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NOTE ON IMPROPER COLORING OF 1-PLANAR GRAPHS

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Abstract. A graph $G = (V, E)$ is called improperly $(d_1, \ldots, d_k)$-colorable if the vertex set V can be partitioned into subsets $V_1, \ldots, V_k$ such that the graph $G[V_i]$ induced by the vertices of V_i has maximum degree at most d_i for all $1 \leqslant i \leqslant k$. In this paper, we mainly study the improper coloring of 1-planar graphs and show that 1-planar graphs with girth at least 7 are $(2, 0, 0, 0)$-colorable.

Keywords: improper coloring; 1-planar graph; discharging method

MSC 2010: 05C15

1. Introduction

Throughout this paper, we only consider simple and undirected graphs. Let G be a finite graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex v in G, a *neighbor* of v is a vertex adjacent to v and the set of neighbors of v is denoted by $N_G(v)$, and the *degree* of v, denoted by $d_G(v)$, is the number of neighbors of v. The minimal degree of the vertices of G is denoted by $\delta(G)$. The *length* of a cycle is the number of its edges, and the *girth* $g(G)$ of a graph G is the length of the shortest cycle. For notations and terminology not given, see e.g., Bondy and Murty in [2].

Let $d_1, \ldots, d_k$ be k nonnegative integers. A graph $G = (V, E)$ is called *improperly $(d_1, \ldots, d_k)$-colorable*, or just *$(d_1, \ldots, d_k)$-colorable*, if the vertex set V can be partitioned into subsets $V_1, \ldots, V_k$ such that the graph $G[V_i]$ induced by the vertices of V_i has maximum degree at most d_i for all $1 \leqslant i \leqslant k$. This notion generalizes the notion of proper k-coloring in which case $d_1 = \ldots = d_k = 0$. The Four Color Theorem (saying that every planar graph is $(0, 0, 0, 0)$-colorable) was proved

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by Appel and Haken (see [1]) using a computer. In 1976, Steinberg proposed that every planar graph without cycles of length 4 or 5 is $(0,0,0)$-colorable. Recently, Cohen-Addad et al. in [11] disproved the conjecture by constructing a planar graph with no cycles of length four and five that is not 3-colorable. Several results about improper coloring of planar graphs without 4-cycles and 5-cycles can be seen in [8] and [10]. Borodin et al. in [7] proved that every planar graph with maximum average degree at most $\frac{14}{5}$ is $(1,1)$-colorable, it follows that every planar graph with girth at least 7 is $(1,1)$-colorable. Similarly, it was shown that every planar graph with girth at least 7 is $(0,4)$-colorable (see [5]).

A graph is 1-*planar* if it can be drawn on the plane so that each edge is crossed by at most one other edge. The notion of 1-planar graphs was introduced by Ringel (see [15]). His conjecture that each 1-planar graph is 6-colorable was confirmed by Borodin (see [3] and [4]), Borodin gave a new simpler proof. The bound 6 is sharp, see the 1-planar drawing of K_6. Borodin also showed that each 1-planar graph is (list) acyclically 20-colorable (see [6]). Zhang et al. in [18], [19], [20], [21] proved several results on edge colorings of 1-planar graphs. On the other hand, the local structure and properties of 1-planar graphs were studied extensively. The further results can be found in [12], [13], [14], [17]. In [13], it was also conjectured that any 1-planar graph of girth at least 6 would be of minimum degree at most 3, hence, it would be $(0,0,0,0)$-colourable. Chen et al. in [9] proved that it is NP-complete to decide whether a given 1-planar graph is $(0,0,0,0)$-colorable.

In this paper, we present the following theorem.

Theorem 1.1. *1-planar graphs with girth at least 7 are $(2,0,0,0)$-colorable.*

2. Preliminaries

In this section, we start with some basic concepts and definitions. Then we will list some lemmas, which will be used in the following sections.

Let G be a graph and v a vertex of G. Call v a *k-vertex*, a *k^+-vertex* or a *k^--vertex* if $d(v)=k$, $d(v)\geqslant k$ or $d(v)\leqslant k$, respectively. A graph G is 1-*planar* if it can be drawn on the plane so that each edge is crossed by at most one other edge. The *associated plane graph G^** of a 1-planar graph G (in the sequel, we will assume that G is already drawn in the plane in 1-plane way) is the plane graph obtained from G by turning each crossing of G into a new 4-vertex, called a *crosser*. One can easily observe that if v is not a crosser (call v an *original vertex*), then $d_{G^*}(v)=d_G(v)$. Therefore in the following, we do not distinguish the two notations $d_{G^*}(v)$ and $d_G(v)$ when v is an original vertex, in which case we only use the brief notation $d(v)$ to represent both $d_{G^*}(v)$ and $d_G(v)$.

Let $F(G^*)$ denote the face set of G^*. For a face $f \in F(G^*)$, the number of edges of f, denoted by $d(f)$, is called the *degree* of f. The *k-face* and *k^+-face* can be defined similarly. For two original vertices u and v, we say u is *true adjacent* to v in G^*, or u is a *true neighbor* of v if u is adjacent to v in G. An original k-vertex v is called a *k^i-vertex* if it is incident with i 3-faces. We say that a 4^2-vertex v is special if v is incident with one 4-face and one k^+-face ($k \geqslant 6$). And a 4^1-vertex u is special if u is incident with two 4-faces and one k^+-face ($k \geqslant 5$).

Now, we discuss some properties of a 1-planar graph G and its associated plane graph G^*, which can be found in [16] and [18].

Lemma 2.1 ([16]). *Let G be a triangle-free 1-planar graph and G^* be its associated plane graph. Then for every vertex $v \in V(G)$ if $d(v) \geqslant 4$, then v is incident with at most $\lfloor \frac{2}{3} d_G(v) \rfloor$ 3-faces in G^*.*

Lemma 2.2 ([18]). *Let G be a 1-planar graph and G^* be the associated plane graph of G. Then for any two crossers u and v in G^*, $uv \notin E(G^*)$.*

3. Structural properties of the minimal counterexample and its associated plane graph

Let $C = \{1, 2, 3, 4\}$ denote the color set with four colors. Suppose G is a minimal 1-planar graph with girth at least 7 which is not $(2, 0, 0, 0)$-colorable. Thus, G is connected. Moreover, every subgraph G' of G has a $(2, 0, 0, 0)$-coloring using color set C. In other words, $V(G')$ is partitioned into four subsets V_1, V_2, V_3 and V_4 such that $\Delta(G[V_1]) \leqslant 2$, $\Delta(G[V_i]) = 0$ $(i = 2, 3, 4)$. To properly color a vertex v means to color v with a color which has not been assigned to any neighbor of v. Now suppose that the vertices in $G[V_i]$ are colored with i, where $i = 1, 2, 3, 4$. Let G^* be the associated plane graph of G. We will give some properties of G and G^* as follows.

Property 3.1. $\delta(G^*) \geqslant 4$.

Proof. Suppose to the contrary that G contains a 3^--vertex v such that $v_1, \ldots, v_k$ $(1 \leqslant k \leqslant 3)$ are neighbors of v. Let $G' = G - v$. By the minimality of G, G' has a $(2, 0, 0, 0)$-coloring φ using color set C. We may color v with a color in $C \setminus \{\varphi(v_i),\ i = 1, 2, \ldots, k\}$. This contradicts the choice of G, so the property holds. $\square$

Property 3.2. *Every 5^--vertex in G is adjacent to at least one 6^+-vertex.*

P r o o f. Since the proof for 4-vertex is similar to the proof of 5-vertex, we only prove that every 5-vertex in G is adjacent to at least one 6^+-vertex.

Let u_1, u_2, u_3, u_4 and u_5 be the five neighbors of a 5-vertex u. Suppose that all of u_1, u_2, u_3, u_4 and u_5 are 5^--vertices. Let $G' = G - u$. By the minimality of G, G' has a $(2,0,0,0)$-coloring φ using color set C.

Case 1: If $\{\varphi(u_1), \varphi(u_2), \varphi(u_3), \varphi(u_4), \varphi(u_5)\} \neq C$, then we may color u with a color in $C \setminus \{\varphi(u_i),\ i = 1,2,3,4,5\}$, a contradiction.

Case 2: If $\{\varphi(u_1), \varphi(u_2), \varphi(u_3), \varphi(u_4), \varphi(u_5)\} = C$, then we may assume that u_1 and u_2 are colored with 1, u_i is colored with $i-1$, where $i = 3,4,5$. Then u_1 and u_2 have at most one neighbor colored by 1, otherwise we can properly recolor u_1 and u_2, back to Case 1. So we can extend the coloring to G by coloring u with 1, a contradiction.

The property holds. □

4. Proof of Theorem 1.1

In this section, we will use discharging method to complete the proof of Theorem 1.1. We firstly define the discharging rules, and then discuss the final charge of all vertices and faces in G^*.

To begin with, we define an initial charge μ on $V(G^*) \cup F(G^*)$ by setting $\mu(x) = d(x) - 4$ for every $x \in V(G^*) \cup F(G^*)$. By Euler's formula $|V(G^*)| - |E(G^*)| + |F(G^*)| = 2$ and $\sum\limits_{v \in V(G^*)} d(v) = \sum\limits_{f \in F(G^*)} d(f) = 2|E(G^*)|$, we can easily deduce that

$$\sum_{v \in V(G^*)} (d(v) - 4) + \sum_{f \in F(G^*)} (d(f) - 4) = -8.$$

Next, we define our discharging rules as follows:

(R1) Every original vertex sends $\frac{1}{2}$ to every incident 3-face.

(R2) Every 6^+-vertex sends $\frac{1}{8}$ to every true adjacent 5^--vertex.

(R3) Every special 4^2-vertex gets $\frac{7}{8}$ from every incident 6^+-face.

(R4) Suppose v is a non-special 4^2-vertex.

- (R4.1) Vertex v gets $\frac{9}{16}$ from every incident 7^+-face with one crossing vertex or every incident 5^+-face with at least two crossers.
- (R4.2) Vertex v gets $\frac{5}{16}$ from every incident 5-face or 6-face with one crosser.
- (R4.3) Vertex v gets $\frac{3}{8}$ from every incident 7^+-face with no crosser.

(R5) Every special 4^1-vertex gets $\frac{3}{8}$ from every incident 5^+-face.

(R6) Every non-special 4^1-vertex gets $\frac{3}{16}$ from every incident 5^+-face.

(R7) Every 5^3-vertex gets $\frac{3}{8}$ from every incident 6^+-face.

(R8) Every 6^4-vertex gets $\frac{3}{8}$ from every incident 6^+-face.
(R9) Every 6^3-vertex gets $\frac{1}{8}$ from every incident 5^+-face.

Since any discharging procedure preserves the total charge of G^*, the above defined discharging rules transform the initial charge μ to the final charge μ^* for every $x \in V(G^*) \cup F(G^*)$ such that

$$-8 = \sum_{x \in V(G^*) \cup F(G^*)} \mu(x) = \sum_{x \in V(G^*) \cup F(G^*)} \mu^*(x) \geqslant 0.$$

This will be a contradiction.

Now, we prove that $\mu^*(x) \geqslant 0$ for all $x \in V(G^*) \cup F(G^*)$ in the above defined discharging rules in the following two ways.

We first consider the discharge of the vertices in $V(G^*)$. Throughout the paper, the white vertices represent crossers.

Case 1: $d(v) = 4$. If v is a crosser, then $\mu^*(v) = \mu(v) = 0$. If v is an original vertex, then it is incident with at most two 3-faces by Lemma 2.1.

Subcase 1.1: Vertex v is incident with two 3-faces. If v is a special 4^2-vertex, see Figure 1, then by (R1), (R2), (R3) and Property 3.2,

$$\mu^*(v) \geqslant \mu(v) - 2 \times \frac{1}{2} + \frac{1}{8} + \frac{7}{8} = 0.$$

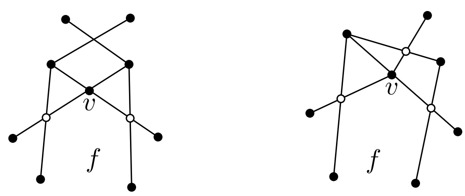

Figure 1. Two cases of the special 4^2-vertex.

If v is not a special 4^2-vertex, then there exist three cases, see Figure 2. In Figure 2 (a), since $g(G) \geqslant 7$, face f_1 is a 6-face with three crossers, or a 7^+-face with at least two crossers and f_2 is a 5^+-face with at least one crossers. In Figure 2 (b),

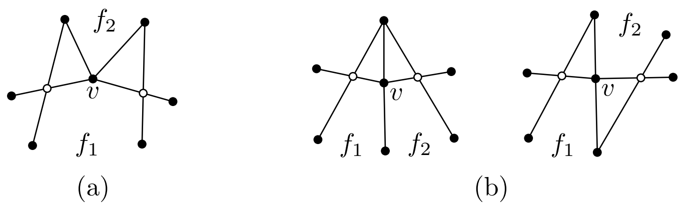

Figure 2. Three cases of the non-special 4^2-vertex.

both f_1 and f_2 are 5^+-faces with at least two crossers, or 7^+-faces with one crosser. According to (R1), (R2), (R4.1), (R4.2) and Property 3.2,

$$\mu^*(v) \geqslant \mu(v) - 2 \times \frac{1}{2} + \frac{1}{8} + \frac{5}{16} + \frac{9}{16} = 0.$$

Subcase 1.2: Vertex v is incident with one 3-face. If v is a special 4^1-vertex, see Figure 3, then by (R1), (R2), (R5) and Property 3.2,

$$\mu^*(v) \geqslant \mu(v) - \frac{1}{2} + \frac{1}{8} + \frac{3}{8} = 0.$$

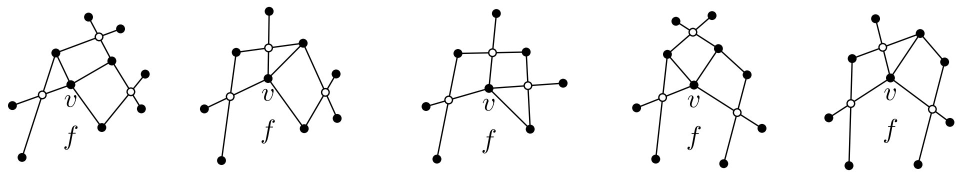

Figure 3. Five cases of the special 4^1-vertex.

If v is not a special 4^1-vertex, see Figure 4, then at least two of f_1, f_2 and f_3 are 5^+-faces. In fact, since $g(G) \geqslant 7$, face f_3 must be a 5^+-face and then v is incident with at most one 4-face. According to (R1), (R2), (R6) and Property 3.2,

$$\mu^*(v) \geqslant \mu(v) - \frac{1}{2} + \frac{1}{8} + 2 \times \frac{3}{16} = 0.$$

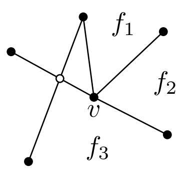

Figure 4. Non-special 4^1-vertex.

Subcase 1.3: Vertex v is not incident with any 3-face. Then by (R2) and Property 3.2, $\mu^*(v) \geqslant \mu(v) + \frac{1}{8} > 0$.

Case 2: $d(v) = 5$. According to Lemma 2.1, vertex v is incident with at most three 3-faces.

Subcase 2.1: Vertex v is incident with three 3-faces. In Figure 5, since $g(G) \geqslant 7$, face f_1 is a 6^+-face. Thus, by (R1), (R2), (R7) and Property 3.2,

$$\mu^*(v) \geqslant \mu(v) - 3 \times \frac{1}{2} + \frac{1}{8} + \frac{3}{8} = 0.$$

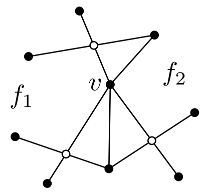

Figure 5. 5^3-vertex.

Subcase 2.2: Vertex v is incident with at most two 3-faces. Then by (R1), (R2) and Property 3.2,

$$\mu^*(v) \geqslant \mu(v) - 2 \times \frac{1}{2} + \frac{1}{8} = \frac{1}{8} > 0.$$

Case 3: $d(v) = 6$. According to Lemma 2.1, vertex v is incident with at most four 3-faces.

Subcase 3.1: Vertex v is incident with four 3-faces, see Figure 6. Since $g(G) \geqslant 7$, both f_1 and f_2 are 6^+-faces. Thus, by (R1), (R2) and (R8),

$$\mu^*(v) \geqslant \mu(v) - 4 \times \frac{1}{2} - 6 \times \frac{1}{8} + 2 \times \frac{3}{8} = 0.$$

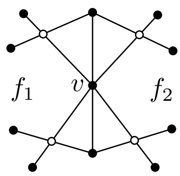

Figure 6. 6^4-vertex.

Subcase 3.2: Vertex v is incident with three 3-faces. Since $g(G) \geqslant 7$, vertex v is incident with at least two 5^+-faces, see Figure 7, both f_1 and f_2 are 5^+-faces. Thus, by (R1), (R2) and (R9),

$$\mu^*(v) \geqslant \mu(v) - 3 \times \frac{1}{2} - 6 \times \frac{1}{8} + 2 \times \frac{1}{8} = 0.$$

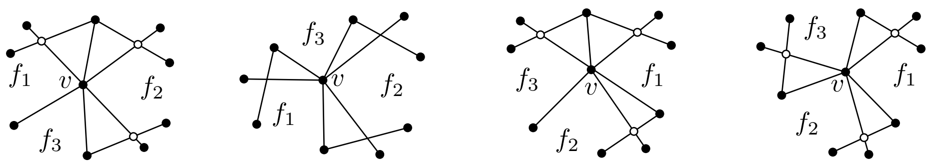

Figure 7. Four cases of the 6^3-vertex.

Subcase 3.3: Vertex v is incident with at most two 3-faces. Then by (R1) and (R2),

$$\mu^*(v) \geqslant \mu(v) - 2 \times \frac{1}{2} - 6 \times \frac{1}{8} = \frac{1}{4} > 0.$$

Case 4: $d(v) = 7$. According to Lemma 2.1, vertex v is incident with at most four 3-faces. Thus, by (R1) and (R2),

$$\mu^*(v) \geqslant \mu(v) - 4 \times \frac{1}{2} - 7 \times \frac{1}{8} = \frac{1}{8} > 0.$$

Case 5: $d(v) \geqslant 8$. According to Lemma 2.1, vertex v is incident with at most $\left\lfloor \frac{2}{3} d(v) \right\rfloor$ 3-faces. Thus, according to (R1) and (R2),

$$\begin{aligned} \mu^*(v) &\geqslant \mu(v) - d(v) \times \frac{1}{8} - \frac{1}{2} \times \left\lfloor \frac{2d(v)}{3} \right\rfloor \\ &\geqslant d(v) - \frac{d(v)}{8} - \frac{d(v)}{3} - 4 = \frac{13d(v)}{24} - 4 > 0. \end{aligned}$$

Next, we consider the discharge of the faces in G^*.

Case 1: $d(f) = 3$. According to Lemma 2.2, face f is incident with one crosser, furthermore, by (R1), we have $\mu^*(f) = \mu(f) + 2 \times \frac{1}{2} = 0$.

Case 2: $d(f) = 4$. Since the discharging procedure does not involve 4-faces, $\mu^*(f) = d(f) - 4 = 0$.

Case 3: $d(f) = 5$. Since special 4^2-vertices, 5^3-vertices and 6^4-vertices receive no charge from f by (R3), (R7) and (R8), we do not need to consider them. According to Lemma 2.2, face f is incident with at most two crossers.

Subcase 3.1: Face f is incident with two crossers. In this case, we firstly prove that the existence of some original vertices that f is incident with, and then we will discuss the specific classification depicted in Figure 8.

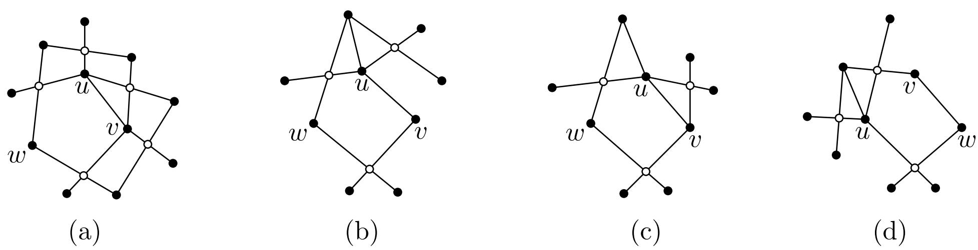

Figure 8. 5-face with two crossers.

Let u and v be the adjacent original incident vertices and w be the other original incident vertex of f. Then w is not a special 4^1-vertex. In fact, suppose w is a special

4^1-vertex. There are two ways to place the 3-face and 4-face incident with w, but in any way it is easy to see that the face of $G^* \setminus w$ whose interior contains w will correspond to a cycle of length at most 6 in G, a contradiction. Hence, assume that u is a special 4^1-vertex. Then none of v and w is a 4^2-vertex. Otherwise, the face of $G^* \setminus u$ whose interior contains u will correspond to a cycle of length at most 5 in G, a contradiction. Thus, a 4^2-vertex and a special 4^1-vertex can not exist on f at the same time.

In Figure 8 (a), both u and v are special 4^1-vertices. Since $g(G) \geqslant 7$, w can only be a 4^1-vertex, a 6^3-vertex or other vertex which receive no charge from f. Thus, by (R5), (R6) and (R9),

$$\mu^*(f) \geqslant \mu(f) - 2 \times \frac{3}{8} - \frac{3}{16} = \frac{1}{16} > 0.$$

In Figures 8 (b)–(d), u is a 4^2-vertex, v and w can only be 4^1-vertices, 6^3-vertices or other vertices which receive no charge from f. Thus, by (R4.1), (R6) and (R9),

$$\mu^*(f) \geqslant \mu(f) - \frac{9}{16} - 2 \times \frac{3}{16} = \frac{1}{16} > 0.$$

Subcase 3.2: Face f is incident with one crosser. There is no special 4^1-vertex. In fact, let u, v, x and y be the four consecutive original incident vertices of f. We can assume that u is a special 4^1-vertex. Then in any way to place the 3-face and 4-face incident with u will contradict $g(G) \geqslant 7$. So u is not a special 4^1-vertex. By a similar argument, v, x and y are not special 4^1-vertices. Moreover, u and y are not 4^2-vertices. Thus, there are at most two 4^2-vertices. According to (R4.2), (R6) and (R9), the worst case is that v and x are 4^2-vertices, u and y are 4^1-vertices, see Figure 9. Thus,

$$\mu^*(f) \geqslant \mu(f) - 2 \times \frac{5}{16} - 2 \times \frac{3}{16} = 0.$$

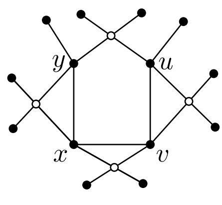

Figure 9. 5-face with one crosser.

Case 4: $d(f) = 6$. According to Lemma 2.2, face f is incident with at most three crossers.

Subcase 4.1: Face f is incident with three crossers. If f is incident with one special 4^2-vertex, then the other two original incident vertices of f can only be

4^1-vertices, 6^3-vertices or other vertices which receive no charge from f. Otherwise, it will contradict $g(G) \geqslant 7$. According to (R3), (R6) and (R9), the worst case is that v is a special 4^2-vertex, u and w are 4^1-vertices, see Figures 10 (a)–(b). Thus,

$$\mu^*(f) = \mu(f) - \frac{7}{8} - 2 \times \frac{3}{16} = \frac{3}{4} > 0.$$

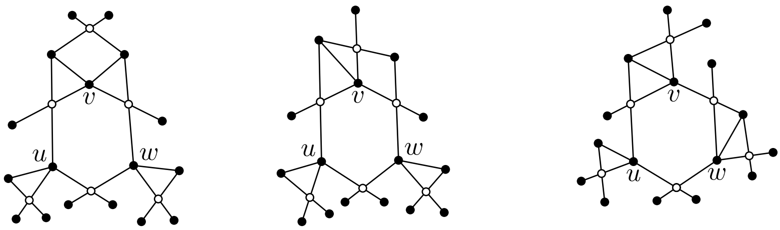

Figure 10. 6-face with three crossers.

If there is no special 4^2-vertex, then f is incident with at most two 4^2-vertices. Otherwise, it will contradict $g(G) \geqslant 7$. According to (R4.1), (R6) and (R9), in the worst case, v and w are 4^2-vertices, u is a 4^1-vertex, see Figure 10 (c). Thus,

$$\mu^*(f) = \mu(f) - 2 \times \frac{9}{16} - \frac{3}{16} = \frac{11}{16} > 0.$$

Subcase 4.2: Face f is incident with two crossers. Since $g(G) \geqslant 7$, face f is not incident with any special 4^2-vertex. Furthermore, there exist at most two 4^2-vertices, see Figure 11. According to (R4)–(R9), in the worst case, see Figures 11 (a)–(b), u and v are 4^2-vertices, x and y are 4^1-vertices. Thus,

$$\mu^*(f) = \mu(f) - 2 \times \frac{9}{16} - 2 \times \frac{3}{16} = \frac{1}{2} > 0.$$

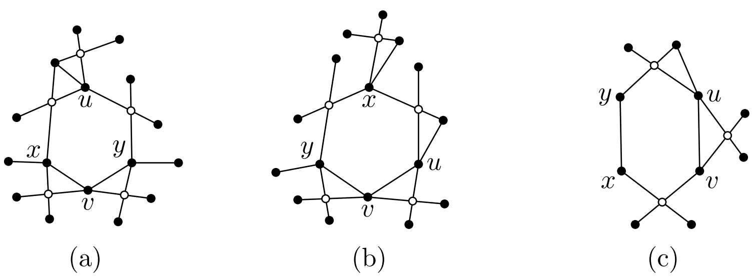

Figure 11. 6-face with two crossers.

Subcase 4.3: Face f is incident with one crosser. Since $g(G) \geqslant 7$, f is not incident with any special 4^2-vertex, 5^3-vertex and 6^4-vertex. Furthermore, there is at most one special 4^1-vertex.

If f is incident with one special 4^1-vertex, then there are at most two incident 4^2-vertices. According to (R4.2), (R5), (R6) and (R9), in the worst case, see Figure 12 (a), v is a special 4^1-vertex, u and w are 4^2-vertices, x and y are 4^1-vertices. Thus,

$$\mu^*(f) = \mu(f) - \frac{3}{8} - 2 \times \frac{5}{16} - 2 \times \frac{3}{16} = \frac{5}{8} > 0.$$

If there is no special 4^1-vertex, then f is incident with at most three 4^2-vertices. According to R4.2, R6 and R9, in the worst case, see Figure 12 (b), u, v and w are 4^2-vertices, x and y are 4^1-vertices. Thus,

$$\mu^*(f) = \mu(f) - 3 \times \frac{5}{16} - 2 \times \frac{3}{16} = \frac{11}{16} > 0.$$

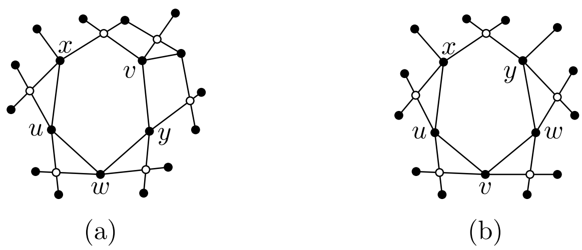

Figure 12. 6-face with one crosser.

Case 5: $d(f) \geqslant 7$.

Subcase 5.1: Face f is not incident with any crosser. Since $g(G) \geqslant 7$, there is no special 4^2-vertex on f. According to (R4)–(R9), in the worst case, see Figure 13, all incident vertices of f are 4^2-vertices. Thus,

$$\mu^*(f) = \mu(f) - d(f) \times \frac{3}{8} = \frac{5d(f)}{8} - 4 > 0.$$

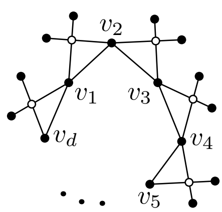

Figure 13. 7^+-face without any crosser.

Subcase 5.2: Face f is incident with one crosser. Since $g(G) \geqslant 7$, there is no special 4^2-vertex on f. Furthermore, f is incident with at most $(d(f)-2)$ 4^2-vertices.

According to (R4)–(R9), in the worst case, see Figure 14, $v_1, \ldots, v_{d-2}$ are 4^2-vertices, u is a 4^1-vertex. Thus,

$$\mu^*(f) = \mu(f) - (d(f) - 2) \times \frac{9}{16} - \frac{3}{16} = \frac{7d(f) - 49}{16} \geqslant 0.$$

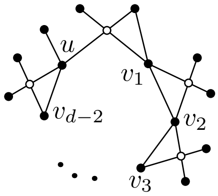

Figure 14. 7^+-face with one crosser.

Subcase 5.3: Face f is incident with two crossers. Since $g(G) \geqslant 7$, face f is incident with at most one special 4^2-vertex.

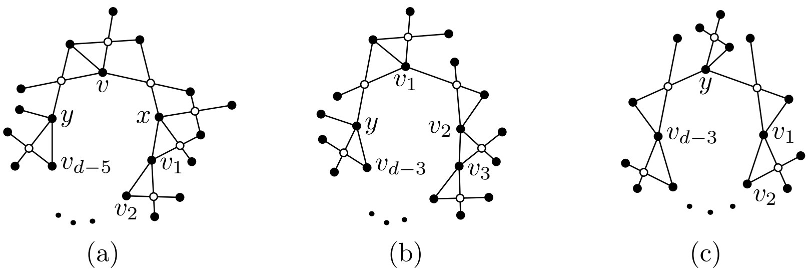

Figure 15. 7^+-face with two crossers.

If f is incident with one special 4^2-vertex, then there exist at most $(d(f) - 5)$ incident 4^2-vertices. According to (R4)–(R9), in the worst case, see Figure 15 (a), v is a special 4^2-vertex, $v_1, \ldots, v_{d-5}$ are 4^2-vertices, x is a special 4^1-vertex, y is a 4^1-vertex. Thus,

$$\mu^*(f) = \mu(f) - \frac{7}{8} - (d(f) - 5) \times \frac{9}{16} - \frac{3}{8} - \frac{3}{16} = \frac{7d(f) - 42}{16} > 0.$$

If there is no special 4^2-vertex, f is incident with at most $(d(f) - 3)$ 4^2-vertices. According to (R4)–(R9), in the worst case, see Figures 15 (b)–(c), $v_1, \ldots, v_{d-3}$ are 4^2-vertices, y is a 4^1-vertex. Thus,

$$\mu^*(f) = \mu(f) - (d(f) - 3) \times \frac{9}{16} - \frac{3}{16} = \frac{7d(f) - 40}{16} > 0.$$

Subcase 5.4: Face f is incident with at least three crossers. Before discussing the case, we give a definition as follows. Let $v_1, \ldots, v_k$ be the k incident vertices of f, with edges v_iv_{i+1} ($i \bmod k$), where $k \geqslant 7$. We call a vertex v_{i-1} is a *false* 2-*distance vertex* of v_{i+1}, if v_{i-1} and v_{i+1} are original vertices and v_i is a crosser.

Assume that v is an incident special 4^2-vertex of f. Then at least one false 2-distance vertex of v is a 4^1-vertex, 6^3-vertex or other vertices which receive no charge from f, since the girth of G is at least 7. According to (R4)–(R9), we can easily see that the worst case is that all of original incident vertices of f are 4^2-vertices. Thus,

$$\mu^*(f) \geqslant \mu(f) - (d(f) - 3) \times \frac{9}{16} = \frac{7d(f) - 37}{16} > 0.$$

The proof of Theorem 1.1 is completed. $\square$

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