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THE DUALITY OF AUSLANDER-REITEN QUIVER OF PATH ALGEBRAS

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Abstract. Let Q be a finite union of Dynkin quivers, $G \subseteq \mathrm{Aut}(\Bbbk Q)$ a finite abelian group, $\widehat{Q}$ the generalized McKay quiver of (Q, G) and Γ_Q the Auslander-Reiten quiver of $\Bbbk Q$. Then G acts functorially on the quiver Γ_Q. We show that the Auslander-Reiten quiver of $\Bbbk\widehat{Q}$ coincides with the generalized McKay quiver of (Γ_Q, G).

Keywords: Auslander-Reiten quiver; generalized McKay quiver; duality

MSC 2010: 16G10, 16G20, 16G70

1. Introduction

Let $Q = (I, E)$ be a quiver, let $\mathrm{Aut}(Q)$, $\mathrm{Aut}(\Bbbk Q)$ be the automorphism groups of Q and the path algebra $\Bbbk Q$, respectively. For the skew group algebra $\Bbbk Q * G$ corresponding to the pair (Q, G) with $G \subseteq \mathrm{Aut}(Q)$, there has been a lot of literature on $\Bbbk Q * G$ (for example see [8], [10], [11], [15], [17]).

It is shown in [15] that if Q has no oriented cycles and $G \subseteq \mathrm{Aut}(Q)$ is a cyclic group, then the skew group algebra $\Bbbk Q * G$ is Morita equivalent to the path algebra of another quiver Γ. The authors illustrate this through several examples. In [10], [11], Hubery showed the duality of (Q, G), that is, there exists an action of G on Γ such that $\Bbbk\Gamma * G$ is Morita equivalent to $\Bbbk Q$. More generally, for an arbitrary finite group G and an action of G on the path algebra $\Bbbk Q$ permuting the set of primitive idempotents and stabilizing the vector space spanned by the arrows, Demonet in [3] defined a quiver $\widehat{Q}$ (we call it the generalized McKay quiver) and proved that the

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skew group algebra $\Bbbk Q * G$ is Morita equivalent to $\Bbbk\widehat{Q}$. Obviously, if $G \subseteq \mathrm{Aut}(Q)$ is a cyclic group, the generalized McKay quiver $\widehat{Q}$ coincides with the Γ constructed in [10], [11], [15].

For the relationship between Q-representations and $\Bbbk Q*G$-modules, the paper [17] gives a detailed description whenever $G \subseteq \mathrm{Aut}(Q)$ is cyclic. By a similar technique, for a quiver Q with relations in $\mathscr{R}$ and a finite abelian group $G \subseteq \mathrm{Aut}(Q)$ preserving the relations in $\mathscr{R}$, we gave in [8] the condition for a $(Q, \mathscr{R})$-representation to be a $\Lambda * G$-module and determined the number of non-isomorphic indecomposable $\Lambda * G$-modules which are induced from the same $(Q, \mathscr{R})$-representation, where $\Lambda = \Bbbk Q/\langle \mathscr{R} \rangle$. In the paper [9], we discussed the duality of (Q, G) in the case that $G \subseteq \mathrm{Aut}(\Bbbk Q)$ is finite abelian, and by the duality, gave the correspondence between the indecomposable $\widehat{Q}$-representations and the positive roots of the valued graph of (Q, G). In this paper, we consider the duality of the Auslander-Reiten quiver of $\Bbbk Q$.

The Auslander-Reiten quiver Γ_Q of $\Bbbk Q$ codifies the structure of the category of finitely generated $\Bbbk Q$-modules. Vertices are the indecomposable $\Bbbk Q$-modules, arrows are the irreducible morphisms between them. Note that an automorphism $\sigma \in \mathrm{Aut}(\Bbbk Q)$ also acts functorially on the category of Q-representations and this determines an action on the set of isomorphism classes. That is to say, σ induces a quiver automorphism of the Auslander-Reiten quiver Γ_Q of $\Bbbk Q$. If $\Bbbk$ is the algebraic closure of a finite field F_q and F is the Frobenius morphism induced by σ, Deng and Du have shown that the Auslander-Reiten quiver of the fixed point algebra $(\Bbbk Q)^F$ is just the F_q-species associated to (Γ_Q, σ) (see [4], [5]). If Q is a connected Dynkin quiver, the order of σ is only $1, 2$, or 3. In this case, Zhang showed that the generalized McKay quiver $\widehat{\Gamma_Q}$ of (Γ_Q, σ) is just the Auslander-Reiten quiver $\Gamma_{\widehat{Q}}$ of $\widehat{Q}$ via case-by-case analysis, where $\widehat{Q}$ is the generalized McKay quiver of (Q, σ) (see [16]). Here, we will give a uniform proof for this result whenever Q is a finite union of Dynkin quivers and $G \subseteq \mathrm{Aut}(\Bbbk Q)$ is a finite abelian group.

Let Q be a finite union of Dynkin quivers, $G \subseteq \mathrm{Aut}(\Bbbk Q)$ a finite abelian group, $\widehat{Q}$ and Γ_Q the generalized McKay quiver of (Q, G) and the Auslander-Reiten quiver of $\Bbbk Q$. Then G also acts functorially on the quiver Γ_Q. By the duality of (Q, G) discussed in [9], there is an action of G on $\widehat{Q}$ so that it also induces an action on the quiver $\Gamma_{\widehat{Q}}$. Our main result is:

Theorem 1.1. *Let $\widehat{\Gamma_Q}$ and $\widehat{\Gamma_{\widehat{Q}}}$ be the generalized McKay quivers of Γ_Q and $\Gamma_{\widehat{Q}}$, respectively. Then*

$$\Gamma_{\widehat{Q}} = \widehat{\Gamma_Q} \quad \textit{and} \quad \Gamma_Q = \widehat{\Gamma_{\widehat{Q}}}.$$

That is, the group action also induces a dual for the Auslander-Reiten quiver of $\Bbbk Q$ and $\Bbbk\widehat{Q}$. Since path algebra $\Bbbk\widehat{Q}$ is Morita equivalent to $\Bbbk Q*G$, we identify $\Gamma_{\widehat{Q}}$ with the Auslander-Reiten quiver of $\Bbbk Q * G$. Based on the understanding of the relationship between indecomposable $\Bbbk Q$-modules and indecomposable $\Bbbk Q * G$-modules, and the relationship between the almost split sequences in the category of $\Bbbk Q$-modules and in the category of $\Bbbk Q * G$-modules, we give a proof of this theorem.

This paper is organized as follows. In Section 1, we shortly review some basic concepts of representations of quivers, Auslander-Reiten quivers and generalized McKay quivers. In Section 2, we discuss the relationship between indecomposable $\Bbbk Q$-modules and indecomposable $\Bbbk Q * G$-modules. In fact, similarly to [8], Section 2, we show that all finite dimensional $\Bbbk Q * G$-modules can be obtained from $\Bbbk Q$-modules, and the number of non-isomorphic indecomposable $\Bbbk Q * G$-modules induced from the same indecomposable G-invariant $\Bbbk Q$-module can be determined. In Section 3, we apply the results of Section 2 and Reiten and Riedtmann's results about the almost split sequences in categories of $\Bbbk Q$-modules and $\Bbbk Q * G$-modules to give the proof of our main theorem. In the last section, we use an interesting example to show the duality of (Q, G), (Γ_Q, G) and the valued quiver corresponding to (Q, G), respectively.

Throughout this paper, G will denote a finite group, $\Bbbk$ denotes an algebraic closed field whose characteristic does not divide the order of G, mod-Λ denotes the category of finite-dimensional right Λ-modules for any Artin algebra Λ. Unless otherwise stated all modules we consider are finite-dimensional and $\otimes := \otimes_{\Bbbk}$.

2. Preliminaries

We recall in this section some basic facts about quivers and their representations, Auslander-Reiten quivers and generalized McKay quivers.

A quiver $Q = (I, E)$ is an oriented graph with I the set of vertices and E the set of arrows. Quiver Q is called finite if I and E are finite sets. For any given quiver Q, we have an associative $\Bbbk$-algebra $\Bbbk Q$, called the path algebra of Q (see [1], [2]). A representation $X = (X_i, X_\alpha)$ of a quiver Q over $\Bbbk$ consists of a family of $\Bbbk$-vector spaces X_i for $i \in I$, together with a family of $\Bbbk$-linear maps $X_\alpha\colon X_i \to X_j$ for $\alpha\colon i \to j$ in E. A morphism $\varphi\colon X \to Y$ between two representations X and Y is given by $\Bbbk$-linear maps $\varphi_i\colon X_i \to Y_i$ for all $i \in I$, satisfying $\varphi_j \circ X_\alpha = Y_\alpha \circ \varphi_i$ for each arrow $\alpha\colon i \to j$. It is well-known that the category of finite-dimensional Q-representations over $\Bbbk$ is naturally equivalent to the category mod-$\Bbbk Q$. Thus in this paper, we identify a Q-representation with a $\Bbbk Q$-module. For background on the representation theory of quivers, the reader is referred to [1], [2] and [6].

The important notion of Auslander-Reiten quivers was introduced in the 70's by Auslander and Reiten and since then it has played an essential role in the representation theory of Artin algebras. Recall firstly that a homomorphism $f\colon X \to Y$ in mod-$\Bbbk Q$ is called irreducible if f is neither a section nor a retraction, but for any factorization $f = f_1 f_2$ either f_2 is a section or f_1 is a retraction. If Q has no oriented cycles, then the Auslander-Reiten quiver Γ_Q of path algebra $\Bbbk Q$ is defined as follows: the vertices of Γ_Q are the isomorphism classes $[X]$ of finitely generated indecomposable $\Bbbk Q$-modules X; for two vertices $[X]$ and $[Y]$ in Γ_Q, the arrows $[X] \to [Y]$ are in bijective correspondence with a basis of $\Bbbk$-vector space $\mathrm{Irr}(X,Y)$, where $\mathrm{Irr}(X,Y)$ is the set of all irreducible morphisms from X to Y. It is well-known that the quiver Γ_Q for a connected quiver Q is a finite quiver if and only if Q is a Dynkin quiver of type A_n ($n \geqslant 1$), D_n ($n \geqslant 4$), E_6, E_7 or E_8, and then Γ_Q contains no multiple edges.

Assume that Λ is a $\Bbbk$-algebra and G acts on Λ; the skew group algebra of Λ under the action of G is by definition the $\Bbbk$-algebra whose underlying $\Bbbk$-vector space is $\Lambda \otimes_\Bbbk \Bbbk[G]$ and whose multiplication is linearly generated by

$$(\lambda \otimes g)(\lambda' \otimes g') = \lambda g(\lambda') \otimes gg'$$

for all $\lambda, \lambda' \in \Lambda$ and $g, g' \in G$ (see [15]). For convenience, we denote this algebra by $\Lambda * G$ and denote the element $\lambda \otimes g$ in $\Lambda * G$ by λg. One sees that Λ and $\Bbbk[G]$ can be viewed as subalgebras of $\Lambda * G$.

Let $\Lambda = \Bbbk Q$ be the path algebra of the quiver $Q = (I, E)$. We consider an action of G on $\Bbbk Q$ permuting the set of primitive idempotents $\{e_i\colon i \in I\}$ and stabilizing the vector space spanned by the arrows. Let $\mathscr{I}$ be a set of representatives of the orbits of I under the action of G. For any $i \in I$, there exists $g \in G$ such that $g^{-1}(i) \in \mathscr{I}$. We fix such a g and denote it by κ_i. For $(i,j) \in \mathscr{I}^2$, G acts on $\mathscr{O}_i \times \mathscr{O}_j$ diagonally, where $\mathscr{O}_i$ and $\mathscr{O}_j$ are the orbits of i and j under the action of G. A set of representatives of the classes of this action will be denoted by $\mathscr{F}_{ij}$.

For $i, j \in I$, define $E_{ij} \subseteq \Bbbk Q$ to be the vector space spanned by the arrows from i to j. Let G_i be the subgroup of G stabilizing e_i. We regard E_{ij} as a left and right $\Bbbk[G_{ij}] := \Bbbk[G_i \cap G_j]$-module by restricting the action of G. In [3] Demonet defined the quiver $\widehat{Q} = (\hat{I}, \widehat{E})$ as

$$\hat{I} = \bigcup_{i \in \mathscr{I}} \{i\} \times \operatorname{irr} G_i,$$

where $\operatorname{irr} G_i$ is a set of representatives of isomorphism classes of irreducible representations of G_i. The set of arrows of $\widehat{Q}$ from (i, ϱ) to (j, σ) is a basis of

$$\bigoplus_{(i',j') \in \mathscr{F}_{ij}} \mathrm{Hom}_{\Bbbk[G_{i'j'}]}((\varrho \cdot \kappa_{i'})|_{G_{i'j'}}, (\sigma \cdot \kappa_{j'})|_{G_{i'j'}} \otimes_\Bbbk E_{i'j'}),$$

where the representation $\varrho \cdot \kappa_{i'}$ of $G_{i'}$ is the same as ϱ as a $\Bbbk$-vector space, and $(\varrho \cdot \kappa_{i'})g = \varrho\kappa_{i'} g \kappa_{i'}^{-1}$ for $g \in G_{i'} = \kappa_{i'}^{-1} G_i \kappa_{i'}$. Furthermore, Demonet proved the following theorem.

Theorem 2.1 (see [3]). *The category* mod-$\Bbbk\widehat{Q}$ *is equivalent to the category* mod-$\Bbbk Q * G$.

In particular, if the quiver Q is a singular vertex with m loops, we can view G as a subgroup of $\mathrm{GL}_m(\Bbbk)$. Then the quiver $\widehat{Q}$ is just the McKay quiver of G, see [7], [14]. Thus, we view the quiver $\widehat{Q}$ as a generalized McKay quiver and call it the generalization of the McKay quiver of (Q, G). Moreover, for any factor algebra $\Bbbk Q/J$, it is easy to see that the skew group algebra $(\Bbbk Q/J) * G$ is Morita equivalent to a factor algebra of $\Bbbk\widehat{Q}$. That is to say, the generalized McKay quiver can realize the Gabriel quiver of $\Lambda * G$ for any basic algebra Λ.

3. Constituting $\Bbbk Q * G$-modules

Let $Q = (I, E)$ be a finite quiver, $G \subseteq \mathrm{Aut}(\Bbbk Q)$ a finite abelian group. In this section, we show that all finite dimensional $\Bbbk Q * G$-modules can be obtained from $\Bbbk Q$-modules, and the number of non-isomorphic indecomposable $\Bbbk Q * G$-modules induced from the same indecomposable G-invariant $\Bbbk Q$-modules can be determined.

Let X be a $\Bbbk Q$-module, $g \in G$. We define a twisted $\Bbbk Q$-module ${}^gX$ on X by taking the same underlying vector space as X with the action $x \cdot \lambda = xg^{-1}(\lambda)$ for $x \in X$ and $\lambda \in \Bbbk Q$. Then, for each $g \in G$, we have an additive autoequivalence functor

$$\begin{aligned} F_g\colon \quad \text{mod-}\Bbbk Q &\to \text{mod-}\Bbbk Q \\ X &\mapsto {}^gX, \end{aligned}$$

where ${}^g\psi := F_g(\psi) = \psi$ for any morphism $\psi\colon X \to Y$ in mod-$\Bbbk Q$.

Consider the subspace

$$X \otimes g := \{x \otimes g\colon x \in X\}$$

of $X \otimes_{\Bbbk Q} \Bbbk Q * G$. Then $X \otimes g$ has a natural $\Bbbk Q$-module structure given by $(x \otimes g)\lambda = xg^{-1}(\lambda) \otimes g$ for any $x \otimes g \in X \otimes g$ and $\lambda \in \Bbbk Q$. It is easy to see that ${}^gX \cong X \otimes g$ as $\Bbbk Q$-modules.

Recall that a $\Bbbk Q$-module X is said to be G-invariant if $F_g(X) \cong X$ for any $g \in G$; a G-invariant $\Bbbk Q$-module X is said to be indecomposable G-invariant if X is nonzero and X cannot be written as the direct sum of two nonzero G-invariant $\Bbbk Q$-modules. For each $X \in$ mod-$\Bbbk Q$, let

$$H_X = \{g \in G\colon F_g(X) \cong X \text{ as } \Bbbk Q\text{-modules}\}.$$

Clearly, H_X is a subgroup of G. We denote by G_X a complete set of left coset representatives of H_X in G. Then one can see that any indecomposable G-invariant $\Bbbk Q$-module has the form

$$\bigoplus_{g \in G_X} {}^{g}X$$

for some indecomposable $X \in \text{mod-}\Bbbk Q$, and the full subcategory of mod-$\Bbbk Q$ generated by the G-invariant $\Bbbk Q$-modules is a Krull-Schmidt category.

For the G-invariant $\Bbbk Q$-modules and the $\Bbbk Q * G$-modules, we have:

Proposition 3.1. *A $\Bbbk Q$-module X is a $\Bbbk Q * G$-module if and only if X is G-invariant.*

Proof. Let X be a $\Bbbk Q * G$-module. We first show that X is G-invariant, i.e, ${}^{g}X \cong X$ for any $g \in G$. For each $g \in G$, we define a map $f_g \colon {}^{g}X \to X$ by $f_g(x) = xg^{-1}$ for all $x \in X$. Then, f_g is a $\Bbbk Q$-module isomorphism since

$$f_g(x \cdot \lambda) = (x \cdot \lambda)g^{-1} = (xg^{-1}(\lambda))g^{-1} = (xg^{-1})\lambda = f_g(x)\lambda$$

for all $\lambda \in \Bbbk Q$ and $x \in X$.

Conversely, if X is a G-invariant $\Bbbk Q$-module, that is, there exists a module isomorphism $\theta_g \colon {}^{g}X \to X$ for any $g \in G$. Then, as observed in [??], page 95, there exists a $\Bbbk Q$-module isomorphism $\varphi_g \colon {}^{g}X \to X$ such that ${}^{g^{1-|g|}}\varphi_g \circ \ldots \circ {}^{g^{-1}}\varphi_g \circ \varphi_g = \mathrm{id}_{{}^{g}Y}$, where $|g|$ is the order of g. We define an action of $\Bbbk Q * G$ on X by $x \cdot \lambda g = \varphi_{g^{-1}}(x\lambda)$ for any $\lambda g \in \Bbbk Q * G$ and $x \in X$. One can check that X is a $\Bbbk Q * G$-module under this action. $\square$

For a given G-invariant $\Bbbk Q$-module X, the map φ_g is not unique in general. Thus, it is possible that there are many $\Bbbk Q * G$-module structure on X induced by different maps φ_g, $g \in G$. How many non-isomorphic $\Bbbk Q * G$-module structures are induced on a given G-invariant $\Bbbk Q$-module? We can give an answer by the following lemmas.

Note that H_X is an abelian group. It follows that the regular representation $\Bbbk H_X$ can be decomposed as

$$\Bbbk H_X = \bigoplus_{i=1}^{r} \varrho_i,$$

where all the ϱ_i are one dimensional irreducible H_X-representations, $r = |H_X|$ is the order of H_X, and $\varrho_i \ncong \varrho_j$ if $i \neq j$.

Since X is a natural H_X-invariant $\Bbbk Q$-module, X has a $\Bbbk Q * H_X$-module structure by Proposition 3.1. Therefore, $\varrho_i \otimes X$ is also a $\Bbbk Q * H_X$-module defined by

$$(l \otimes x)\lambda g = lg \otimes x \cdot \lambda g$$

for any $\lambda g \in \Bbbk Q * H_X$ and $l \otimes x \in \varrho_i \otimes X$. Consequently, $\mathrm{Hom}_{\Bbbk Q}(X, \varrho_i \otimes X)$ is a $\Bbbk H_X$-module given by

$$(f \triangleleft g)(x) = f(x) \cdot g$$

for $f \in \mathrm{Hom}_{\Bbbk Q}(X, \varrho_i \otimes X)$, $g \in H_X$, and $x \in X$; $\varrho_i \otimes \mathrm{End}_{\Bbbk Q}(X)$ is a $\Bbbk H_X$-module given by

$$(l \otimes f)g = lg \otimes f \triangleleft g$$

for $l \otimes f \in \varrho_i \otimes \mathrm{End}_{\Bbbk Q}(X)$ and $g \in H_X$. Note that all the representations ϱ_i are one dimensional as $\Bbbk$-vector spaces, one can check that

$$\mathrm{Hom}_{\Bbbk Q}(X, \varrho_i \otimes X) \cong \varrho_i \otimes \mathrm{End}_{\Bbbk Q}(X)$$

as $\Bbbk H_X$-modules. Therefore, we have:

Lemma 3.2. *Let X be an indecomposable $\Bbbk Q$-module. Then*

(1) *$\varrho_i \otimes X \cong X$ as $\Bbbk Q$-modules and $\varrho_i \otimes X$ is indecomposable as a $\Bbbk Q * H_X$-module for each $i \in \{1, 2, \ldots, r\}$;*

(2) *$\varrho_i \otimes X \ncong \varrho_j \otimes X$ as $\Bbbk Q * H_X$-modules if $i \neq j$;*

(3) *$X \otimes_{\Bbbk Q} \Bbbk Q * H_X \cong \bigoplus_{i=1}^{r} \varrho_i \otimes X$ as $\Bbbk Q * H_X$-modules;*

(4) *for any $\Bbbk Q * H_X$-module Y, if $Y \cong X$ as $\Bbbk Q$-modules, then there exists a unique $i \in \{1, 2, \ldots, r\}$ such that $Y \cong \varrho_i \otimes X$ as $\Bbbk Q * H_X$-modules. Hence there are r non-isomorphic $\Bbbk Q * H_X$-modules induced from X.*

Proof. (1) Note that for each $0 \neq l \in \varrho_i$, there is a $\Bbbk Q$-module isomorphism $f\colon X \to \varrho_j \otimes X$ given by $x \mapsto l \otimes x$. We obtain that $\varrho_i \otimes X$ is an indecomposable $\Bbbk Q$-module, and hence an indecomposable $\Bbbk Q * H_X$-module.

(2) If $\varrho_i \otimes X \cong \varrho_j \otimes X$, we have $\varrho_i \otimes \mathrm{End}_{\Bbbk Q}(X) \cong \varrho_j \otimes \mathrm{End}_{\Bbbk Q}(X)$. Since $\mathrm{End}_{\Bbbk Q}(X)/\mathrm{radEnd}_{\Bbbk Q}(X) \cong \Bbbk$ and $\mathrm{radEnd}_{\Bbbk Q}(X)$ is closed under the action of H_X, we have

$$\varrho_i \otimes \mathrm{End}_{\Bbbk Q}(X)/\mathrm{radEnd}_{\Bbbk Q}(X) \cong \varrho_j \otimes \mathrm{End}_{\Bbbk Q}(X)/\mathrm{radEnd}_{\Bbbk Q}(X).$$

This means $\varrho_i \cong \varrho_j$ as $\Bbbk H_X$-modules and we get a contradiction.

(3) By [13], Lemma 3.2.1, $(\varrho_i \otimes X) \otimes X \mid (\varrho_i \otimes X) \otimes_{\Bbbk Q} \Bbbk Q * H_X$, that is, $\varrho_i \otimes X$ is a direct summand of $(\varrho_i \otimes X) \otimes_{\Bbbk Q} \Bbbk Q * H_X$ as $\Bbbk Q * H_X$-modules. Then we have $\varrho_i \otimes X \mid X \otimes_{\Bbbk Q} \Bbbk Q * H_X$, since $\varrho_i \otimes X \cong X$ as $\Bbbk Q$-modules. Note that $\varrho_i \otimes X \ncong \varrho_j \otimes X$ if $i \neq j$, hence we get that $\left(\bigoplus_{i=1}^{r} \varrho_i \otimes X\right) \Big| X \otimes_{\Bbbk Q} \Bbbk Q * H_X$, so that $X \otimes_{\Bbbk Q} \Bbbk Q * H_X \cong \bigoplus_{i=1}^{r} \varrho_i \otimes X$ by [15], Proposition 1.8.

(4) Let Y be a $\Bbbk Q * H_X$-module such that $Y \cong X$ as $\Bbbk Q$-modules. Then Y is an indecomposable $\Bbbk Q * H_X$-module. Since $Y \mid Y \otimes_{\Bbbk Q} \Bbbk Q * H_X \cong X \otimes_{\Bbbk Q} \Bbbk Q * H_X$, it is easy to see that there exists a unique $i \in \{1, 2, \ldots, r\}$ such that $Y \cong \varrho_i \otimes X$. $\square$

Lifting to the $\Bbbk Q * G$-module, we have:

Lemma 3.3. *Let X be an indecomposable $\Bbbk Q$-module. Then*

(1) $(\varrho_i \otimes X) \otimes_{\Bbbk Q * H_X} \Bbbk Q * G \cong \bigoplus_{g \in G_X} {}^g X$ *as $\Bbbk Q$-modules;*

(2) $(\varrho_i \otimes X) \otimes_{\Bbbk Q * H_X} \Bbbk Q * G$ *is an indecomposable $\Bbbk Q * G$-module;*

(3) $(\varrho_i \otimes X) \otimes_{\Bbbk Q * H_X} \Bbbk Q * G \ncong (\varrho_j \otimes X) \otimes_{\Bbbk Q * H_X} \Bbbk Q * G$ *as $\Bbbk Q * G$-modules if $i \neq j$;*

(4) $X \otimes_{\Bbbk Q} \Bbbk Q * G \cong \bigoplus_{i=1}^{r} (\varrho_i \otimes X) \otimes_{\Bbbk Q * H_X} \Bbbk Q * G$ *as $\Bbbk Q * G$-modules.*

Proof. (1) Note that $(\varrho_i \otimes X) \otimes_{\Bbbk Q * H_X} \Bbbk Q * G \cong \bigoplus_{g \in G_X} \varrho_i \otimes X \otimes g$ and $\varrho_i \otimes X \cong X$ as $\Bbbk Q$-modules, so we have $(\varrho_i \otimes X) \otimes_{\Bbbk Q * H_X} \Bbbk Q * G \cong \bigoplus_{g \in G_X} X \otimes g \cong \bigoplus_{g \in G_X} {}^g X$.

(2) The result follows from the fact that $(\varrho_i \otimes X) \otimes_{\Bbbk Q * H_X} \Bbbk Q * G \cong \bigoplus_{g \in G_X} {}^g X$ is an indecomposable G-invariant $\Bbbk Q$-module.

(3) Suppose that $(\varrho_i \otimes X) \otimes_{\Bbbk Q * H_X} \Bbbk Q * G \cong (\varrho_j \otimes X) \otimes_{\Bbbk Q * H_X} \Bbbk Q * G$. We have that $\varrho_i \otimes X \otimes e \mid (\varrho_j \otimes X) \otimes_{\Bbbk Q * H_X} \Bbbk Q * G \cong \bigoplus_{g \in G_X} \varrho_j \otimes X \otimes g$ for the unit e of G. If $\varrho_i \otimes X \otimes e \cong \varrho_j \otimes X \otimes e$, then $\varrho_i \otimes X \cong \varrho_j \otimes X$ as $\Bbbk Q * H_X$-modules. This is a contradiction. If $\varrho_i \otimes X \otimes e \cong \varrho_j \otimes X \otimes g$ for some $e \neq g \in G_X$, we have $X \cong {}^g X$ as $\Bbbk Q$-modules. This is also a contradiction.

(4) Note that $(\varrho_i \otimes X) \otimes_{\Bbbk Q * H_X} \Bbbk Q * G \mid (\varrho_i \otimes X) \otimes_{\Bbbk Q * H_X} \Bbbk Q * G \otimes_{\Bbbk Q} \Bbbk Q * G$, by the statement (1) we have $(\varrho_i \otimes X) \otimes_{\Bbbk Q * H_X} \Bbbk Q * G \Big| \Big(\bigoplus_{g \in G_X} {}^g X\Big) \otimes_{\Bbbk Q} \Bbbk Q * G$ and $(\varrho_i \otimes X) \otimes_{\Bbbk Q * H_X} \Bbbk Q * G \mid X \otimes_{\Bbbk Q} \Bbbk Q * G$ for any $i \in \{1, 2, \ldots, r\}$. Thus, $\Big(\bigoplus_{i=1}^{r} (\varrho_i \otimes X) \otimes_{\Bbbk Q * H_X} \Bbbk Q * G\Big) \Big| X \otimes_{\Bbbk Q} \Bbbk Q * G$, so that $X \otimes_{\Bbbk Q} \Bbbk Q * G \cong \bigoplus_{i=1}^{r} (\varrho_i \otimes X) \otimes_{\Bbbk Q * H_X} \Bbbk Q * G$ by [15], Proposition 1.8. □

By the above discussion, we get the main result of this section.

Theorem 3.4. *Let $G \subseteq \operatorname{Aut}(\Bbbk Q)$ be a finite abelian group. For any indecomposable $\Bbbk Q$-module X and $\Bbbk Q * G$-module Y such that $Y \cong \bigoplus_{g \in G_X} {}^g X$ as $\Bbbk Q$-modules, there exists a unique $i \in \{1, 2, \ldots, r\}$ such that $Y \cong (\varrho_i \otimes X) \otimes_{\Bbbk Q * H_X} \Bbbk Q * G$. That is, there are r non-isomorphic $\Bbbk Q * G$-modules induced from the indecomposable G-invariant $\Bbbk Q$-module $\bigoplus_{g \in G_X} {}^g X$.*

Therefore, a finite dimensional $\Bbbk Q$-module Y is an indecomposable $\Bbbk Q * G$-module if and only if Y is an indecomposable G-invariant $\Bbbk Q$-module.

Proof. Let Y be a $\Bbbk Q * G$-module such that $Y \cong \bigoplus_{g \in G_X} {}^g X$ for some indecomposable $\Bbbk Q$-module X. Then Y is an indecomposable $\Bbbk Q * G$-module. Note that since $Y \mid Y \otimes_{\Bbbk Q} \Bbbk Q * G \cong \Big(\bigoplus_{g \in G_X} {}^g X\Big) \otimes_{\Bbbk Q} \Bbbk Q * G$ and ${}^g X \otimes_{\Bbbk Q} \Bbbk Q * G \cong X \otimes_{\Bbbk Q} \Bbbk Q * G$ for any $g \in G$, we have $Y \mid X \otimes_{\Bbbk Q} \Bbbk Q * G$. Thus there exists a unique $i \in \{1, 2, \ldots, r\}$ such that $Y \cong (\varrho_i \otimes X) \otimes_{\Bbbk Q * H_X} \Bbbk Q * G$.

Following from Proposition 3.1, we get that an indecomposable G-invariant $\Bbbk Q$-module Y is a $\Bbbk Q * G$-module and indecomposable. Conversely, for an indecomposable $\Bbbk Q * G$-module Y, we have $Y \cong \bigoplus_{j=1}^{s} \Big(\bigoplus_{g \in G_{X_j}} {}^{g}X_j \Big)$ with some indecomposable $\Bbbk Q$-modules $X_1, X_2, \ldots, X_s$. Since $Y \mid Y \otimes_{\Bbbk Q} \Bbbk Q * G \cong \bigoplus_{j=1}^{s} \bigoplus_{g \in G_{X_j}} {}^{g}X_j \otimes_{\Bbbk Q} \Bbbk Q * G$, there exists j such that $Y \mid X_j \otimes_{\Bbbk Q} \Bbbk Q * G$. We denote by $\Bbbk H_{X_j} = \bigoplus_{i=0}^{r_j} \varrho_i^j$ the irreducible decomposition of $\Bbbk H_{X_j}$ as H_{X_j}-representations. Then there exists a unique ϱ_i^j such that $Y \cong (\varrho_i^j \otimes X_j) \otimes_{\Bbbk Q * H_{X_j}} \Bbbk Q * G \cong \bigoplus_{g \in G_{X_j}} {}^{g}X_j$ as $\Bbbk Q$-modules, so that Y is indecomposable. □

Following from this theorem, for any indecomposable $\Bbbk Q$-module X there are $|H_X|$ indecomposable $\Bbbk Q * G$-module structures on $\bigoplus_{g \in G_X} {}^{g}X$ which are $\{(\varrho_i \otimes X) \otimes_{\Bbbk Q * H_X} \Bbbk Q * G \colon 1 \leqslant i \leqslant |H_X|\}$. And all the irreducible $\Bbbk Q * G$-modules can be obtain in this way.

For convenience, we denote

$$\mathscr{X}^i := (\varrho_i \otimes X) \otimes_{\Bbbk Q * H_X} \Bbbk Q * G$$

for all $i \in \{1, 2, \ldots, |H_X|\}$.

4. Proof of main theorem

Let Q be a finite union of Dynkin quivers, let $G \subseteq \mathrm{Aut}(\Bbbk Q)$ be a finite abelian group. In this section, we discuss the structure of the quivers $\widehat{\Gamma_Q}$ and $\Gamma_{\widehat{Q}}$, and show the duality of the Auslander-Reiten quiver Γ_Q of $\Bbbk Q$.

For any $g \in G$, we have obtained in Section 2 an autoequivalence functor $F_g\colon \operatorname{mod}\text{-}\Bbbk Q \to \operatorname{mod}\text{-}\Bbbk Q$, $X \mapsto {}^{g}X$. Therefore, for any finite dimensional $\Bbbk Q$-modules X, Y and Z,

(1) $X \xrightarrow{\alpha} Y$ is an irreducible morphism if and only if ${}^{g}X \xrightarrow{{}^{g}\alpha} {}^{g}Y$ is;
(2) $X \xrightarrow{\alpha} Y$ is a (minimal) left (or right) almost split morphism if and only if ${}^{g}X \xrightarrow{{}^{g}\alpha} {}^{g}Y$ is;
(3) a short exact sequence $0 \to X \xrightarrow{\alpha} Z \xrightarrow{\beta} Y \to 0$ is an almost split sequence if and only if $0 \to {}^{g}X \xrightarrow{{}^{g}\alpha} {}^{g}Z \xrightarrow{{}^{g}\beta} {}^{g}Y \to 0$ is.

Denote by Γ_Q the Auslander-Reiten quiver of $\Bbbk Q$. Note that the quiver Γ_Q contains no multiple edges, $F_g \circ F_{g'} = F_{gg'}$ and $F_{g^{-1}} \circ F_g = \mathrm{Id}_{\operatorname{mod}\text{-}\Bbbk Q}$ for any $g, g' \in G$, there is a natural action of G on Γ_Q given by

$$g([X]) = [{}^gX], \quad g([X] \to [Y]) = [{}^gX] \to [{}^gY],$$

such that $G \subseteq \mathrm{Aut}(\Gamma_Q)$. Thus we obtain the generalized McKay quiver $\widehat{\Gamma_Q}$ of (Γ_Q, G) by the definition.

Let $\mathbf{I}$ denote the vertex set of Γ_Q, i.e., $\mathbf{I} = \{[X]\colon X \text{ is an indecomposable } \Bbbk Q\text{-module}\}$; let $\mathfrak{I}$ denote the set of representatives of the classes of $\mathbf{I}$ under the action of G; let $G_{\mathbf{i}}$ denote the subgroup of G stabilizing $\mathbf{i}$, for each $\mathbf{i} \in \mathbf{I}$. Obviously,

$$G_{\mathbf{i}} = H_X = \{g \in G\colon\ F_g(X) \cong X \text{ as } \Bbbk Q\text{-modules}\},$$

if $\mathbf{i} = [X]$ for an indecomposable $\Bbbk Q$-module X. By the definition, the vertex set $\hat{\mathbf{I}}$ of $\widehat{\Gamma_Q}$ is

$$\bigcup_{\mathbf{i}\in\mathfrak{I}} \{\mathbf{i}\} \times \operatorname{irr} G_{\mathbf{i}} = \{(\mathbf{i}, \varrho)\colon\ \mathbf{i} \in \mathfrak{I}, \varrho \in \operatorname{irr} G_{\mathbf{i}}\},$$

where $\operatorname{irr} G_{\mathbf{i}}$ is the set of representatives of isomorphism classes of irreducible representations of $G_{\mathbf{i}}$. Now, we write G as the product of some finite cyclic group, i.e.,

$$G = \langle g_1 \rangle \times \langle g_2 \rangle \times \ldots \times \langle g_n \rangle,$$

where the order of g_l is m_l for $1 \leqslant l \leqslant n$. Then, each $G_{\mathbf{i}}$ has the form

$$G_{\mathbf{i}} = \langle g_1^{d_{\mathbf{i}_1}} \rangle \times \langle g_2^{d_{\mathbf{i}_2}} \rangle \times \ldots \times \langle g_n^{d_{\mathbf{i}_n}} \rangle,$$

where $\nu_{\mathbf{i}_l} := |\langle g_j^{d_{\mathbf{i}_l}} \rangle| = m_l/d_{\mathbf{i}_l}$, $1 \leqslant l \leqslant n$, so that

$$d_{\mathbf{i}} := |\mathscr{O}_{\mathbf{i}}| = \frac{|G|}{|G_{\mathbf{i}}|} = d_{\mathbf{i}_1} \times d_{\mathbf{i}_2} \times \ldots \times d_{\mathbf{i}_n}.$$

For each $l \in \{1, 2, \ldots, n\}$, we assume that ξ_l is a primitive m_lth root of unity. Let $e_{(\mathbf{i}, s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_n})}$ be

$$\frac{1}{|G_{\mathbf{i}}|} \sum_{j_1=0}^{\nu_{\mathbf{i}_1}-1} \sum_{j_2=0}^{\nu_{\mathbf{i}_2}-1} \cdots \sum_{j_n=0}^{\nu_{\mathbf{i}_n}-1} \xi_1^{d_{\mathbf{i}_1} j_1 s_{\mathbf{i}_1}} \xi_2^{d_{\mathbf{i}_2} j_2 s_{\mathbf{i}_2}} \ldots \xi_n^{d_{\mathbf{i}_n} j_n s_{\mathbf{i}_n}} g_1^{d_{\mathbf{i}_1} j_1} g_2^{d_{\mathbf{i}_2} j_2} \ldots g_n^{d_{\mathbf{i}_n} j_n}.$$

Then one can check that $\{e_{(\mathbf{i}, s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_n})}\colon\ s_{\mathbf{i}_l} \in \mathbb{Z}/\nu_{\mathbf{i}_l}\mathbb{Z} \text{ for all } 1 \leqslant l \leqslant n\}$ is a complete set of primitive orthogonal idempotents of $\Bbbk[G_{\mathbf{i}}]$. Note that each $e_{(\mathbf{i}, s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_n})}$ corresponding to a unique irreducible representation ϱ of $G_{\mathbf{i}}$ is defined by the group homomorphism $\varphi_\varrho\colon\ G_{\mathbf{i}} \to \Bbbk$, $g_j^{d_{\mathbf{i}_l}} \mapsto \xi^{d_{\mathbf{i}_l} s_{\mathbf{i}_l}}$, $1 \leqslant l \leqslant n$; we reindex $\hat{\mathbf{I}}$ by

$$\hat{\mathbf{I}} = \{(\mathbf{i}, s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_n})\colon\ \mathbf{i} \in \mathfrak{I},\ s_{\mathbf{i}_l} \in \mathbb{Z}/\nu_{\mathbf{i}_l}\mathbb{Z} \text{ for all } 1 \leqslant l \leqslant n\}.$$

Obviously, $|\hat{\mathbf{I}}| = \sum_{\mathbf{i}\in\mathfrak{I}} |G_{\mathbf{i}}|$.

For any $\mathbf{i}=[X]$, $\mathbf{j}=[Y]\in\mathfrak{I}$, we consider the group $G_{\mathbf{ij}}=G_{\mathbf{i}}\cap G_{\mathbf{j}}=\langle g_1^{t_1}\rangle\times\langle g_2^{t_2}\rangle\times\ldots\times\langle g_n^{t_n}\rangle$, where t_l is the least common multiple of $d_{\mathbf{i}_l}$ and $d_{\mathbf{j}_l}$ for $1\leqslant l\leqslant n$. Note that the vector space $E_{\mathbf{ij}}$ spanned by arrows $\alpha\colon\ \mathbf{i}\to\mathbf{j}$ in Γ_Q is a $\Bbbk[G_{\mathbf{ij}}]$-bimodule and is 1-dimensional as a $\Bbbk$-vector space, the action of $g=g_1^{t_1}g_2^{t_2}\ldots g_n^{t_n}$ on $E_{\mathbf{ij}}$ is an identity.

Next, we calculate

$$\begin{aligned}&e_{(\mathbf{j},s_{\mathbf{j}_1},s_{\mathbf{j}_2},\ldots,s_{\mathbf{j}_n})}\alpha e_{(\mathbf{i},s_{\mathbf{i}_1},s_{\mathbf{i}_2},\ldots,s_{\mathbf{i}_n})}\\&\quad=\frac{d_{\mathbf{i}}d_{\mathbf{j}}}{|G|^2}\sum_{p_1=0}^{\nu_{\mathbf{i}_1}-1}\cdots\sum_{p_n=0}^{\nu_{\mathbf{i}_n}-1}\sum_{q_1=0}^{\nu_{\mathbf{j}_1}-1}\cdots\sum_{q_n=0}^{\nu_{\mathbf{j}_n}-1}\xi_1^{d_{\mathbf{i}_1}p_1s_{\mathbf{i}_1}+d_{\mathbf{j}_1}q_1s_{\mathbf{j}_1}}\ldots\xi_n^{d_{\mathbf{i}_n}p_ns_{\mathbf{i}_n}+d_{\mathbf{j}_n}q_ns_{\mathbf{j}_n}}\\&\qquad\qquad g_1^{d_{\mathbf{j}_1}q_1}\ldots g_n^{d_{\mathbf{j}_n}q_n}(\alpha)g_1^{d_{\mathbf{i}_1}p_1+d_{\mathbf{j}_1}q_1}\ldots g_n^{d_{\mathbf{i}_n}p_n+d_{\mathbf{j}_n}q_n}.\end{aligned}$$

We write

$$\begin{aligned}d_{\mathbf{i}_l}p_l&=P_lt_l+d_{\mathbf{i}_l}p'_l, &&\text{where } 0\leqslant P_l<\frac{m_l}{t_l},\ 0\leqslant p'_l<\frac{t_l}{d_{\mathbf{i}_l}},\\ d_{\mathbf{j}_l}q_l&=P'_lt_l+d_{\mathbf{j}_l}q'_l, &&\text{where } 0\leqslant P'_l<\frac{m_l}{t_l},\ 0\leqslant q'_l<\frac{t_l}{d_{\mathbf{j}_l}},\\ d_{\mathbf{i}_l}k_l&\equiv(P_l+P'_l)t_l+d_{\mathbf{i}_l}p'_l \bmod m_l, &&\text{where } 0\leqslant k_l<\nu_{\mathbf{i}_l}\end{aligned}$$

for all $0\leqslant l\leqslant n$. Then the right-hand side of the equation becomes

$$\begin{aligned}&\frac{d_{\mathbf{i}}d_{\mathbf{j}}}{|G|^2}\sum_{P'_1=0}^{m_1/t_1-1}\xi_1^{P'_1t_1(s_{\mathbf{j}_1}-s_{\mathbf{i}_1})}\cdots\sum_{P'_n=0}^{m_n/t_n-1}\xi_n^{P'_nt_n(s_{\mathbf{j}_n}-s_{\mathbf{i}_n})}\\&\quad\sum_{k_1=0}^{\nu_{\mathbf{i}_1}-1}\cdots\sum_{k_n=0}^{\nu_{\mathbf{i}_n}-1}\sum_{q'_1=0}^{t_1/d_{\mathbf{j}_1}-1}\cdots\sum_{q'_n=0}^{t_n/d_{\mathbf{j}_n}-1}\xi_1^{d_{\mathbf{i}_1}k_1s_{\mathbf{i}_1}+d_{\mathbf{j}_1}q'_1s_{\mathbf{j}_1}}\ldots\xi_n^{d_{\mathbf{i}_n}k_ns_{\mathbf{i}_n}+d_{\mathbf{j}_n}q'_ns_{\mathbf{j}_n}}\\&\qquad\qquad g_1^{d_{\mathbf{j}_1}q'_1}\ldots g_n^{d_{\mathbf{j}_n}q'_n}(\alpha)g_1^{d_{\mathbf{i}_1}k_1+d_{\mathbf{j}_1}q'_1}\ldots g_n^{d_{\mathbf{i}_n}k_n+d_{\mathbf{i}_n}q'_n}.\end{aligned}$$

It is easy to see that

$$\Big\{g_1^{d_{\mathbf{j}_1}q'_1}\ldots g_n^{d_{\mathbf{j}_n}q'_n}(\alpha)g_1^{d_{\mathbf{i}_1}k_1+d_{\mathbf{j}_1}q'_1}\ldots g_n^{d_{\mathbf{i}_n}k_n+d_{\mathbf{j}_n}q'_n}\colon\ 0\leqslant k_l<\nu_{\mathbf{i}_l},\ 0\leqslant q'_l<\frac{t_l}{d_{\mathbf{j}_l}}\ \text{for } 1\leqslant l\leqslant n\Big\}$$

is a linearly independent set. Thus $e_{(\mathbf{j},s_{\mathbf{j}_1},s_{\mathbf{j}_2},\ldots,s_{\mathbf{j}_n})}\alpha e_{(\mathbf{i},s_{\mathbf{i}_1},s_{\mathbf{i}_2},\ldots,s_{\mathbf{i}_n})}\neq 0$ if and only if $s_{\mathbf{i}_l}\equiv s_{\mathbf{j}_l}\bmod m_l/t_l$ for all $0\leqslant l\leqslant n$. It follows that, for any arrow $\mathbf{i}\to\mathbf{j}$ in Γ_Q, we get an arrow $(\mathbf{i},s_{\mathbf{i}_1},s_{\mathbf{i}_2},\ldots,s_{\mathbf{i}_n})\to(\mathbf{j},s_{\mathbf{j}_1},s_{\mathbf{j}_2},\ldots,s_{\mathbf{j}_n})$ in $\widehat{\Gamma_Q}$ for each sequence

$(s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_n}, s_{\mathbf{j}_1}, s_{\mathbf{j}_2}, \ldots, s_{\mathbf{j}_n})$ satisfying $s_{\mathbf{i}_l} \equiv s_{\mathbf{j}_l} \bmod m_l/t_l$ for all $0 \leqslant l \leqslant n$. And all the arrows in $\widehat{\Gamma_Q}$ can be got in this way.

In particular, if $G_{\mathbf{i}} \supseteq G_{\mathbf{j}}$, there are $|G|/d_{\mathbf{i}}$ arrows from $(\mathbf{i}, s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_n})$ to $(\mathbf{j}, s_{\mathbf{j}_1}, s_{\mathbf{j}_2}, \ldots, s_{\mathbf{j}_n})$ in $\widehat{\Gamma_Q}$, for any irreducible morphism $\mathbf{i} \to \mathbf{j}$. More precisely, if $|G_{\mathbf{i}}/G_{\mathbf{j}}| = k$, i.e., $\sum\limits_{l=1}^{n} d_{\mathbf{j}_l}/d_{\mathbf{i}_l} = k$. Then, for any fixed vertex $(\mathbf{j}, s_{\mathbf{j}_1}, s_{\mathbf{j}_2}, \ldots, s_{\mathbf{j}_n})$ in $\widehat{\Gamma_Q}$, there are k vertices $(\mathbf{i}, s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_n})$ satisfying $s_{\mathbf{i}_l} \equiv s_{\mathbf{j}_l} \bmod m_l/d_{\mathbf{j}_l}$ for all $0 \leqslant l \leqslant n$. Thus we can reindex the set $\big\{(\mathbf{i}, s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_n})\colon\ s_{\mathbf{i}_l} \in \mathbb{Z}/\nu_{\mathbf{i}_l}\mathbb{Z}$ for all $1 \leqslant l \leqslant n\big\}$ by

$$\{(\mathbf{j}, s_{\mathbf{j}_t}, s_{\mathbf{j}_2}, \ldots, s_{\mathbf{j}_n})^t\colon\ 1 \leqslant t \leqslant k \text{ and } s_{\mathbf{j}_l} \in \mathbb{Z}/\nu_{\mathbf{j}_l}\mathbb{Z} \text{ for all } 1 \leqslant l \leqslant n\},$$

so that there is an arrow $(\mathbf{j}, s_{\mathbf{j}_1}, s_{\mathbf{j}_2}, \ldots, s_{\mathbf{j}_n})^t \to (\mathbf{j}, s_{\mathbf{j}_1}, s_{\mathbf{j}_2}, \ldots, s_{\mathbf{j}_n})$ for all t.

Similarly, if $|G_{\mathbf{j}}/G_{\mathbf{i}}| = k$, we can reindex the set $\{(\mathbf{j}, s_{\mathbf{j}_1}, s_{\mathbf{j}_2}, \ldots, s_{\mathbf{j}_n})\colon\ s_{\mathbf{j}_l} \in \mathbb{Z}/\nu_{\mathbf{j}_l}\mathbb{Z}$ for all $1 \leqslant l \leqslant n\}$ by

$$\{(\mathbf{i}, s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_n})^t\colon\ 1 \leqslant t \leqslant k \text{ and } s_{\mathbf{i}_l} \in \mathbb{Z}/\nu_{\mathbf{i}_l}\mathbb{Z} \text{ for all } 1 \leqslant l \leqslant n\},$$

so that there is an arrow $(\mathbf{i}, s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_n}) \to (\mathbf{i}, s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_n})^t$ for all t.

On the other hand, we consider the Auslander-Reiten quiver $\Gamma_{\widehat{Q}}$ of $\Bbbk\widehat{Q}$, where we identify $\Gamma_{\widehat{Q}}$ with the Auslander-Reiten quiver of $\Bbbk Q * G$. Following from the result in Section 2, all the indecomposable $\Bbbk Q * G$-modules (up to isomorphism) are

$$\mathbb{I} := \{\mathscr{X}^i\colon\ [X] \in \mathfrak{I},\ 1 \leqslant i \leqslant |H_X|\}.$$

Obviously, the vertex set $\mathbb{I}$ of $\Gamma_{\widehat{Q}}$ satisfies $|\mathbb{I}| = \sum\limits_{[\mathbf{X}] \in \mathfrak{I}} |H_X| = \sum\limits_{\mathbf{i} \in \mathfrak{I}} |G_{\mathbf{i}}| = |\hat{\mathbf{I}}|$.

Before characterizing the arrows in $\Gamma_{\widehat{Q}}$, we need some facts.

Lemma 4.1 (see [15])**.** *Let X, Y be indecomposable $\Bbbk Q$-modules, let X', Y' be indecomposable $\Bbbk Q * G$-modules. Then*

1. *if $X \to Y$ is a minimal left (or right) almost split morphism in* mod-$\Bbbk Q$*, then $X \otimes_{\Bbbk Q} \Bbbk Q * G \to Y \otimes_{\Bbbk Q} \Bbbk Q * G$ is the direct sum of some minimal left (or right) almost split morphisms in* mod-$\Bbbk Q * G$*;*
2. *if $X' \to Y'$ is a minimal left (or right) almost split morphism in* mod-$\Bbbk Q * G$*, then $X' \to Y'$ in* mod-$\Bbbk Q$ *is the direct sum of some minimal left (or right) almost split morphisms.*

Lemma 4.2 (see [2])**.** *Assume that $X, Y, Z, Z' \in$* mod-$\Bbbk Q$ *and X, Y are indecomposable. Then*

(1) *a morphism* $\beta\colon Z \to Y$ *is irreducible if and only if there exists a morphism* $\beta'\colon Z' \to Y$ *such that* $(\beta, \beta')\colon Z \oplus Z' \to Y$ *is a minimal right almost split morphism in* mod-$\Bbbk Q$*;*

(2) *a morphism* $\alpha\colon X \to Z$ *is irreducible if and only if there exists a morphism* $\alpha'\colon X \to Z'$ *such that* $\binom{\alpha}{\alpha'}\colon X \to Z \oplus Z'$ *is a minimal left almost split morphism in* mod-$\Bbbk Q$.

Let $Q = (I, E)$ be a finite quiver. For any Q-representation $X = (X_i, X_\alpha)$, we denote $I_X := \{i \in I\colon X_i \neq 0\}$ and call it the support of X.

Lemma 4.3. *Let* $Q = (I, E)$ *be a finite union of Dynkin quivers and* $G \subseteq \mathrm{Aut}(\Bbbk Q)$ *a finite abelian group. For any indecomposable* Q*-representations* $X = (X_i, X_\alpha)$ *and* $Y = (Y_i, Y_\alpha)$*, if the supports of* X *and* Y *are in the same connected component of* Q*, then we have*

$$H_X \subseteq H_Y \quad \text{or} \quad H_Y \subsetneq H_X.$$

Proof. Let vertices i and j be in the same connected component of Q. Suppose that there exists an arrow between i and j and $|\mathscr{O}_i| \geqslant |\mathscr{O}_j|$. Note that the connected component is Dynkin, and there are at most three edges connections in Dynkin diagram; we get $|\mathscr{O}_i| = n|\mathscr{O}_j|$, where $n = 1, 2$, or 3. Clearly, $G_i = G_j$ if $|\mathscr{O}_i| = |\mathscr{O}_j|$, and $G_i \subset G_j$ if $|\mathscr{O}_i| > |\mathscr{O}_j|$. By induction, we have $G_i \subseteq G_j$ or $G_j \subsetneq G_i$ for any two vertices i and j in the same connected component of Q.

Moreover, it is easy to see that

$$H_X = \bigcap_{i \in I_X} G_i \quad \text{and} \quad H_Y = \bigcap_{i \in I_Y} G_i.$$

Hence $H_X = G_i$ for some $i \in I_X$ and $H_Y = G_j$ for some $j \in I_Y$. We get the lemma. □

Now, we consider the arrows in $\Gamma_{\widehat{Q}}$. Let $X \to Y$ be an irreducible morphism in mod-$\Bbbk Q$. We suppose that $H_X \supseteq H_Y$ and denote $H_X/H_Y := \{g_1 + H_Y, g_2 + H_Y, \ldots, g_k + H_Y\}$. By Lemma 4.2, there exists a $\Bbbk Q$-module M satisfying ${}^gY$ is not a summand of M, such that

$$X \to \left(\bigoplus_{i=1}^{k} {}^{g_i}Y\right) \oplus M$$

is a minimal left almost split sequence in mod-$\Bbbk Q$. By Lemma 4.1,

$$X \otimes_{\Bbbk Q} \Bbbk Q * G \to \left(\bigoplus_{i=1}^{k} {}^{g_i}Y \otimes_{\Bbbk Q} \Bbbk Q * G\right) \oplus M \otimes_{\Bbbk Q} \Bbbk Q * G,$$

i.e., $\mathscr{X}^1 \oplus \ldots \oplus \mathscr{X}^{|H_X|} \to (\mathscr{Y}^1)^{\oplus k} \oplus \ldots \oplus (\mathscr{Y}^{|H_Y|})^{\oplus k} \oplus M \otimes_{\Bbbk Q} \Bbbk Q * G$ is the direct sum of some minimal left almost split sequence in mod-$\Bbbk Q * G$, where $(\mathscr{Y}^i)^{\oplus k} := \underbrace{\mathscr{Y}^i \oplus \ldots \oplus \mathscr{Y}^i}_{k \text{ fold}}$ for $1 \leqslant i \leqslant |H_Y|$.

Thus, there exist $|H_X|$ arrows in $\Gamma_{\widehat{Q}}$ corresponding to the irreducible morphism $X \to Y$. More precisely, by Lemma 4.1, there exists a permutation ω on $\{1, 2, \ldots, |H_X|\}$ such that for each i there are irreducible morphisms $\mathscr{X}^{\omega(j)} \to \mathscr{Y}^i$ for all $ik \leqslant j \leqslant (i+1)k - 1$.

For the case $H_X \subseteq H_Y$ and $|H_Y/H_X| = k$, we can get a similar conclusion: there exists a permutation ω on $\{1, 2, \ldots, |H_Y|\}$ such that there is an irreducible morphism $\mathscr{X}^i \to \mathscr{Y}^{\omega(j)}$ for all $1 \leqslant i \leqslant |H_X|$ and $ik \leqslant j \leqslant (i+1)k - 1$.

We are now in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Define a map $\Phi\colon \Gamma_{\widehat{Q}} \to \widehat{\Gamma_Q}$ as follows. For each irreducible morphism $X \to Y$ in mod-$\Bbbk Q$, by Lemma 4.3, $H_X = G_{\mathbf{i}} \supseteq G_{\mathbf{j}} = H_Y$ or $H_X = G_{\mathbf{i}} \subsetneq G_{\mathbf{j}} = H_Y$.

(1) If $H_X = G_{\mathbf{i}} \supseteq G_{\mathbf{j}} = H_Y$ and $|H_X/H_Y| = k$, $k \geqslant 1$, then

$$\begin{aligned} \Phi\colon \qquad \mathscr{Y}^i &\mapsto (\mathbf{j}, s_{\mathbf{j}_1}, s_{\mathbf{j}_2}, \ldots, s_{\mathbf{j}_n}) \\ \mathscr{X}^{\omega(j)} &\mapsto (\mathbf{i}, s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_n}) = (\mathbf{j}, s_{\mathbf{j}_1}, s_{\mathbf{j}_2}, \ldots, s_{\mathbf{j}_n})^t, \end{aligned}$$

where $t \equiv j \bmod ki$ and $e_{(\mathbf{j}, s_{\mathbf{j}_1}, s_{\mathbf{j}_2}, \ldots, s_{\mathbf{j}_n})}$ is the idempotent of $\Bbbk[G_{\mathbf{j}}]$ corresponding to the irreducible representation ϱ_i of $G_{\mathbf{j}}$. Note that $|H_X| = |G_{\mathbf{i}}|$ and $|H_Y| = |G_{\mathbf{j}}|$; Φ defines two one-to-one correspondences between $\{\mathscr{X}^j\colon 1 \leqslant j \leqslant |H_X|\}$, $\{\mathscr{Y}^i\colon 1 \leqslant i \leqslant |H_Y|\}$ and $\{(\mathbf{i}, s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_n})\colon s_{\mathbf{i}_l} \in \mathbb{Z}/\nu_{\mathbf{i}_l}\mathbb{Z}$ for all $1 \leqslant l \leqslant n\}$, $\{(\mathbf{j}, s_{\mathbf{j}_1}, s_{\mathbf{j}_2}, \ldots, s_{\mathbf{j}_n})\colon s_{\mathbf{j}_l} \in \mathbb{Z}/\nu_{\mathbf{j}_l}\mathbb{Z}$ for all $1 \leqslant l \leqslant n\}$, respectively.

In this case, for the irreducible morphism $X \to Y$, there are an arrow $\mathscr{X}^{\omega(j)} \to \mathscr{Y}^i$ in $\Gamma_{\widehat{Q}}$ and an arrow $(\mathbf{j}, s_{\mathbf{j}_1}, s_{\mathbf{j}_2}, \ldots, s_{\mathbf{j}_n})^t \to (\mathbf{j}, s_{\mathbf{j}_1}, s_{\mathbf{j}_2}, \ldots, s_{\mathbf{j}_n})$ in $\widehat{\Gamma_Q}$. Let Φ map $\mathscr{X}^{\omega(j)} \to \mathscr{Y}^i$ to $(\mathbf{j}, s_{\mathbf{j}_1}, s_{\mathbf{j}_2}, \ldots, s_{\mathbf{j}_n})^t \to (\mathbf{j}, s_{\mathbf{j}_1}, s_{\mathbf{j}_2}, \ldots, s_{\mathbf{j}_n})$.

(2) if $H_X = G_{\mathbf{i}} \subsetneq G_{\mathbf{j}} = H_Y$ and $|H_Y/H_X| = k$, $k > 1$, then

$$\begin{aligned} \Phi\colon \qquad \mathscr{X}^i &\mapsto (\mathbf{i}, s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_n}) \\ \mathscr{Y}^{\omega(j)} &\mapsto (\mathbf{j}, s_{\mathbf{j}_1}, s_{\mathbf{j}_2}, \ldots, s_{\mathbf{j}_n}) = (\mathbf{i}, s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_n})^t, \end{aligned}$$

where $t \equiv j \bmod ki$ and $e_{(\mathbf{i}, s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_n})}$ is the idempotent of $\Bbbk[G_{\mathbf{i}}]$ corresponding to the irreducible representation ϱ_i of $G_{\mathbf{i}}$. Similarly, Φ defines also two one-to-one correspondences between the vertices in $\Gamma_{\widehat{Q}}$ and $\widehat{\Gamma_Q}$ corresponding to X and Y, respectively.

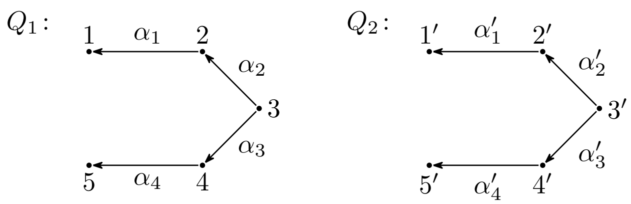
In this case, for the irreducible morphism $X \to Y$, there are an arrow $\mathscr{X}^i \to \mathscr{Y}^{\omega(j)}$ in $\Gamma_{\widehat{Q}}$ and an arrow $(\mathbf{i}, s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_n}) \to (\mathbf{i}, s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_n})^t$ in $\widehat{\Gamma_Q}$. Let Φ map $\mathscr{X}^i \to \mathscr{Y}^{\omega(j)}$ to $(\mathbf{i}, s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_n}) \to (\mathbf{i}, s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_n})^t$.

Then, it is easy to see that $\Phi\colon \Gamma_{\widehat{Q}} \to \widehat{\Gamma_Q}$ is a quiver isomorphism, and $\Gamma_{\widehat{Q}} = \widehat{\Gamma_Q}$. By [9], Propsoition 3.6, there is an action of G on $\widehat{\Gamma_Q}$ such that $\widehat{\Gamma_{\widehat{Q}}} = \widehat{\widehat{\Gamma_Q}} = \Gamma_Q$. The proof is completed. □

5. An example

In the end of this paper, we use an example to show the duality of (Q, G), (Γ_Q, G) and the valued quiver corresponding to (Q, G), whenever Q is a finite union of Dynkin quivers and $G \subseteq \mathrm{Aut}(\Bbbk Q)$ is abelian.

Let $Q = (I, E) = Q_1 \cup Q_2$ be the quiver

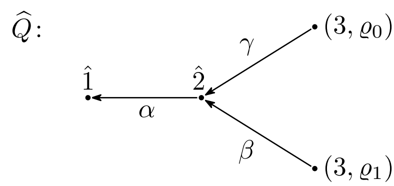


and consider the group $G = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We consider an action of G on $\Bbbk Q$ as follows:

	e_1	e_2	e_3	e_4	e_5	$e_{1'}$	$e_{2'}$	$e_{3'}$	$e_{4'}$	$e_{5'}$
a	e_5	e_4	e_3	e_2	e_1	$e_{5'}$	$e_{4'}$	$e_{3'}$	$e_{2'}$	$e_{1'}$
b	$e_{1'}$	$e_{2'}$	$e_{3'}$	$e_{4'}$	$e_{5'}$	e_1	e_2	e_3	e_4	e_5

	α_1	α_2	α_3	α_4	α'_1	α'_2	α'_3	α'_4
a	$-\alpha_4$	$-\alpha_3$	$-\alpha_2$	$-\alpha_1$	$-\alpha'_4$	$-\alpha'_3$	$-\alpha'_2$	$-\alpha'_1$
b	α'_1	α'_2	α'_3	α'_4	α_1	α_2	α_3	α_4

where e_i is the idempotent element of $\Bbbk Q$ corresponding to a vertex i, $i \in I$. Then one can calculate directly that the generalized McKay quiver of (Q, G) is



where we take $\mathscr{I} = \{1, 2, 3\}$ and ϱ_0, ϱ_1 are the non-isomorphism irreducible representations of $G_3 = \langle a \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

Since G is abelian, all the character of G are linear, i.e., the group homomorphism $\chi\colon G \to \Bbbk$. The group of all the characters of G with multiplication $\chi\chi'(g) = \chi(g)\chi'(g)$, $g \in G$, is also an abelian group, denoted by $\widetilde{G}$. Setting $\varphi\colon G \to \widetilde{G}$ as $\varphi(g) = \chi_g$, $\chi_g(g') = (-1)^{t_1s_1+t_2s_2}$ if $g = a^{t_1}b^{t_2}$ and $g' = a^{s_1}b^{s_2}$, where $t_1, t_2, s_1, s_2 \in \{0, 1\}$, then φ is a group isomorphism.

Following from [15], we can define a linear action of G on $\Bbbk Q * G$ by $g(\lambda h) = \chi_g(h)\lambda h$, $g \in G$, $\lambda h \in \Bbbk Q * G$. Then $G \subseteq \mathrm{Aut}(\Bbbk Q * G)$ and under this action, we can prove that $(\Bbbk Q * G) * G$ is Morita equivalent to $\Bbbk Q$ (see [9], Proposition 3.7). Let $e := \sum\limits_{i\in\mathscr{I}} e_i \in \Bbbk Q \subseteq \Bbbk Q * G$. Since $e\Bbbk Q * Ge \cong \Bbbk\widehat{Q}$ (see [3], Theorem 1) and the action of G on $\Bbbk Q * G$ stabilizes e, the action of G on $\Bbbk Q * G$ naturally induces an action of G on $\Bbbk\widehat{Q}$ as follows:

	$e_{\widehat{1}}$	$e_{\widehat{2}}$	$e_{(3,\varrho_0)}$	$e_{(3,\varrho_1)}$	α	β	γ
a	$e_{\widehat{1}}$	$e_{\widehat{2}}$	$e_{(3,\varrho_1)}$	$e_{(3,\varrho_0)}$	$\xi_1\alpha$	$\xi_2\gamma$	$\xi_3\beta$
b	$e_{\widehat{1}}$	$e_{\widehat{2}}$	$e_{(3,\varrho_0)}$	$e_{(3,\varrho_1)}$	$\xi_4\alpha$	$\xi_5\beta$	$\xi_6\gamma$

where $\xi_1, \xi_4, \xi_5, \xi_6 \in \{1, -1\}$, the idempotent element e_i corresponds to the vertex i, $i \in \{\widehat{1}, \widehat{2}, (3, \varrho_0), (3, \varrho_1)\}$ and $\xi_2, \xi_3 \in \Bbbk$ satisfy $\xi_3\xi_4 = 1$. Then, one can check that the generalized McKay quiver $\widehat{\widehat{Q}}$ of $(\widehat{Q}, G)$ is just the quiver Q.

For any quiver Q with an action of G on the path algebra $\Bbbk Q$, we can construct a symmetric matrix $B = (b_{ij})$ indexed by $\mathscr{I}$ by setting

$$b_{ij} = \begin{cases} 2|\mathscr{O}_i|, & i = j; \\ -|\{\text{edges between vertices in } \mathscr{O}_i \text{ and } \mathscr{O}_j\}|, & i \neq j. \end{cases}$$

Let $d_i := \frac{1}{2}b_{ii} = |\mathscr{O}_i|$ and $D = \mathrm{diag}(d_i)$. Then $C = (c_{ij}) = D^{-1}B$ is a symmetrizable generalized Cartan matrix indexed by $\mathscr{I}$. We write Γ for the corresponding valued graph, that is, Γ has the vertex set $\mathscr{I}$ and we draw an edge i—j equipped with the ordered pair $(|c_{ji}|, |c_{ij}|)$ whenever $c_{ij} \neq 0$.

For our quivers Q and $\widehat{Q}$, we denote by Γ and $\widehat{\Gamma}$ the corresponding valued graphs (Q, G) and $(\widehat{Q}, G)$, respectively. By direct calculation, it is easy to see that the generalized Cartan matrices of Γ and $\widehat{\Gamma}$ are

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix} \quad \text{and} \quad \widehat{C} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}.$$

Obviously, $\widehat{C}$ is the transposed matrix of C. Therefore Γ and $\widehat{\Gamma}$ are dual valued graphs, in the sense of [12].

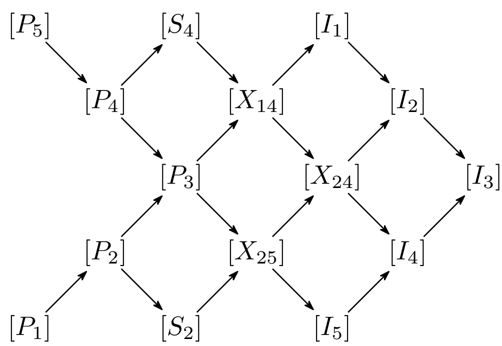
Let $\widehat{\Gamma_Q}$ denote the generalized McKay quiver of (Γ_Q, G) and $\Gamma_{\widehat{Q}}$ the Auslander-Reiten quiver of $\Bbbk\widehat{Q}$. For the quiver $Q = Q_1 \cup Q_2$ and the action of $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ given as above, we now show that $\widehat{\Gamma_Q} = \Gamma_{\widehat{Q}}$ and $\widehat{\Gamma_{\widehat{Q}}} = \Gamma_Q$. First, we consider all indecomposable representations of Q_1. For $1 \leqslant i \leqslant j \leqslant 5$, we denote by $X_{ij} = (X_i, X_\alpha)$ the representation of Q_1 with

$$X_l = \begin{cases} \Bbbk, & \text{if } i \leqslant l \leqslant j; \\ 0, & \text{otherwise,} \end{cases} \qquad X_\alpha = \begin{cases} 1, & \text{if } \alpha\colon\ k \to l, \text{ where } i \leqslant k,\, l \leqslant j; \\ 0, & \text{otherwise.} \end{cases}$$

We denote by P_i, I_i and S_i the projective, injective and simple representation corresponding to vertex i, respectively. It is well-known that all the indecomposable Q_1-representations are

$$\begin{array}{lllll} X_{11} = P_1 = S_1, & X_{12} = P_2, & X_{13} = I_1, & X_{14}, & X_{15} = P_3, \\ X_{22} = S_2, & X_{23} = I_2, & X_{24}, & X_{25}, & X_{33} = I_3 = S_3, \\ X_{34} = I_4, & X_{35} = I_5, & X_{44} = S_4, & X_{45} = P_4, & X_{55} = P_5 = S_5 \end{array}$$

and the Auslander-Reiten quiver Γ_{Q_1} is



Thus the Auslander-Reiten quiver Γ_Q is a double copy of Γ_{Q_1}.

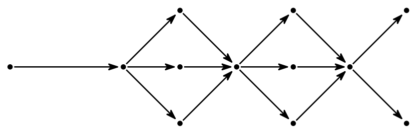
Secondly, following from the action of G on $\Bbbk Q$, the action of $G = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is as follows: $b(X_{ij}) = X'_{ij}$,

	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{22}	X_{23}	X_{24}	X_{25}	X_{33}	X_{34}	X_{35}	X_{44}	X_{45}	X_{55}
a	X_{55}	X_{45}	X_{35}	X_{25}	X_{15}	X_{44}	X_{34}	X_{24}	X_{14}	X_{33}	X_{23}	X_{13}	X_{22}	X_{12}	X_{11}

and the the action of a on X'_{ij} is given by $a(X'_{ij}) = X'_{kl}$ if $a(X_{ij}) = X_{kl}$, where X'_{ij} is the indecomposable Q_2-representation defined similarly to X_{ij}. It is easy to see that the action of G commutes with the Auslander-Reiten translate.

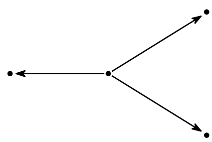
By direct calculation, we have

$\widehat{\Gamma_Q}$:



This quiver coincides with the Auslander-Reiten quiver $\Gamma_{\widehat{Q}}$ of $\Bbbk\widehat{Q}$, so that $\widehat{\Gamma_{\widehat{Q}}} = \Gamma_Q$.

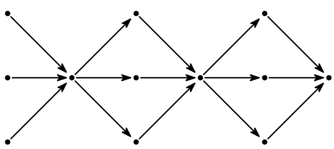
At last, we remark that if the group is non-abelian, the conclusion is not true in general. For example, let Q be the quiver



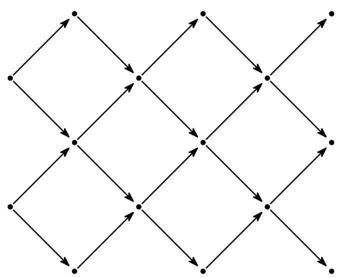
We consider the action of S_3, the quiver automorphism group of Q. Accordingly, we obtain the generalized McKay quiver $\widehat{Q}$ of (Q, S_3) as follows:

It is well-known that the Auslander-Reiten quivers of $\Bbbk Q$ and $\Bbbk\widehat{Q}$ are

Γ_Q:



$\Gamma_{\widehat{Q}}$:



One can check that there exists no subgroup G' of $\mathrm{Aut}(\Bbbk\widehat{Q})$ such that the generalized McKay quiver of $(\widehat{Q}, G')$ is Q, there exists no subgroup G' of $\mathrm{Aut}(\Bbbk\widehat{Q})$ such that the generalized McKay quiver of (Γ_Q, G') is $\Gamma_{\widehat{Q}}$.

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