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GEOMETRIC PROPERTIES OF LIE HYPERSURFACES IN A COMPLEX HYPERBOLIC SPACE

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Abstract. We study homogeneous real hypersurfaces having no focal submanifolds in a complex hyperbolic space. They are called Lie hypersurfaces in this space. We clarify the geometry of Lie hypersurfaces in terms of their sectional curvatures, the behavior of the characteristic vector field and their holomorphic distributions.

Keywords: complex hyperbolic space; homogeneous real hypersurface; Lie hypersurface; homogeneous ruled real hypersurface; equidistant hypersurface; horosphere; sectional curvature; shape operator; integral curve of the characteristic vector field; holomorphic distributions; homogeneous curve

MSC 2010: 53B25, 53C40

1. Introduction

In Riemannian submanifold theory it is one of the most interesting objects to investigate geometric properties of homogeneous submanifolds M^n in an ambient Riemannian manifold $\widetilde{M}^{n+p}$. Here, by definition a homogeneous submanifold M is expressed as an orbit of some closed subgroup of the full isometry group of $\widetilde{M}$. In this paper, we adopt Lie hypersurfaces and a complex hyperbolic space as homogeneous submanifolds M and an ambient space $\widetilde{M}$, respectively.

We denote by $\mathbb{C}H^n(c)$ a complex n ($n \geqslant 2$) dimensional complete and simply connected complex hyperbolic space of constant holomorphic sectional curvature c ($c < 0$). It is known that $\mathbb{C}H^n(c)$ is a Lie group itself as well as a Riemannian symmetric space of rank one. In fact, we can identify a hyperbolic space with the solvable part of the Iwasawa decomposition of the identity component of the isometry group of the hyperbolic space. A *Lie hypersurface* of a Lie group is defined as an

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orbit of a closed subgroup with codimension one (see [2]). A real hypersurface M in $\mathbb{C}H^n(c)$ is a Lie hypersurface if and only if M is homogeneous and has no focal submanifolds in $\mathbb{C}H^n(c)$. The family of all such hypersurfaces is parametrized by some interval (see also [2]).

It is known that every Lie hypersurface in $\mathbb{C}H^n(c)$ is congruent to one of the horoshere HS, the homogeneous ruled real hypersurface HR and an equidistant hypersurface M_r of HR at distance r $(0 < r < \infty)$ (cf. [2], [3]). We here remark that $\lim_{r\to 0} M_r = \mathrm{HR}$ and $\lim_{r\to\infty} M_r = \mathrm{HS}$, that is, the above parametrization gives a deformation of the homogeneous ruled real hypersurface HR to the horosphere HS through equidistant hypersurfaces M_r. The geometries of the horosphere HS and the homogeneous ruled real hypersurface HR are studied in detail (see [1], [8], [9], [11], [13]). One can see that the former is a great contrast to the latter in some sense. For example, HS is a Hopf hypersurface but HR is a non-Hopf hypersurface (see [14]). In this context, we are interested in Lie hypersurfaces of $\mathbb{C}H^n(c)$, in particular equidistant hypersurfaces M_r of the homogeneous ruled real hypersurface HR.

Our aim of this paper is to clarify geometric properties of equidistant hypersurfaces M_r of HR. We first estimate sectional curvatures of M_r. Needless to say, the sectional curvature is one of the most important and simplest geometric invariants in Riemannian geometry. Our result is an improvement of a problem that was left open in [5]. We determine the maximum and minimum values of sectional curvatures of M_r completely (Theorem 1). We next study the behavior of integral curves of the characteristic vector field of M_r. Particularly, we find an interesting relation between the shape of integral curves of the characteristic vector field and the maximum value of sectional curvatures of M_r (Theorem 2). We also investigate the derivative of the shape operator and the integrability of the holomorphic distribution of M_r.

2. Basic terminology on real hypersurfaces

In this section we summarize some basic materials about real hypersurfaces in $\mathbb{C}H^n(c)$. Let M^{2n-1} be a real hypersurface with unit normal local vector field $\mathcal{N}$ of a complex hyperbolic space $\mathbb{C}H^n(c)$ $(n \geqslant 2)$. Then it is known that an almost contact metric structure (φ, ξ, η, g) on M associated with $\mathcal{N}$ is naturally induced from the Kähler structure J of the ambient space $\mathbb{C}H^n(c)$. They are defined as

$$\xi := -J\mathcal{N}, \quad \eta(X) := g(\xi, X) \quad \text{and} \quad \varphi X := JX - \eta(X)\mathcal{N},$$

where g denotes the Riemannian metric on M induced from the standard metric g

of $\mathbb{C}H^n(c)$. They satisfy the following:

$$\varphi^2 X = -X + \eta(X)\xi, \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y),$$
$$\eta(\xi) = 1, \quad \varphi\xi = 0 \quad \text{and} \quad \eta(\varphi X) = 0$$

for all vectors $X, Y \in TM$. We call φ, ξ and η *the structure tensor*, *the characteristic vector* and *the contact form* on M, respectively.

Denote by $\widetilde{\nabla}$ and ∇ the Riemannian connections of $\mathbb{C}H^n(c)$ and M, respectively. The relation between them is given by the following formulas of Gauss and Weingarten

$$\begin{cases} \widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\mathcal{N}, \\ \widetilde{\nabla}_X \mathcal{N} = -AX \end{cases}$$

for all vector fields X and Y on M, where A is the shape operator of M in $\mathbb{C}H^n(c)$ associated with $\mathcal{N}$. Then we have

$$\nabla_X \xi = \varphi A X \tag{2.1}$$

for each $X \in TM$.

Eigenvalues and eigenvectors of the shape operator A of M are called *principal curvatures* and *principal curvature vectors* of M in $\mathbb{C}H^n(c)$, respectively. We usually call M a *Hopf hypersurface* if the characteristic vector ξ of M is a principal curvature vector at each point of M.

3. Lie hypersurfaces

In this section, we prepare some fundamental facts on Lie hypersurfaces. Let M_r be an equidistant hypersurface of the homogeneous ruled real hypersurface HR at distance r $(0 < r < \infty)$ in $\mathbb{C}H^n(c)$ $(n \geqslant 2)$. For simplicity of the notation, we put

$$t := \tanh\frac{\sqrt{|c|}r}{2}, \quad s := \operatorname{sech}\frac{\sqrt{|c|}r}{2} \tag{3.1}$$

and

$$\mu(t) := \frac{\sqrt{|c|}}{2}t(3-t^2), \quad \varrho(t) := \frac{\sqrt{|c|}}{2}t^3, \quad \lambda(t) := \frac{\sqrt{|c|}}{2}t, \tag{3.2}$$
$$\mu(s) := \frac{\sqrt{|c|}}{2}s(3-s^2), \quad \varrho(s) := \frac{\sqrt{|c|}}{2}s^3, \quad \lambda(s) := \frac{\sqrt{|c|}}{2}s.$$

We use this convention throughout the paper.

Take a point p of M_r. Then by virtue of [2] and [5] we know that there exists an orthonormal basis $\{A_0, Z_0, X_1, Y_1, \dots, X_{n-1}, Y_{n-1}\}$ of $T_p\mathbb{C}H^n(c)$ satisfying the following (see Section 2, 3 in [5] and Section 4 in [2]):

(1) The complex structure J of $T_p\mathbb{C}H^n(c)$ is given by

$$JA_0 = Z_0, \quad JZ_0 = -A_0, \quad JX_i = Y_i, \quad JY_i = -X_i \quad (1 \leqslant i \leqslant n-1).$$

(2) A unit normal vector $\mathcal{N}$ at p of M_r and the tangent space T_pM_r can be given by

$$\mathcal{N} = tA_0 - sX_1,$$
$$T_pM = \mathrm{Span}_{\mathbb{R}}\{T, Z_0, Y_1, X_2, Y_2, \dots, X_{n-1}, Y_{n-1}\},$$

where

$$T := sA_0 + tX_1. \tag{3.3}$$

(3) The Riemannian connection ∇ of M_r satisfies the following:
   (a) $\nabla_T T = 0$, $\nabla_T Y_1 = \lambda(t)Z_0$, $\nabla_T X = 0$, $\nabla_T Z_0 = -\lambda(t)Y_1$,
   (b) $\nabla_{Y_1} T = -\lambda(s)Y_1 - \lambda(t)Z_0$, $\nabla_{Y_1} Y_1 = \lambda(s)T$, $\nabla_{Y_1} X = 0$, $\nabla_{Y_1} Z_0 = \lambda(t)T$,
   (c) $\nabla_X T = -\lambda(s)X$, $\nabla_X Y_1 = 0$, $\nabla_X Y = \frac{1}{2}\sqrt{|c|}g(JX,Y)Z_0 + \lambda(s)g(X,Y)T$, $\nabla_X Z_0 = -\frac{1}{2}\sqrt{|c|}JX$,
   (d) $\nabla_{Z_0} T = -\lambda(t)Y_1 - 2\lambda(s)Z_0$, $\nabla_{Z_0} Y_1 = \lambda(t)T$, $\nabla_{Z_0} X = -\frac{1}{2}\sqrt{|c|}JX$, $\nabla_{Z_0} Z_0 = 2\lambda(s)T$,

   where T is given in (3.3) and $X, Y \in \mathrm{Span}_{\mathbb{R}}\{X_2, Y_2, \dots, X_{n-1}, Y_{n-1}\}$.

(4) The matrix representation of the shape operator A of M_r with regard to the normal vector $\mathcal{N}$ satisfies

$$A|_{\mathrm{Span}_{\mathbb{R}}\{Z_0, Y_1\}} = \begin{pmatrix} 2\lambda(t) & -\lambda(s) \\ -\lambda(s) & \lambda(t) \end{pmatrix}, \quad A|_{\mathfrak{v}} = \lambda(t)I_{2n-3},$$

where $\mathfrak{v} := \mathrm{Span}_{\mathbb{R}}\{T, X_2, Y_2, \dots, X_{n-1}, Y_{n-1}\}$.

We see from the above facts that the characteristic vector of M_r is written as

$$\xi = -J\mathcal{N} = -tZ_0 + sY_1. \tag{3.4}$$

Put

$$W := -JT = -\varphi T = -sZ_0 - tY_1, \tag{3.5}$$

which is perpendicular to ξ. We then have the following lemma immediately.

Lemma 1. *Let M_r be an equidistant hypersurface of the homogeneous ruled real hypersurface HR at distance r $(0 < r < \infty)$ in $\mathbb{C}H^n(c)$ $(n \geqslant 2)$ and $p \in M_r$. Then there exists an orthonormal basis $\{\xi, W, T, X_2, Y_2, \dots, X_{n-1}, Y_{n-1}\}$ of T_pM_r satisfying the following:*

(1) $\varphi W = T$, $\varphi X_i = Y_i$, $\varphi Y_i = -X_i$ $(2 \leqslant i \leqslant n-1)$.

(2) *The Riemannian connection ∇ of M_r is given by*

(a) $\nabla_\xi \xi = \varrho(s)T$, $\nabla_\xi W = \varrho(t)T$, $\nabla_\xi T = -\varrho(s)\xi - \varrho(t)W$, $\nabla_\xi X_i = \lambda(t)Y_i$, $\nabla_\xi Y_k = -\lambda(t)Y_k$,

(b) $\nabla_W \xi = \varrho(t)T$, $\nabla_W W = \mu(s)T$, $\nabla_W T = -\varrho(t)\xi - \mu(s)W$, $\nabla_W X_i = \lambda(s)Y_i$, $\nabla_W Y_k = -\lambda(s)X_k$,

(c) $\nabla_T \xi = -\lambda(t)W$, $\nabla_T W = \lambda(t)\xi$, $\nabla_T T = \nabla_T X_i = \nabla_T Y_k = 0$,

(d) $\nabla_{X_i} \xi = \lambda(t)Y_i$, $\nabla_{X_i} W = \lambda(s)Y_i$, $\nabla_{X_i} T = -\lambda(s)X_i$, $\nabla_{X_i} X_j = \delta_{ij}\lambda(s)T$, $\nabla_{X_i} Y_k = -\delta_{ik}\{\lambda(t)\xi + \lambda(s)W\}$,

(e) $\nabla_{Y_k} \xi = -\lambda(t)X_k$, $\nabla_{Y_k} W = -\lambda(s)X_k$, $\nabla_{Y_k} T = -\lambda(s)Y_k$, $\nabla_{Y_k} X_i = \delta_{ki}\{\lambda(t)\xi + \lambda(s)W\}$, $\nabla_{Y_k} Y_l = \delta_{kl}\lambda(s)T$.

(3) *The matrix representation of the shape operator A of M_r with respect to an orthogonal decomposition $T_pM_r = \mathrm{Span}_{\mathbb{R}}\{\xi, W\} \oplus \mathfrak{v}$ satisfies*

$$A|_{\mathrm{Span}_{\mathbb{R}}\{\xi,W\}} = \begin{pmatrix} \mu(t) & \varrho(s) \\ \varrho(s) & \varrho(t) \end{pmatrix}, \quad A|_{\mathfrak{v}} = \lambda(t)I_{2n-3},$$

where $\mathfrak{v} := \mathrm{Span}_{\mathbb{R}}\{T, X_2, Y_2, \dots, X_{n-1}, Y_{n-1}\}$.

4. Sectional curvatures

In this section, we study sectional curvatures of every equidistant hypersurface M_r at distance r $(0 < r < \infty)$ from HR in $\mathbb{C}H^n(c)$ $(n \geqslant 2)$. Denoting the curvature tensor of M_r by R, we have the equation of Gauss given by

$$\begin{aligned} g((R(X,Y)Z,U) = \frac{c}{4}\{&g(Y,Z)g(X,U) - g(X,Z)g(Y,U) + g(\varphi Y,Z)g(\varphi X,U) \\ &- g(\varphi X,Z)g(\varphi Y,U) - 2g(\varphi X,Y)g(\varphi Z,U)\} \\ &+ g(AY,Z)g(AX,U) - g(AX,Z)g(AY,U) \end{aligned}$$

for vector fields X, Y, Z, U on M_r. The sectional curvature K of M_r is defined by $K(X,Y) := g(R(X,Y)Y,X)$, where X and Y are orthonormal vectors on M_r. Then the equation of Gauss yields that

$$K(X,Y) = \frac{c}{4}(1 + 3g(\varphi X,Y)^2) + g(AX,X)g(AY,Y) - g(AX,Y)^2. \tag{4.1}$$

Theorem 1. *Let M_r be an equidistant hypersurface of the homogeneous ruled real hypersurface HR at distance r $(0 < r < \infty)$ in $\mathbb{C}H^n(c)$ $(n \geqslant 2)$. Set $t := \tanh \frac{1}{2}\sqrt{|c|}r$. Then the maximum and the minimum values of its sectional curvature K are given as follows:*

$$\max K = \begin{cases} \dfrac{c}{8}\bigl(2 - 3t^2 - t\sqrt{4 - 3t^2}\bigr) & (n \geqslant 3), \\ \dfrac{c}{8}\bigl(5 - 3t^2 - \sqrt{-15t^4 + 22t^2 + 9}\bigr) & (n = 2), \end{cases} \tag{4.2}$$

$$\min K = \frac{c}{8}\bigl(5 - 3t^2 + \sqrt{-15t^4 + 22t^2 + 9}\bigr) \quad (n \geqslant 2). \tag{4.3}$$

P r o o f. Note that $\max K$ for $n \geqslant 2$ and also $\min K$ for $n = 2$ have been calculated in [5]. Hence we have only to determine $\min K$ for $n \geqslant 3$. In the following we will use the notation (3.2) and Lemma 1.

Let P be an arbitrary 2-plane in the tangent space T_pM_r at an arbitrary point $p \in M_r$. Then there exist orthonormal vectors X, Y which are orthogonal to ξ, such that the pair of vectors $\{X', Y\}$, where $X' := (\sin\theta)X + (\cos\theta)\xi$, forms an orthonormal basis of the plane P for some $\theta \in \mathbb{R}$. Since X is orthogonal to ξ, one has $X \in \mathrm{Span}_{\mathbb{R}}\{W\} \oplus \mathfrak{v}$. Thus we can write

$$X = g(X,W)W + (X - g(X,W)W), \quad (X - g(X,W)W) \in \mathfrak{v}.$$

We then see from Lemma 1, case (3) that

$$AX = g(X,W)(\varrho(s)\xi + \varrho(t)W) + \lambda(t)(X - g(X,W)W).$$

We also have a similar expression of AY. By using these, one can calculate $K(X', Y)$ in terms of (4.1). For simplicity we put

$$K := K(X', Y), \quad x := g(X,W), \quad y := g(Y,W), \quad z := g(\varphi X, Y). \tag{4.4}$$

Thus, a straightforward calculation gives

$$\begin{aligned} K = \frac{|c|}{4}\{&-1 + t^2(3 - t^2)\cos^2\theta + t^2\sin^2\theta - t^2(1 - t^2)x^2\sin^2\theta \\ &+ 2t(1 - t^2)^{3/2}x\sin\theta\cos\theta - (1 - t^2)y^2(t^2 + \cos^2\theta) - 3z^2\sin^2\theta\}. \end{aligned} \tag{4.5}$$

We denote the right-hand side of (4.5) by $F = F(x, y, z, t, \theta)$. The variables satisfy

$$|x| \leqslant 1, \ |y| \leqslant 1, \ |z| \leqslant 1, \ 0 < t < 1, \ 0 \leqslant \theta < 2\pi.$$

Needless to say, these variables are not independent. However, we here show that the following inequality holds, even if we think of $F(x,y,z,t,\theta)$ as a function of independent variables x, y, z, t, θ:

$$F(x,y,z,t,\theta) \geqslant \frac{c}{8}\big(5 - 3t^2 + \sqrt{-15t^4 + 22t^2 + 9}\big). \tag{4.6}$$

First, we note that there is only one term of F containing z. Since $|z| \leqslant 1$, one can easily see

$$F \geqslant F|_{z^2=1}.$$

By a direct calculation, one has

$$F|_{z^2=1} = \frac{|c|}{8}\{-5 - y^2 - t^2(-4 + x^2 + y^2) + t^4(-1 + x^2 + 2y^2) + A\cos 2\theta + B\sin 2\theta\},$$

where $A = A(x,y,t)$ and $B = B(x,y,t)$ are defined by

$$A = -\{-3 + t^4(1 + x^2) + y^2 - t^2(2 + x^2 + y^2)\}, \quad B = 2t(1 - t^2)^{3/2}x.$$

Putting $R = R(x,y,t) = \sqrt{A^2 + B^2}$, we have $A\cos 2\theta + B\sin 2\theta \geqslant -R$. Hence one has

$$F|_{z^2=1} \geqslant \frac{|c|}{8}\{-5 - y^2 - t^2(-4 + x^2 + y^2) + t^4(-1 + x^2 + 2y^2) - R\}.$$

Let $G = G(x,y,t)$ denote the right-hand side of this inequality again. We next change the coordinates (x,y) to (u,α) by

$$x = u\cos\alpha, \quad y = u\sin\alpha \quad (0 \leqslant u \leqslant 1,\ 0 \leqslant \alpha < 2\pi).$$

Then one can express $G = G(u,\alpha,t)$ as

$$G = \frac{|c|}{16}\{-10 + 8t^2 - 2t^4 - u^2(1 - t^2)(1 + 3t^2) + u^2(1 - t^4)\cos 2\alpha - 2R\}$$

with $R = R(u,\alpha,t) = \sqrt{A^2 + B^2}$, where

$$A^2 = \frac{1}{4}\{2(t^2 - 3)(t^2 + 1) + u^2(t^2 - 1)^2 + u^2(t^4 - 1)\cos 2\alpha\}^2,$$
$$B^2 = 2u^2t^2(1 - t^2)^3(1 + \cos 2\alpha).$$

Let us set $s = \cos 2\alpha$ afresh. Then, it satisfies $|s| \leqslant 1$, and we get by differentiating the function $G = G(u,s,t)$ with respect to s

$$\frac{\partial G}{\partial s} = \frac{|c|}{16}\Big\{u^2(1 - t^4) - \frac{1}{R}\frac{\partial}{\partial s}(A^2 + B^2)\Big\},$$

where

$$\frac{\partial}{\partial s}(A^2+B^2) = \frac{1}{2}(1-t^2)u^2\{14t^2+2t^6+6(1-t^4)+(-1+s+t^2+st^2)(1-t^4)u^2\}. \tag{4.7}$$

In order to see $\partial G/\partial s \leqslant 0$, we recall that $0 < t < 1$, $0 \leqslant u \leqslant 1$, $|s| \leqslant 1$. Then $u^2(1-t^4) \geqslant 0$ and we have $\partial(A^2+B^2)/\partial s \geqslant 0$ by substituting $s = -1$ in (4.7). After a computation we obtain

$$\{u^2(1-t^4)\}^2 - \left\{\frac{1}{R}\frac{\partial}{\partial s}(A^2+B^2)\right\}^2 = -\frac{4}{R^2}(1-t^2)^4t^2u^4\{2+6t^2+(1-t^4)(1-u^2)\} \leqslant 0.$$

This implies $\partial G/\partial s \leqslant 0$. We hence conclude that the function G takes its minimum with respect to s at $s = 1$. Applying the same procedure to $G|_{s=1}$, one can find the function $G|_{s=1}$ takes its minimum with respect to u at $u = 1$. Thus, we obtain $F \geqslant G|_{s=1,u=1}$, hence the inequality (4.6) follows.

It remains to show that the equality of (4.6) is attained. Let us put $X = W$, $Y = \varphi W$. Then we have $x = 1$, $y = 0$, $z = 1$ from (4.4), so that F becomes

$$F|_{x=1,y=0,z=1} = \frac{|c|}{8}\{-5+3t^2+A\cos 2\theta + B\sin 2\theta\},$$

where $A = -2t^4+3t^2+3$, $B = 2t(1-t^2)^{3/2}$. Since θ is an independent variable of X, Y, the part $A\cos 2\theta + B\sin 2\theta$ of $F|_{x=1,y=0,z=1}$ can take the value $-R = -\sqrt{A^2+B^2} = -\sqrt{-15t^4+22t^2+9}$. This proves the relation (4.3). $\square$

Remark 1. It is easy to see from (4.2) that the maximum values of K are monotone increasing functions of the distance r. By an elementary computation we observe that $\max K = 0$ implies

$$r = \frac{1}{\sqrt{|c|}}\log\frac{2\sqrt{3}+\sqrt{13-\sqrt{73}}}{2\sqrt{3}-\sqrt{13-\sqrt{73}}} \quad \text{for } n = 2$$

and

$$r = \frac{1}{\sqrt{|c|}}\log(2+\sqrt{3}) \quad \text{for } n \geqslant 3.$$

5. The integral curve of the characteristic vector field

We denote by $\mathbb{C}M^n(c)$ a complex n-dimensional complete and simply connected complex space form of constant holomorphic sectional curvature c, that is, $\mathbb{C}M^n(c)$ is a complex projective space $\mathbb{C}P^n(c)$, a complex hyperbolic space $\mathbb{C}H^n(c)$ or a complex Euclidean space $\mathbb{C}^n$ according as c is positive, negative or zero.

First of all, we shall review the real curve theory in $\mathbb{C}M^n(c)$. Let $\gamma = \gamma(s)$ be a smooth real curve parametrized by its arclenth s in $\mathbb{C}M^n(c)$. The curve γ is said to be a *Frenet curve of proper order* d $(2 \leqslant d \leqslant 2n)$ if there exist an orthonormal system $\{V_1 := \dot{\gamma}, V_2, \ldots, V_d\}$ of vector fields along γ and positive smooth functions $\kappa_1(s), \ldots, \kappa_{d-1}(s)$ satisfying:

$$\widetilde{\nabla}_{\dot{\gamma}} V_j(s) = -\kappa_{j-1}(s)V_{j-1}(s) + \kappa_j(s)V_{j+1}(s), \quad j = 1, \ldots, d,$$

where $\kappa_0 V_0 \equiv \kappa_d V_{d+1} \equiv 0$, and $\widetilde{\nabla}_{\dot{\gamma}}$ denotes the covariant differentiation along γ with respect to the standard Riemannian connection $\widetilde{\nabla}$ of $\mathbb{C}M^n(c)$. The functions $\kappa_1, \ldots, \kappa_{d-1}$ and the orthonormal frames $\{V_1, \ldots, V_d\}$ are called the *curvatures* and the *Frenet frame* of the curve γ, respectively. We call a Frenet curve a *helix* when all of its curvatures $\kappa_1, \ldots, \kappa_{d-1}$ are constant functions. A helix of proper order 2, that is to say, a curve which satisfies

$$\widetilde{\nabla}_{\dot{\gamma}} \dot{\gamma} = kV_2, \quad \widetilde{\nabla}_{\dot{\gamma}} V_2 = -k\dot{\gamma}$$

for some positive constant k is called a *circle* of curvature k. We regard a geodesic as a circle with null curvature.

For the Frenet frame $\{V_1, \ldots, V_d\}$ of a Frenet curve γ, we set

$$\tau_{ij}(s) := g(V_i(s), JV_j(s)) \quad (1 \leqslant i < j \leqslant d)$$

and call them the *holomorphic torsions* along γ. In the real curve theory in $\mathbb{C}M^n(c)$, the notion of holomorphic torsions plays an important role. A real curve γ in $\mathbb{C}M^n(c)$ is said to be *homogeneous* if it is an orbit of one-parameter subgroup of the full isometry group $\mathrm{I}(\mathbb{C}M^n(c))$ of $\mathbb{C}M^n(c)$. It is known that there exit many helices which are not homogeneous curves in a nonflat complex space form $\mathbb{C}M^n(c)$ $(c \neq 0)$ (see [10]). We can give the necessary and sufficient condition for a Frenet curve to be homogeneous in $\mathbb{C}M^n(c)$ by using the notion of holomorphic torsions as follows:

Theorem A ([12]). *A Frenet curve γ is homogeneous in a complex space form $\mathbb{C}M^n(c)$ if and only if it is a helix and all of its holomorphic torsions are constant functions.*

For a circle γ with positive curvature k in $\mathbb{C}M^n(c)$, we have just one holomorphic torsion $\tau_{12}(s) = g(V_1(s), JV_2(s))$. By easy computation one can find that the holomorphic torsion τ_{12} of the circle γ is automatically constant. Hence by Theorem A every circle of positive curvature is homogeneous in $\mathbb{C}M^n(c)$.

Now, we study integral curves of the characteristic vector field of Lie hypersurfaces in a complex hyperbolic space $\mathbb{C}H^n(c)$ $(n \geqslant 2)$. We note that every integral curve of the characteristic vector of an arbitrary Hopf hypersurface M is a circle of common curvature and of common holomorphic torsion $\tau_{12} = \pm 1$ in $\mathbb{C}H^n(c)$. Hence, so it does on a horosphere HS in $\mathbb{C}H^n(c)$. On the other hand, every integral curve of the characteristic vector of the homogeneous ruled real hypersurface HR in $\mathbb{C}H^n(c)$ is a circle of common curvature $\frac{1}{2}\sqrt{|c|}$ and of common holomorphic torsion $\tau_{12} = 0$ (see [1], [8]).

Theorem 2. *Let M_r be an equidistant hypersurface of the homogeneous ruled real hypersurface HR at distance r $(0 < r < \infty)$ in $\mathbb{C}H^n(c)$ $(n \geqslant 2)$. Then the integral curve γ of the characteristic vector field ξ of M_r is a homogeneous curve of proper order 3 or 4 in the ambient space $\mathbb{C}H^n(c)$. Moreover, for $n \geqslant 3$, the curve γ is of proper order 3 if and only if the sectional curvature K of M_r satisfies $\max K = 0$.*

Proof. Let $\widetilde{\nabla}$ and ∇ denote the Riemannian connections of $\mathbb{C}H^n(c)$ and M_r, respectively, and let $\mathcal{N}$ be a unit normal local vector field of M_r in $\mathbb{C}H^n(c)$. In the following calculation we use notation in Section 3 and Lemma 1.

Setting $V_1 := \dot{\gamma} = \xi$, by the formula of Gauss, (2.1) and Lemma 1, case (3) we have

$$\widetilde{\nabla}_{\dot{\gamma}} V_1 = \varphi A\xi + g(A\xi, \xi)\mathcal{N} = \varrho(s)T + \mu(t)\mathcal{N}. \tag{5.1}$$

We here put

$$\kappa_1 := \sqrt{\varrho(s)^2 + \mu(t)^2}\ (> 0) \quad \text{and} \quad V_2 := \frac{1}{\kappa_1}(\varrho(s)T + \mu(t)\mathcal{N}). \tag{5.2}$$

Formulas of Gauss and Weingarten and Lemma 1, the case (3) yield

$$\widetilde{\nabla}_\xi T = -\varrho(s)\xi - \varrho(t)W, \quad \widetilde{\nabla}_\xi \mathcal{N} = -\mu(t)\xi - \varrho(s)W. \tag{5.3}$$

By using (5.3) and the fact that $\mu(t) + \varrho(t) = 3\lambda(t)$, we see

$$\begin{aligned}\widetilde{\nabla}_{\dot{\gamma}} V_2 &= \frac{1}{\kappa_1}(\varrho(s)\widetilde{\nabla}_\xi T + \mu(t)\widetilde{\nabla}_\xi \mathcal{N}) = \frac{1}{\kappa_1}(-(\varrho(s)^2 + \mu(t)^2)\xi - \varrho(s)(\mu(t) + \varrho(t))W) \\ &= -\kappa_1 V_1 - \frac{3\lambda(t)\varrho(s)}{\kappa_1} W.\end{aligned}$$

Set

$$\kappa_2 := \frac{3\lambda(t)\varrho(s)}{\kappa_1}\ (> 0) \quad \text{and} \quad V_3 := -W. \tag{5.4}$$

Then, we have

$$\widetilde{\nabla}_{\dot\gamma}V_3 = -\varrho(t)T - \varrho(s)\mathcal{N} = -\kappa_2 V_2 + \kappa_2 V_2 - \varrho(t)T - \varrho(s)\mathcal{N}$$
$$= -\kappa_2 V_2 + \frac{\kappa_2}{\kappa_1}(\varrho(s)T + \mu(t)\mathcal{N}) - \varrho(t)T - \varrho(s)\mathcal{N}$$
$$= -\kappa_2 V_2 + \Big(\frac{\kappa_2}{\kappa_1}\varrho(s) - \varrho(t)\Big)T + \Big(\frac{\kappa_2}{\kappa_1}\mu(t) - \varrho(s)\Big)\mathcal{N}. \tag{5.5}$$

Here, one finds from (5.4) and (5.2) that

$$\frac{\kappa_2}{\kappa_1}\varrho(s) - \varrho(t) = \frac{3\lambda(t)\varrho(s)^2}{\kappa_1^2} - \varrho(t) = \frac{1}{\kappa_1^2}((\mu(t)+\varrho(t))\varrho(s)^2 - (\mu(t)^2 + \varrho(s)^2)\varrho(t))$$
$$= -\frac{\mu(t)}{\kappa_1^2}(\mu(t)\varrho(t) - \varrho(s)^2)$$

and similarly

$$\frac{\kappa_2}{\kappa_1}\mu(t) - \varrho(s) = \frac{\varrho(s)}{\kappa_1^2}(\mu(t)\varrho(t) - \varrho(s)^2).$$

Thus, (5.5) implies

$$\widetilde{\nabla}_{\dot\gamma}V_3 = -\kappa_2 V_2 + \frac{1}{\kappa_1^2}(\mu(t)\varrho(t) - \varrho(s)^2)(-\mu(t)T + \varrho(s)\mathcal{N}). \tag{5.6}$$

Definition (3.2) gives

$$\mu(t)\varrho(t) - \varrho(s)^2 = \frac{|c|}{4}\Big(3\tanh^2\frac{\sqrt{|c|}r}{2} - 1\Big).$$

Set $r_0 := \big(1/\sqrt{|c|}\big)\log\big(2+\sqrt{3}\big)$. We here consider the following three cases:

(i) The case of $0 < r < r_0$. In this case, we have $0 < \tanh\frac{1}{2}\sqrt{|c|}r < 1/\sqrt{3}$ and $\mu(t)\varrho(t) - \varrho(s)^2 < 0$. Accordingly, from (5.6) we can put

$$\kappa_3 : = -\frac{1}{\kappa_1^2}(\mu(t)\varrho(t) - \varrho(s)^2)\| - \mu(t)T + \varrho(s)\mathcal{N}\| = -\frac{\mu(t)\varrho(t) - \varrho(s)^2}{\kappa_1},$$
$$V_4 : = -\frac{-\mu(t)T + \varrho(s)\mathcal{N}}{\| - \mu(t)T + \varrho(s)\mathcal{N}\|} = -\frac{1}{\kappa_1}(-\mu(t)T + \varrho(s)\mathcal{N}). \tag{5.7}$$

One can then see

$$\widetilde{\nabla}_{\dot\gamma}V_4 = \frac{1}{\kappa_1}(-\mu(t)\varrho(t) + \varrho(s)^2)W = -\kappa_3 V_3$$

by use of (5.3), (5.4) and (5.7). Since all the curvatures κ_1, κ_2, κ_3 are constant, the curve γ in this case is a helix of proper order 4.

(ii) The case of $r = r_0$. In this case, we have $\tanh \frac{1}{2}\sqrt{|c|}r = 1/\sqrt{3}$ and $\mu(t)\varrho(t) - \varrho(s)^2 = 0$. Then, (5.6) reduces to $\widetilde{\nabla}_{\dot\gamma}V_3 = -\kappa_2 V_2$, so that the curve γ is a helix of proper order 3.

(iii) The case of $r > r_0$. We have $\mu(t)\varrho(t) - \varrho(s)^2 > 0$. Then, in the same way as in case (i), we find $\kappa_3 = (\mu(t)\varrho(t) - \varrho(s)^2)/\kappa_1$, $V_4 = (-\mu(t)T + \varrho(s)\mathcal{N})/\kappa_1$ and γ is a helix of proper order 4.

We need to verify that all holomorphic torsions $\tau_{ij} = g(V_i, JV_j)$ are constant along γ in each case. But this is clear because $\lambda(t)$, $\mu(t)$, $\varrho(t)$, $\varrho(s)$ are all constant. Therefore, by Theorems A the integral curve γ of the characteristic vector field ξ of M_r is a homogeneous curve in $\mathbb{C}H^n(c)$ of proper order 3 or 4.

The last assertion of Theorem 2 clearly follows from Remark 1. $\square$

6. The shape operator and the holomorphic distribution

In this section we will investigate the length of the derivative of the shape operator restricted to the holomorphic distribution and also investigate the integrability of the holomorphic distribution of Lie hypersurfaces M in $\mathbb{C}H^n(c)$. The *holomorphic distribution* $\mathfrak{D}^0$ is defined as:

$$\mathfrak{D}^0 := \{X \in TM\colon\ \eta(X) = 0\},$$

where η is the contact form on M.

First of all, we recall some facts. There exist no real hypersurfaces with parallel shape operator A in a nonflat complex space form $\mathbb{C}M^n(c)$ ($c \neq 0$). Kimura and the second author [6] introduced the notion of η-parallelism and classified Hopf hypersurfaces having η-parallel shape operator in a complex projective space $\mathbb{C}P^n(c)$. The shape operator A of M is said to be *η-parallel* if $g((\nabla_X A)Y, Z) = 0$ for all vectors X, Y and Z in $\mathfrak{D}^0$. Besides, for Hopf hypersurfaces M in a nonflat complex space form $\mathbb{C}M^n(c)$ ($n \geqslant 2$), the holomorphic distribution $\mathfrak{D}^0$ on M is not integrable (see [4]). It is known that the holomorphic distribution $\mathfrak{D}^0$ is integrable if and only if $g((\varphi A + A\varphi)X, Y)$ vanishes for any $X, Y \in \mathfrak{D}^0$. Motivated by these facts, we establish the following.

Proposition 1. *Let M_r be an equidistant hypersurface of the homogeneous ruled real hypersurface HR at distance r ($0 < r < \infty$) in $\mathbb{C}H^n(c)$ ($n \geqslant 2$), and let $\{e_1, \ldots, e_{2n-2}\}$ be an orthonormal basis of the holomorphic distribution $\mathfrak{D}^0$ of M_r. Denote by A the shape operator of M_r in $\mathbb{C}H^n(c)$. Then we have the following.*

(1) *Set*

$$\|\nabla^0 A\| := \sqrt{\sum_{i,j,k} g((\nabla_{e_i} A)e_j, e_k)^2}.$$

Then

$$\|\nabla^0 A\| = \frac{\sqrt{3}|c|}{2}\operatorname{sech}^3 \frac{\sqrt{|c|}r}{2} \tanh \frac{\sqrt{|c|}r}{2}, \tag{6.1}$$

which takes its maximum value when $r = (\log 3)/\sqrt{|c|}$. *Moreover, we have* $\|\nabla^0 A\| > 0$ *and* $\lim\limits_{r\to 0} \|\nabla^0 A\| = \lim\limits_{r\to\infty} \|\nabla^0 A\| = 0$, *so that both the horosphere HS and the homogeneous ruled real hypersurface HR are* η*-parallel, but it is not so for* M_r.

(2) *Set*

$$\|\Psi^0\| := \sqrt{\sum_{i,j} g((\varphi A + A\varphi)e_i, e_j)^2}\,.$$

Then

$$\|\Psi^0\| = \sqrt{\frac{|c|}{2}} \tanh \frac{\sqrt{|c|}r}{2} \sqrt{\Big(\tanh^2 \frac{\sqrt{|c|}r}{2} + 1\Big)^2 + 4(n-2)}, \tag{6.2}$$

which is a monotone increasing function of the distance r *from HR. Furthermore, we have* $\lim\limits_{r\to 0} \|\Psi^0\| = 0$, *so that the holomorphic distribution* $\mathfrak{D}^0$ *of the homogeneous ruled real hypersurface HR is integrable, but it is not so for the other Lie hypersurfaces in* $\mathbb{C}H^n(c)$.

P r o o f. We employ $\{W, T, X_2, Y_2, \ldots, X_{n-1}, Y_{n-1}\}$, which is given in Lemma 1, as an orthonormal basis of $\mathfrak{D}^0$. Then we obtain (6.1) and (6.2) by straightforward calculations. Let us set $t := \tanh \frac{1}{2}\sqrt{|c|}r$. Then it satisfies $0 < t < 1$, and one can write (6.1) as

$$\|\nabla^0 A\| = \frac{\sqrt{3}|c|}{2} t\sqrt{(1-t^2)^3}.$$

Denoting the right-hand side of this relation by $f = f(t)$, we get

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\sqrt{3}|c|}{2}(1-4t^2)\sqrt{1-t^2}.$$

Equation $1 - 4t^2 = 0$ implies $r = (\log 3)/\sqrt{|c|}$. This proves the case (1) of our proposition. The rest is easy. $\square$

Remark 2. Kon and Loo obtained the complete classification of η-parallel real hypersurfaces in a nonflat complex space form $\mathbb{C}M^n(c)$ for $n \geqslant 3$. See [7].

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