 (1-t)^{i+s-1} t^{m/2+s-2} D^{2i+2s-2} G(tx)\,\mathrm{d}t.
\end{aligned}$$

By substituting the resulting expression into (3.22), we have

$$\begin{aligned}
T_1(x) &= \sum_{i=1}^\infty \sum_{s=1}^\infty \frac{(-1)^s x^{2i+2s-2}}{4^{i+s-1}(i-1)!} \int_0^1 \frac{(1-t)^{i+s-1} t^{m/2+s-2}}{(s+i-1)!\,(s-1)!} D^{2i+2s-2} G(tx)\,\mathrm{d}t \\
&= \sum_{j=2}^\infty \sum_{s=1}^{j-1} \frac{(-1)^s x^{2j-2}}{4^{j-1}(j-s-1)!} \int_0^1 \frac{(1-t)^{j-1} t^{s-1}}{(j-1)!\,(s-1)!} t^{m/2-1} D^{2j-2} G(tx)\,\mathrm{d}t \\
&= -\sum_{j=2}^\infty \frac{x^{2j-2}}{4^{j-1}} \int_0^1 \sum_{s=0}^{j-2} \frac{(-1)^s t^s}{(j-s-2)!\,s!} \frac{(1-t)^{j-1}}{(j-1)!} t^{m/2-1} D^{2j-2} G(tx)\,\mathrm{d}t \\
&= -\sum_{j=2}^\infty \frac{x^{2j-2}}{4^{j-1}} \int_0^1 \frac{(1-t)^{j-2}}{(j-2)!} \frac{(1-t)^{j-1}}{(j-1)!} t^{m/2-1} D^{2j-2} G(tx)\,\mathrm{d}t \\
&= -\sum_{j=1}^\infty \frac{x^{2j}}{4^{j}} \int_0^1 \frac{(1-t)^{j-1}}{(j-1)!} \frac{(1-t)^{j}}{j!} t^{m/2-1} D^{2j} G(tx)\,\mathrm{d}t.
\end{aligned}$$

In a similar way, we also calculate the fifth, seventh and eighth sums in (3.21). Thus, the expression (3.21) turns into

$$\begin{aligned}
G(x) &- \sum_{i=1}^\infty \frac{(-1)^i x^{2i-1}}{2\cdot 4^{i-1}(i-1)!\,(i-1)!} \int_0^1 (1-\beta)^{i-1} \beta^{i+m/2-2} D^{2i-1} G(\beta x)\,\mathrm{d}\beta \\
&+ \sum_{i=1}^\infty \frac{(-1)^i x^{2i}}{4^i i!\,(i-1)!} \int_0^1 (1-\beta)^{i-1} \beta^{i+m/2-1} D^{2i} G(\beta x)\,\mathrm{d}\beta \\
&- \sum_{i=1}^\infty \frac{x^{2i-1}}{2\cdot 4^{i-1}(i-1)!\,(i-1)!} \int_0^1 (1-\beta)^{i-1} \beta^{m/2-1} D^{2i-1} G(\beta x)\,\mathrm{d}\beta \\
&- \sum_{i=1}^\infty \frac{x^{2i}}{4^i (i-1)!\,i!} \int_0^1 (1-\beta)^{2i-1} \beta^{m/2-1} D^{2i} G(\beta x)\,\mathrm{d}\beta \\
&- \sum_{i=1}^\infty \frac{x^{2i-1}}{2\cdot 4^{i-1}} \int_0^1 \frac{(1-\beta)^{2i-2} - (1-\beta)^{i-1}}{(i-1)!\,(i-1)!} \beta^{m/2-1} D^{2i-1} G(\beta x)\,\mathrm{d}\beta \\
&+ \sum_{i=1}^\infty \frac{x^{2i}}{4^i i!\,(i-1)!} \int_0^1 (1-\beta)^{i-1} \beta^{m/2-1} D^{2i} G(\beta x)\,\mathrm{d}\beta
\end{aligned}$$

$$+\sum_{i=1}^{\infty}\frac{x^{2i-1}}{2\cdot 4^{i-1}}\int_0^1\frac{(1-\beta)^{2i-2}-(-\beta)^{i-1}(1-\beta)^{i-1}}{(i-1)!\,(i-1)!}\beta^{m/2-1}D^{2i-1}G(\beta x)\,\mathrm{d}\beta$$
$$+\sum_{i=1}^{\infty}\frac{x^{2i}}{4^i}\int_0^1\frac{(1-\beta)^{2i-1}-(1-\beta)^{i-1}-(-\beta)^{i}(1-\beta)^{i-1}}{(i-1)!i!}\beta^{m/2-1}D^{2i}G(\beta x)\,\mathrm{d}\beta,$$

which equals $G(x)$. This means that identity (3.8) holds for the functions $f_j(x)$.

Therefore, we complete the proof. □

4. SOLUTIONS OF GENERALIZED DIRAC EQUATIONS IN CLIFFORD ANALYSIS

4.1. Solutions of the modified Dirac equation. Now we consider the modified Dirac equation in Clifford analysis

$$(D+\lambda)F(x)=0, \tag{4.1}$$

where λ is a real number.

Theorem 4.1. *If $G(x)$ is a real analytic function defined in Ω_0, then the solution of the equation* (4.1) *can be written as*

$$F(x)=f_0(x)+\sum_{i=1}^{\infty}\frac{(\lambda x)^{2i}}{4^i i!\,(i-1)!}\int_0^1(1-\alpha)^{i-1}\alpha^{m/2-1}f_0(\alpha x)\,\mathrm{d}\alpha \tag{4.2}$$
$$+\sum_{i=1}^{\infty}\frac{(\lambda x)^{2i-1}}{2\cdot 4^{i-1}(i-1)!\,(i-1)!}\int_0^1(1-\alpha)^{i-1}\alpha^{m/2-1}f_0(\alpha x)\,\mathrm{d}\alpha,$$

where

$$f_0(x)=G(x)-\sum_{s=1}^{\infty}\frac{(-1)^s x^{2s-1}}{2\cdot 4^{s-1}}\int_0^1\frac{(1-\beta)^{s-1}\beta^{s+m/2-2}}{(s-1)!\,(s-1)!}D^{2s-1}G(\beta x)\,\mathrm{d}\beta \tag{4.3}$$
$$+\sum_{s=1}^{\infty}\frac{(-1)^s x^{2s}}{4^s s!\,(s-1)!}\int_0^1(1-\beta)^{s-1}\beta^{s+m/2-1}D^{2s}G(\beta x)\,\mathrm{d}\beta.$$

Proof. Assume that $G(x)$ is a real analytic function defined in Ω_0. Then $Df_0(x)=0$ by Theorem 3.2. Furthermore, we have

$$D_0\Big(\int_0^1(1-\alpha)^{i-1}\alpha^{m/2-1}f_0(\alpha x)\,\mathrm{d}\alpha\Big)=0.$$

From Lemma 3.2, we can see that

$$D\left[x^{2i}\int_0^1(1-\alpha)^{i-1}\alpha^{m/2-1}f_0(\alpha x)\,\mathrm{d}\alpha\right]=-2ix^{2i-1}\int_0^1(1-\alpha)^{i-1}\alpha^{m/2-1}f_0(\alpha x)\,\mathrm{d}\alpha,$$

and

$$\begin{aligned}D&\left[x^{2i-1}\int_0^1(1-\alpha)^{i-1}\alpha^{m/2-1}f_0(\alpha x)\,\mathrm{d}\alpha\right]\\&\qquad=-2x^{2(i-1)}E_{m/2+i-1}\int_0^1(1-\alpha)^{i-1}\alpha^{m/2-1}f_0(\alpha x)\,\mathrm{d}\alpha.\end{aligned}$$

Differentiating both sides of the equation (4.2), we have

$$\begin{aligned}DF(x)=&-\sum_{i=1}^{\infty}\frac{\lambda^{2i}x^{2i-1}}{2\cdot 4^{i-1}(i-1)!\,(i-1)!}\int_0^1(1-\alpha)^{i-1}\alpha^{m/2-1}f_0(\alpha x)\,\mathrm{d}\alpha\\&-\sum_{i=1}^{\infty}\frac{\lambda^{2i-1}x^{2(i-1)}}{4^{i-1}(i-1)!\,(i-1)!}E_{m/2+i-1}\int_0^1(1-\alpha)^{i-1}\alpha^{m/2-1}f_0(\alpha x)\,\mathrm{d}\alpha.\end{aligned}$$

The second sum in the above expression can be written as

$$\begin{aligned}&-\lambda E_{m/2}\int_0^1\alpha^{m/2-1}f_0(\alpha x)\,\mathrm{d}\alpha\\&\qquad-\sum_{i=2}^{\infty}\frac{\lambda^{2i-1}x^{2(i-1)}}{4^{i-1}(i-1)!}E_{m/2+i-1}\int_0^1\frac{(1-\alpha)^{i-1}}{(i-1)!}\alpha^{m/2-1}f_0(\alpha x)\,\mathrm{d}\alpha\\&=-\lambda f_0(x)-\sum_{i=2}^{\infty}\frac{\lambda^{2i-1}x^{2(i-1)}}{4^{i-1}(i-1)!\,(i-2)!}\int_0^1(1-\alpha)^{i-2}\alpha^{m/2-1}f_0(\alpha x)\,\mathrm{d}\alpha\\&\qquad-\lambda f_0(x)-\sum_{i=1}^{\infty}\frac{\lambda^{2i+1}x^{2i}}{4^{i}i!\,(i-1)!}\int_0^1(1-\alpha)^{i-1}\alpha^{m/2-1}f_0(\alpha x)\,\mathrm{d}\alpha.\end{aligned}$$

To sum up, we conclude that the function $F(x)$ is a solution of the equation (4.1). $\square$

4.2. Solutions of the inhomogeneous Dirac equation. Now we consider the inhomogeneous Dirac equation in Clifford analysis

$$Dg=f(x), \tag{4.4}$$

where $f(x)\in C^{\infty}(\Omega_0,R_{0,m})$ is a real analytic function.

Theorem 4.2. *Suppose that $f(x)$ is a real analytic function defined on Ω_0. Then the function $G(x)$ given by*

$$G(x) = \sum_{i=0}^{\infty} \frac{(-1)^{i+1} x^{2i+1}}{2 \cdot 4^i \cdot i! \cdot i!} \int_0^1 (1-\alpha)^i \alpha^{m/2+i-1} D^{2i} f(\alpha x) \, \mathrm{d}\alpha + \sum_{i=0}^{\infty} \frac{(-1)^i x^{2(i+1)}}{4^{i+1} \cdot (i+1)! \cdot i!} \int_0^1 (1-\alpha)^i \alpha^{m/2+i} D^{2i+1} f(\alpha x) \, \mathrm{d}\alpha \tag{4.5}$$

is a solution of the equation (4.4).

Proof. We seek a real analytic solution of the inhomogeneous Dirac equation (4.4) in Ω_0. Suppose that Ω_0 is a star domain with center 0. Then it follows by Theorem 3.2 that

$$G(x) = f_0(x) - \sum_{i=1}^{\infty} \frac{x^{2i-1}}{2 \cdot 4^{i-1} (i-1)! \, (i-1)!} \int_0^1 (1-\alpha)^{i-1} \alpha^{m/2-1} f_{2i-1}(\alpha x) \, \mathrm{d}\alpha + \sum_{i=1}^{\infty} \frac{x^{2i}}{4^i i! \, (i-1)!} \int_0^1 (1-\alpha)^{i-1} \alpha^{m/2-1} f_{2i}(\alpha x) \, \mathrm{d}\alpha, \tag{4.6}$$

where $f_j(x)$ are monogenic functions in Ω_0 given by the relation

$$f_j(x) = D^j G(x) - \sum_{s=1}^{\infty} \frac{(-1)^s x^{2s-1}}{2 \cdot 4^{s-1}} \int_0^1 \frac{(1-\beta)^{s-1} \beta^{s-1}}{(s-1)! \, (s-1)!} \beta^{m/2-1} D^{j+2s-1} G(\beta x) \, \mathrm{d}\beta + \sum_{s=1}^{\infty} \frac{(-1)^s x^{2s}}{4^s} \int_0^1 \frac{(1-\beta)^{s-1} \beta^s}{s! \, (s-1)!} \beta^{m/2-1} D^{j+2s} G(\beta x) \, \mathrm{d}\beta. \tag{4.7}$$

Using (4.6) and (4.7), we obtain

$$\begin{aligned} G(x) - f_0(x) &= \sum_{s=1}^{\infty} \frac{(-1)^s x^{2s-1}}{2 \cdot 4^{s-1}} \int_0^1 \frac{(1-\beta)^{s-1} \beta^{s-1}}{(s-1)! \, (s-1)!} \beta^{m/2-1} D^{2s-1} G(\beta x) \, \mathrm{d}\beta \\ &\quad - \sum_{s=1}^{\infty} \frac{(-1)^s x^{2s}}{4^s} \int_0^1 \frac{(1-\beta)^{s-1} \beta^s}{s! \, (s-1)!} \beta^{m/2-1} D^{2s} G(\beta x) \, \mathrm{d}\beta \\ &= \sum_{s=0}^{\infty} \frac{(-1)^{s+1} x^{2s+1}}{2 \cdot 4^s} \int_0^1 \frac{(1-\beta)^s \beta^s}{(s)! \, (s)!} \beta^{m/2-1} D^{2s+1} G(\beta x) \, \mathrm{d}\beta \\ &\quad + \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+1)}}{4^s} \int_0^1 \frac{(1-\beta)^s \beta^{s+1}}{s! \, (s-1)!} \beta^{m/2-1} D^{2(s+1)} G(\beta x) \, \mathrm{d}\beta. \\ &= \sum_{s=0}^{\infty} \frac{(-1)^{s+1} x^{2s+1}}{2 \cdot 4^s \cdot s! \cdot s!} \int_0^1 (1-\alpha)^s \alpha^{m/2+s-1} D^{2s} f(\alpha x) \, \mathrm{d}\alpha \\ &\quad + \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+1)}}{4^{s+1} \cdot (s+1)! \cdot s!} \int_0^1 (1-\alpha)^s \alpha^{m/2+s} D^{2s+1} f(\alpha x) \, \mathrm{d}\alpha. \end{aligned}$$

The left-hand side of the resulting relation is a solution of the inhomogeneous Dirac equation (4.4); therefore, its right-hand is a solution as well. Thus, we complete the proof. ☐

4.3. Solutions of inhomogeneous poly-Dirac equations. In this section, we investigate the inhomogeneous poly-Dirac equation

$$D^k g = f(x) \tag{4.8}$$

where $f(x) \in C^\infty(\Omega_0, R_{0,m})$ is a real analytic function.

Applying Theorem 4.2, we obtain the following theorem by induction.

Theorem 4.3. *Suppose that $f(x) \in C^\infty(\Omega_0, R_{0,m})$ is a real analytic function. Then the function $G(x)$ given by*

$$\begin{aligned} G(x) = &\sum_{i=0}^{\infty} \frac{(-1)^{i+k} x^{2i+k}}{2^k \cdot 4^i \cdot i! \cdot (i+k-1)!} \int_0^1 (1-\alpha)^{i+k-1} \alpha^{m/2+i-1} D^{2i} f(\alpha x)\, \mathrm{d}\alpha \\ &+ \sum_{i=0}^{\infty} \frac{(-1)^{i+k-1} x^{2i+k+1}}{2^{k+1} \cdot 4^i \cdot (i+k)! \cdot i!} \int_0^1 (1-\alpha)^{i+k-1} \alpha^{m/2+i} D^{2i+1} f(\alpha x)\, \mathrm{d}\alpha \end{aligned}$$

is a solution of the equation (4.8).

Acknowledgement. The author would like to thank Valery V. Karachik for helpful discussions on the method of normalized systems of functions.

References

[1] *Z. R. Al-Yasiri, K. Gürlebeck*: On a boundary value problem for a p-Dirac equation. Math. Methods Appl. Sci. *39* (2016), 4056–4068. zbl MR doi

[2] *N. Aronszajn, T. M. Creese, L. J. Lipkin*: Polyharmonic Functions. Oxford Mathematical Monographs, Oxford University Press, Oxford, 1983. zbl MR

[3] *B. A. Bondarenko*: Operator Algorithms in Differential Equations. Izdatel'stvo Fan Uzbekskoj SSR, Tashkent, 1984. (In Russian.) zbl MR

[4] *F. Brackx, R. Delanghe, F. Sommen*: Clifford Analysis. Research Notes in Mathematics 76, Pitman Advanced Publishing Program, Boston, 1982. zbl MR

[5] *F. Brackx, H. De Schepper, D. Eelbode, V. Souček*: Explicit formulae for monogenic projections. Int. Conf. on Numerical Analysis and Applied Mathematics 2008 (T. Simos et al., eds.). AIP Conference Proceedings 1048, American Institute of Physics, Melville, 2008, pp. 697–700. zbl doi

[6] *D. Constales, D. Grob, R. S. Kraußhar*: Reproducing kernel functions of solutions to polynomial Dirac equations in the annulus of the unit ball in $\mathbb{R}^n$ and applications to boundary value problems. J. Math. Anal. Appl. *358* (2009), 281–293. zbl MR doi

[7] *R. Delanghe, F. Sommen, V. Souček*: Clifford Algebra and Spinor-Valued Functions. A Function Theory for the Dirac Operator. Mathematics and Its Applications 53, Kluwer Academic Publishers, Dordrecht, 1992. zbl MR doi

[8] *R. Howe*: Remarks on classical invariant theory. Trans. Am. Math. Soc. *313* (1989), 539–570. zbl MR doi

[9] *U. Kähler*: Clifford analysis and the Navier-Stokes equations over unbounded domains. Adv. Appl. Clifford Algebr. *11* (2001), 305–318. zbl MR doi

[10] *V. V. Karachik*: Polynomial solutions to systems of partial differential equations with constant coefficients. Yokohama Math. J. *47* (2000), 121–142. zbl MR

[11] *V. V. Karachik*: Normalized system of functions with respect to the Laplace operator and its applications. J. Math. Anal. Appl. *287* (2003), 577–592. zbl MR doi

[12] *V. V. Karachik*: Method of Normalized Systems of Functions. Izd. Tsentr Yuzhno-Ural'skiĭ Gosudarstvennyĭ Universitet, Chelyabinsk, 2014. (In Russian.) zbl

[13] *V. V. Karachik*: Solution of the Dirichlet problem with polynomial data for the polyharmonic equation in a ball. Differ. Equ. *51* (2015), 1033–1042 (In English. Russian original.); translation from Differ. Uravn. *51* (2015), 1038–1047. zbl MR doi

[14] *V. V. Karachik, B. Turmetov*: Solvability of some Neumann-type boundary value problems for biharmonic equations. Electron. J. Differ. Equ. *217* (2017), Paper No. 218, 17 pages. zbl MR

[15] *M. Ku, D. Wang*: Solutions to the polynomial Dirac equations on unbounded domains in Clifford analysis. Math. Methods Appl. Sci. *34* (2011), 418–427. zbl MR doi

[16] *J. Ryan*: Cauchy-Green type formulae in Clifford analysis. Trans. Am. Math. Soc. *347* (1995), 1331–1341. zbl MR doi

[17] *F. Sommen, B. Jancewicz*: Explicit solutions of the inhomogeneous Dirac equation. J. Anal. Math. *71* (1997), 59–74. zbl MR doi

[18] *H. F. Yuan*: Dirichlet type problems for Dunkl-Poisson equations. Bound. Value Probl. *2016* (2016), Article ID 222, 16 pages. zbl MR doi

[19] *H. F. Yuan*: Solutions of the Poisson equation and related equations in super spinor space. Comput. Methods Funct. Theory *16* (2016), 699–715. zbl MR doi

[20] *H. F. Yuan, V. V. Karachik*: Dunkl-Poisson equation and related equations in superspace. Math. Model. Anal. *20* (2015), 768–781. MR doi

Author's address: Hongfen Yuan, School of Mathematics and Physics, Hebei University of Engineering, Guangming South Street 199, Handan, Hebei Prov., 056038, P. R. China, e-mail: `yhf0609@163.com`.