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INVARIANT HARMONIC UNIT VECTOR FIELDS ON THE OSCILLATOR GROUPS

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Abstract. We find all the left-invariant harmonic unit vector fields on the oscillator groups. Besides, we determine the associated harmonic maps from the oscillator group into its unit tangent bundle equipped with the associated Sasaki metric. Moreover, we investigate the stability and instability of harmonic unit vector fields on compact quotients of four dimensional oscillator group $G_1(1)$.

Keywords: harmonic vector field; harmonic map; oscillator group

MSC 2010: 53C25, 53C43

1. Introduction

Recall that a unit vector field V on a Riemannian manifold (M, g) determines a map from (M, g) to its unit tangent bundle (T_1M, g_S) equipped with the Sasaki metric g_S. When M is closed and orientable, the *energy* of V is the energy of the corresponding map. V is said to be a *harmonic* vector field if it determines a critical point for the energy functional. This kind of vector fields have also been studied in [8], where similar notions are introduced when M is non-compact and non-orientable. It should be noted that harmonic vector fields do not necessarily yield harmonic maps.

Several examples related to the harmonicity of a unit vector field and of the corresponding map are provided in [1], [2], [10] and [16]. In addition, in [17], Vanhecke and González-Dávila have studied the existence and classification of invariant harmonic unit vector fields on some Lie groups equipped with left invariant metrics. They proved that every unimodular Lie group admits a left invariant harmonic unit

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vector field, and this is also true for any odd-dimensional Lie group. However, it has been an open problem whether the above assertion holds for an even-dimensional non-unimodular Lie group. Furthermore, they showed that every harmonic left invariant unit vector field determines a harmonic map into its unit tangent bundle on a Lie group with bi-invariant metric. They also proved that on the Damek-Ricci spaces there does not exist any invariant unit vector field such that the corresponding map into the unit tangent bundle is harmonic, although harmonic invariant unit vector fields always exist. Besides, all left invariant unit vector fields determine a harmonic map of $(G, g = -B)$ into its unit tangent bundle (T_1G, g_S) on a compact and semisimple Lie group G with Killing form B (also see [17]). And, they have investigated the stability and instability of harmonic unit vector fields for the energy functional on compact quotients of three dimensional unimodular Lie groups (see [11]).

On the other hand, the study of oscillator groups have many applications both in geometry and physics. For instance, in [13], Medina proved that oscillator groups are, except for direct extensions with Euclidean groups, the only non-commutative simple connected solvable Lie groups which admit a bi-invariant Lorentzian metric. Moreover, the reductive pairs determined by the homogeneous Lorentzian structures on the four-dimensional oscillator group equipped with a bi-invariant Lorentzian metric provide four solutions to the Einstein-Yang-Mills equations (see [6], [12]). Recently, Boucetta and Medina determined the solutions of the generalized classical Yang-Baxter equation and the classical Yang-Baxter equation on a generic class of oscillator Lie algebras (see [4]). In [7], Gadea and Oubiña obtained all the homogeneous pseudo-Riemannian structures on the oscillator groups equipped with a family of left invariant Lorentzian metrics. They also determined all the corresponding reductive decompositions and groups of isometries in the 4-dimensional case. More recently, Onda has surveyed the main results about algebraic Ricci solitons on these groups endowed with left invariant pseudeo-Riemannian metrics (see [15]).

The oscillator group $G_n(\lambda) = G(\lambda_1, \dots, \lambda_n)$ is the connected simply connected solvable non-nilpotent Lie group whose Lie algebra $\mathfrak{g}_n(\lambda)$ is the oscillator algebra $\mathfrak{g}_n(\lambda) = \mathfrak{g}(\lambda_1, \dots, \lambda_n)$ which is linearly spanned by $(2n+2)$-elements

$$P, X_1, \dots, X_n, Y_1, \dots, Y_n, Q$$

with the following non-vanishing Lie brackets:

$$[X_i, Y_j] = \delta_{ij} P, \quad [Q, X_j] = \lambda_j Y_j, \quad [Q, Y_j] = -\lambda_j X_j, \quad 1 \leqslant i, j \leqslant n. \tag{1.1}$$

The aim of this paper is to give a complete description of the set of left-invariant harmonic unit vector fields on oscillator groups. We will also determine all the left-invariant vector fields such that the corresponding maps into the tangent bundle are harmonic. Moreover, we study the stability and instability of harmonic unit vector fields on compact quotients of four dimensional oscillator group $G_1(1)$. The main results of this article are the following:

Theorem 1.1. *Let $G_n(\lambda) = G(\lambda_1, \dots, \lambda_n)$ be the oscillator group equipped with a left invariant Riemannian metric and let $\{P, X_1, \dots, X_n, Y_1, \dots, Y_n, Q\}$ be an orthonormal basis of Lie algebra satisfying* (1.1). *Then the set of left-invariant harmonic unit vector fields on the oscillator group $G_n(\lambda) = G(\lambda_1, \dots, \lambda_n)$ is given by*

$$\{\pm Q\} \cup \left(\mathcal{S} \cap \left\{\sum_{j=1}^{n}(a_j X_j + a_{n+j} Y_j)\right\}\right) \cup \left(\mathcal{S} \cap \left\{\sum_{j\in A}^{n}(a_j X_j + a_{n+j} Y_j) + a_{2n+1}P\right\}\right),$$

where for $\lambda_i^2 \neq \lambda_j^2$, $(a_i^2 + a_{n+i}^2)(a_j^2 + a_{n+j}^2) = 0$, $a_j, a_{n+j}, a_{2n+1} \in \mathbb{R}$, $1 \leqslant i, j \leqslant n$, $A = \{j \in B \colon n - 1 - 2\lambda_j^2 = 0\}$, $B = \{1, 2, \dots, n\}$, $\mathcal{S}$ is the unit sphere of the Lie algebra $\mathfrak{g}_n(\lambda)$ of the Lie group $G_n(\lambda)$.

In particular, if $\lambda_1 = \lambda_2 = \ldots = \lambda_n = \lambda$, $n - 1 - 2\lambda \neq 0$, then the set of left invariant harmonic unit vector fields on the oscillator group $G_n(\lambda) = G(\lambda_1, \dots, \lambda_n)$ is given by

$$\{\pm P\} \cup \{\pm Q\} \cup \left(\mathcal{S} \cap \left\{\sum_{j=1}^{n}(a_j X_j + a_{n+j} Y_j)\right\}\right).$$

Theorem 1.2. *Keep the above assumptions and notations. Then the set of left-invariant unit vector fields on the oscillator group $G_n(\lambda) = G(\lambda_1, \dots, \lambda_n)$, such that the corresponding maps into the unit tangent bundles are harmonic, is given by*

$$\{\pm P\} \cup \{\pm Q\} \cup \left(\mathcal{S} \cap \left\{\sum_{j=1}^{n}(a_j X_j + a_{n+j} Y_j)\right\}\right),$$

where for $\lambda_i^2 \neq \lambda_j^2$, $(a_i^2 + a_{n+i}^2)(a_j^2 + a_{n+j}^2) = 0$.

Theorem 1.3. *Let $G_1(1)$ be the four dimensional oscillator Lie group equipped with a left invariant Riemannian metric and let $\{P, X, Y, Q\}$ be an orthonormal basis of Lie algebra $\mathfrak{g}_1(1)$. Let Γ be a discrete subgroup such that $\Gamma \setminus G_1(1)$ is compact. Then*

(i) *the vector fields $\pm Q$ are stable critical points for the energy on $\Gamma \setminus G_1(1)$,*
(ii) *the vector fields $\pm P$ are unstable critical points for the energy on $\Gamma \setminus G_1(1)$ with index at least 1;*
(iii) *each vector field $V \in \mathcal{S} \cap \{X, Y\}_{\mathbb{R}}$ is an unstable critical point for the energy on $\Gamma \setminus G_1(1)$ with index at least 2.*

Remark 1.4. In Theorem 1.3, we denote left invariant vector fields on $G_1(1)$ and their corresponding projections on $M = \Gamma \setminus G_1(1)$ by the same letter (see Section 5).

In Section 2, we give some basic notions and facts on harmonic unit vector fields. The definition and fundamental properties of the oscillator group $G_n(\lambda) = G(\lambda_1, \ldots, \lambda_n)$ in the above theorems will be given in Section 3. Theorems 1.1, 1.2 and 1.3 are proved in Sections 3, 4 and 5, respectively.

2. LEFT-INVARIANT HARMONIC UNIT VECTOR FIELDS ON LIE GROUPS

Let (M, g) be an n-dimensional connected Riemannian manifold and (T_1M, g_S) be its unit tangent bundle sphere equipped with the associated Sasaki metric g_S (see [5]). Denote by ∇ the Levi-Civita connection and by R the corresponding Riemannian curvature tensor which is defined as $R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ for all smooth vector fields X, Y. Moreover, we assume that the set $\mathfrak{X}^1(M)$ of the unit vector fields on M is non-empty. We put $A_V = -\nabla V$ for $V \in \mathfrak{X}^1(M)$. Given a smooth vector field V on M, the energy of a smooth vector field $V\colon (M, g) \to (TM, g_S)$ on M is defined by

$$E(V) = \frac{n}{2}\mathrm{vol}(M, g) + \frac{1}{2}\int_M \|A_V\|^2 \,\mathrm{d}v. \tag{2.1}$$

Here, $\mathrm{d}v$ denotes the volume form on (M, g), $B(V) = \int_M \|A_V\|^2 \,\mathrm{d}v$ is the *total bending* of the vector field V (see [18]). We put

$$b(V) = \frac{1}{2}\|A_V\|^2 = \frac{1}{2}\mathrm{tr}(A_V^t A_V).$$

From [17], we know that the unit vector field V is a critical point for the energy functional E if and only if the 1-form ν_V defined by

$$\nu_V(X) = \mathrm{tr}(Z \mapsto (\nabla_Z A_V^t)X)$$

vanishes on the distribution $\mathcal{H}^V$, which is the space of the vector fields orthogonal to V.

Definition 2.1. A unit vector field V on a Riemannian manifold (M, g) is called *harmonic* if $\nu_V(X) = 0$ for all $X \in \mathcal{H}^V$.

A unit Killing vector field V is harmonic if and only if it is an eigenvector of the Ricci operator (see [8]). Moreover, the map $V\colon (M, g) \to (TM, g_S)$ is a *harmonic map* if and only if V is a harmonic unit vector field such that the one form $\widetilde{\nu}_V$, defined by

$$\widetilde{\nu}_V(X) = \mathrm{tr}(Z \mapsto R(A_V Z, V)X)), \tag{2.2}$$

vanishes for all vectors X (see [8]).

A vector field $V \in \mathfrak{X}^1(M)$ is called *normal* if $g(R(X,Y)Z,V) = 0$ for all $X,Y,Z \in \mathcal{H}^V$. We say that $V \in \mathfrak{X}^1(M)$ is a *strongly normal* vector field if $g((\nabla_X A_V)Y,Z) = 0$ for all $X,Y,Z \in \mathcal{H}^V$. Because of the equation

$$R_{XY}V = (\nabla_X A_V)Y - (\nabla_Y A_V)X,$$

it is easy to see that each strongly normal vector field is also normal. Furthermore, a unit Killing vector field V is strongly normal if and only if it is normal, and we can see that V is harmonic and determines a harmonic map in this case (see [17]). From [9] we know that a unit vector field on a 3-dimensional Riemannian manifold is normal if and only if it is an eigenvector of the Ricci operator. Recall that every unit killing vector field V is a geodesic vector field. We have the following result.

Proposition 2.2 ([17]). *Every strongly normal geodesic vector field $V \in \mathfrak{X}^1(M)$ is harmonic. Moreover, the corresponding map is harmonic if and only if $\widetilde{\nu}_V(V) = 0$.*

For $V \in \mathfrak{X}^1(M)$ harmonic, the *Hessian form for the energy* at V is the quadratic form $(\operatorname{Hess} E)_V$ on $T_V\mathfrak{X}^1(M)$ given by

$$(\operatorname{Hess} E)_V(X) = \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} B(\gamma(t)), \quad X \in T_V\mathfrak{X}^1(M) = \mathcal{H}^V,$$

where $\gamma\colon I \to \mathfrak{X}^1(M)$, $t \mapsto \gamma(t)$, is a smooth curve in $\mathfrak{X}^1(M)$, I an open interval of $\mathbb{R}$ such that $0 \in I$ and $\gamma(0) = V$, $\gamma'(0) = X$.

On a closed and oriented Riemannian manifold M, the Hessian form $(\operatorname{Hess} E)_V$ at a unit harmonic vector field $V \in \mathfrak{X}^1(M)$ can be expressed as [18]

$$(\operatorname{Hess} E)_V(X) = \int_M (\|\nabla X\|^2 - \|X\|^2\|A_V\|^2)\,\mathrm{d}v, \tag{2.3}$$

where $X \in \mathcal{H}^V$.

We say a unit harmonic vector field V is *stable* if $(\operatorname{Hess} E)_V(X) \geqslant 0$ for all $X \in \mathcal{H}^V$ or, equivalently, the associated bilinear symmetric map, that is the *Hessian* of E at V, is positive semidefinite. The *index* (or *nullity*) of V is the index (nullity) of this bilinear map. Note that if $(\operatorname{Hess} E)_V$ is semidefinite, then $\{X \in \mathcal{H}^V\colon (\operatorname{Hess} E)_V(X) = 0\}$ is the subspace $\{X \in \mathcal{H}^V\colon (\operatorname{Hess} E)_V(X,W) = 0 \ \forall\, W \in \mathcal{H}^V\}$ and its dimension coincides with the nullity of V.

Now we consider left-invariant harmonic unit vector fields on a Lie group G equipped with a left invariant metric g. The left invariant metric g determines an associated inner product $\langle , \rangle$ on the Lie algebra $\mathfrak{g}$. Then by the invariance with respect to the left translation, the function b defined above can be viewed as a function on the unit sphere $\mathcal{S}$ of the Lie algebra $\mathfrak{g}$. For $V \in \mathcal{S}$, the distribution $\mathcal{H}^V$ can be identified with the orthogonal complement $V^\perp$ of V in $\mathfrak{g}$. $V^\perp$ can be naturally identified with the tangent space $T_V\mathcal{S}$ of $\mathcal{S}$ at V. Thus, it is easy to see that a left invariant unit vector field V is harmonic if and only if the 1-form ν_V on $\mathfrak{g}$ vanishes on $V^\perp \cong T_V\mathcal{S}$. In [17], it is shown that

$$\nu_V(X) = \mathrm{d}b_V(X) - \mathrm{tr}(\mathrm{ad}_{A_V^t X}), \quad X \in T_V\mathcal{S}.$$

So, V is harmonic if and only if $\mathrm{d}b_V(X) = \mathrm{tr}(\mathrm{ad}_{A_V^t X})$ for all $X \in T_V\mathcal{S}$. Recall that a Lie group G is called *unimodular* if $\mathrm{tr}(\mathrm{ad}_X) = 0$ for all $X \in g$ (see [14]). We have the following:

Proposition 2.3 ([17]). *A left invariant unit vector field V on a unimodular Lie group G is harmonic if and only if V is a critical point of the function b on $\mathcal{S}$.*

For a non-unimodular Lie group G, we consider its unimodular kernel $\mathcal{U}$ defined by

$$\mathcal{U} = \{X \in g\colon\ \mathrm{tr}(\mathrm{ad}_X) = 0\}.$$

Since $\mathrm{tr}(\mathrm{ad}_X)$ is a linear functional, $\mathcal{U}$ is an ideal of codimension 1. For a unit vector H orthogonal to $\mathcal{U}$, it is obvious that the linear transformation ad_H, which is restricted to $\mathcal{U}$, is a derivation of $\mathcal{U}$. And, we have the following:

Proposition 2.4 ([17]). *A left invariant unit vector field V on a non-unimodular Lie group is harmonic if and only if*

$$\mathrm{d}b_V(X) = \mathrm{tr}(\mathrm{ad}_H)\langle A_V H, X\rangle$$

for all $X \in T_V\mathcal{S}$. Moreover, if $\mathrm{ad}_{H|\mathcal{U}}$ is a symmetrical endomorphism of $\mathcal{U}$ with respect to $\langle,\rangle$, then V is harmonic if and only if it is a critical point of the function b on $\mathcal{S}$.

3. HARMONIC VECTOR FIELDS ON THE OSCILLATOR GROUP $G_n(\lambda)$

Oscillator algebra $\mathfrak{g}_n(\lambda) = \mathfrak{g}(\lambda_1, \ldots, \lambda_n)$ is linearly spanned by $(2n+2)$-elements

$$P, X_1, \ldots, X_n, Y_1, \ldots, Y_n, Q$$

with the following non-vanishing Lie brackets:

$$[X_i, Y_j] = \delta_{ij}P, \quad [Q, X_j] = \lambda_j Y_j, \quad [Q, Y_j] = -\lambda_j X_j, \quad 1 \leqslant i, j \leqslant n. \tag{3.1}$$

From the definition, it is easily seen that $\mathfrak{g}_n(\lambda)$ is the semidirect product of the Heisenberg algebra $\mathfrak{h}_n$ generated by $(P, X_1, \ldots, X_n, Y_1, \ldots, Y_n)$, and a one-dimensional abelian Lie subalgebra spanned by Q, under the homomorphism $\mathrm{ad}|_{\mathfrak{h}_n} \colon \langle Q\rangle \to \mathrm{Der}(\mathfrak{h}_n)$. And the corresponding connected simply connected Lie group is called *oscillator group* $G_n(\lambda) = G(\lambda_1, \ldots, \lambda_n)$.

On the oscillator group $G_n(\lambda) = G(\lambda_1, \ldots, \lambda_n)$, we consider the left invariant Riemannian metric for which the $(2n+2)$-elements $\{P, X_1, \ldots, X_n, Y_1, \ldots, Y_n, Q\}$ form an orthonormal basis at each point. By (3.1), it is easy to see that $G_n(\lambda) = G(\lambda_1, \ldots, \lambda_n)$ is a unimodular Lie group. We shall use Proposition 2.3 to find all the left invariant harmonic unit vector fields on the oscillator group $G_n(\lambda) = G(\lambda_1, \ldots, \lambda_n)$.

Denote by δ_{jk} the Kronecker symbol, $1 \leqslant j, k \leqslant n$, using (3.1) and the well-known Koszul formula, one can determine the Levi-Civita connection on $G_n(\lambda) = G(\lambda_1, \ldots, \lambda_n)$ as follows:

$$\begin{array}{llll}
\nabla_P P = 0, & \nabla_P Q = 0, & \nabla_P X_j = -\frac{1}{2} Y_j, & \nabla_P Y_j = \frac{1}{2} X_j, \\
\nabla_Q P = 0, & \nabla_Q Q = 0, & \nabla_Q X_j = \lambda_j Y_j, & \nabla_Q Y_j = -\lambda_j X_j, \\
\nabla_{X_j} P = -\frac{1}{2} Y_j, & \nabla_{X_j} Q = 0, & \nabla_{X_j} X_k = 0, & \nabla_{X_j} Y_k = \frac{1}{2}\delta_{jk} P, \\
\nabla_{Y_j} P = \frac{1}{2} X_j, & \nabla_{Y_j} Q = 0, & \nabla_{Y_j} X_k = -\frac{1}{2}\delta_{jk} P, & \nabla_{Y_j} Y_k = 0.
\end{array} \tag{3.2}$$

For a left invariant vector field $V = \sum\limits_{i=1}^{n} (a_i X_i + a_{n+i} Y_i) + a_{2n+1} P + a_{2n+2} Q$ on $G_n(\lambda)$ we have

$$\begin{aligned}
\nabla_{X_j} V &= \frac{1}{2} a_{n+j} P - \frac{1}{2} a_{2n+1} Y_j, \quad j = 1, 2, \ldots, n \\
\nabla_{Y_j} V &= -\frac{1}{2} a_j P + \frac{1}{2} a_{2n+1} X_j, \quad j = 1, 2, \ldots, n \\
\nabla_P V &= \frac{1}{2} \sum_{i=1}^{n} (a_{n+i} X_i - a_i Y_i), \\
\nabla_Q V &= \sum_{i=1}^{n} \lambda_i (a_i Y_i - a_{n+i} X_i).
\end{aligned} \tag{3.3}$$

Thus

$$\begin{aligned}
\nabla V = \frac{1}{2} \sum_{i=1}^{n} \{ &(a_{n+i} P - a_{2n+1} Y_i) \otimes \alpha_i + (a_{2n+1} X_i - a_i P) \otimes \beta_i \\
&+ (a_{n+i} X_i - a_i Y_i) \otimes \gamma + 2\lambda_i (a_i Y_i - a_{n+i} X_i) \otimes \tau \},
\end{aligned}$$

where $\{\alpha_i, \beta_i, \gamma, \tau\}$ is the dual coframe field of $\{X_i, Y_i, P, Q\}$, $1 \leqslant i \leqslant n$. Then the matrix form of ∇V is given by

$$\nabla V = \begin{pmatrix} 0 & \frac{1}{2}a_{2n+1}I_n & A & C \\ -\frac{1}{2}a_{2n+1}I_n & 0 & B & D \\ A^t & B^t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} A &= \Big(\frac{1}{2}a_{n+1}, \frac{1}{2}a_{n+2}, \ldots, \frac{1}{2}a_{2n}\Big)^t, \\ B &= \Big(-\frac{1}{2}a_1, -\frac{1}{2}a_2, \ldots, -\frac{1}{2}a_n\Big)^t, \\ C &= (-a_{n+1}\lambda_1, -a_{n+2}\lambda_2, \ldots, -a_{2n}\lambda_n)^t, \\ D &= (a_1\lambda_1, a_2\lambda_2, \ldots, a_n\lambda_n)^t. \end{aligned}$$

The symbol A^t denotes the transposition of matrix A. From the matrix expression of ∇V, we have the following result.

Proposition 3.1. *A left invariant vector field V on oscillator group $G_n(\lambda)$ is a Killing vector field if and only if $V = k_1P + k_2Q$, $k_1, k_2 \in \mathbb{R}$; a left invariant vector field V on $G_n(\lambda)$ is a parallel vector field if and only if $V = l_1Q$, $l_1 \in \mathbb{R}$.*

By some calculations, we obtain

$$b(V) = \frac{1}{2}\mathrm{tr}(\nabla V^t \nabla V) = \frac{n}{4}a_{2n+1}^2 + \frac{1}{4}\sum_{i=1}^{n}(1 + 2\lambda_i^2)(a_i^2 + a_{n+i}^2).$$

Now we can give the proof of the main result of this paper.

Proof of Theorem 1.1. Applying Proposition 2.3 to the unimodular Lie group $G_n(\lambda) = G(\lambda_1, \ldots, \lambda_n)$, we see that if a left invariant unit vector field

$$V = \sum_{i=1}^{n}(a_iX_i + a_{n+i}Y_i) + a_{2n+1}P + a_{2n+2}Q$$

is harmonic, then it is a critical point of the function b on $\mathcal{S}$. On the other hand, it is proved in [17] that $\mathrm{d}b_V(X) = -\mathrm{tr}(A_V^t \nabla X)$, $X \in T_V\mathcal{S}$. So

$$V = \sum_{i=1}^{n}(a_iX_i + a_{n+i}Y_i) + a_{2n+1}P + a_{2n+2}Q$$

is harmonic if and only if $\operatorname{tr}(A_V^t \nabla X) = 0$, $X \in T_V\mathcal{S}$. Since $T_V\mathcal{S} \cong V^{\perp}$ forms a $(2n+1)$-dimensional vector space, we only need to consider the following vector fields:

$$\begin{array}{llll}
-a_{2n+1}Q + a_{2n+2}P, & -a_{2n+1}X_j + a_jP, & -a_{2n+1}Y_j + a_{n+j}P, & 1 \leqslant j \leqslant n, \\
-a_{2n+2}X_j + a_jQ, & -a_{2n+2}Y_j + a_{n+j}Q, & -a_{n+j}X_k + a_kY_j, & 1 \leqslant j, k \leqslant n, \\
-a_kX_j + a_jX_k, & -a_{n+k}Y_j + a_{n+j}Y_k, & & 1 \leqslant j \neq k \leqslant n.
\end{array}$$

On the other hand, we also have

$$\begin{aligned}
A_VX_i &= -\frac{1}{2}a_{n+i}P + \frac{1}{2}a_{2n+1}Y_i, \quad i = 1, 2, \ldots, n, \\
A_VY_i &= \frac{1}{2}a_iP - \frac{1}{2}a_{2n+1}X_i, \quad i = 1, 2, \ldots, n, \\
A_VP &= \frac{1}{2}\sum_{i=1}^{n}(-a_{n+i}X_i + a_iY_i), \\
A_VQ &= \sum_{i=1}^{n}\lambda_i(-a_iY_i + a_{n+i}X_i).
\end{aligned} \tag{3.4}$$

Now, we only need to consider the following cases:

Case I: $X = -a_{2n+1}Q + a_{2n+2}P$. Then we have

$$A_XX_i = \frac{1}{2}a_{2n+2}Y_i, \quad A_XY_i = -\frac{1}{2}a_{2n+2}X_i, \quad 1 \leqslant i \leqslant n, \quad A_XP = A_XQ = 0.$$

Thus

$$\begin{aligned}
\mathrm{d}b_V(X) &= -\sum_{i=1}^{n}\langle \nabla_{X_i}X, A_VX_i\rangle - \sum_{i=1}^{n}\langle \nabla_{Y_i}X, A_VY_i\rangle - \langle \nabla_PX, A_VP\rangle - \langle \nabla_QX, A_VQ\rangle \\
&= \sum_{i=1}^{n}\langle A_XX_i, A_VX_i\rangle + \sum_{i=1}^{n}\langle A_XY_i, A_VY_i\rangle + \langle A_XP, A_VP\rangle + \langle A_XQ, A_VQ\rangle \\
&= \frac{n}{4}a_{2n+1}a_{2n+2} + \frac{n}{4}a_{2n+1}a_{2n+2} = \frac{n}{2}a_{2n+1}a_{2n+2}.
\end{aligned}$$

Case II: $X = -a_{2n+1}X_j + a_jP$, $1 \leqslant j \leqslant n$. Then we have

$$A_XX_i = \frac{1}{2}a_jY_i, \quad 1 \leqslant i \leqslant n, \quad A_XY_i = -\frac{1}{2}a_jX_i - \frac{1}{2}\delta_{ij}a_{2n+1}P, \quad 1 \leqslant i \leqslant n,$$
$$A_XP = -\frac{1}{2}a_{2n+1}Y_j, \quad A_XQ = a_{2n+1}\lambda_jY_j.$$

Thus

$$\begin{aligned}
\mathrm{d}b_V(X) = {} & \sum_{i=1}^{n}\Big\langle \frac{1}{2}a_{2n+1}Y_i - \frac{1}{2}a_{n+i}P, \frac{1}{2}a_jY_i\Big\rangle \\
& + \Big\langle -\frac{1}{2}a_{2n+1}X_j + \frac{1}{2}a_jP, -\frac{1}{2}a_jX_j - \frac{1}{2}a_{2n+1}P\Big\rangle \\
& + \sum_{i\neq j}^{n}\Big\langle -\frac{1}{2}a_{2n+1}X_i + \frac{1}{2}a_iP, -\frac{1}{2}a_jX_i\Big\rangle \\
& + \Big\langle -\frac{1}{2}\sum_{i=1}^{n}a_{n+i}X_i + \frac{1}{2}\sum_{i=1}^{n}a_iY_i, -\frac{1}{2}a_{2n+1}Y_j\Big\rangle \\
& + \Big\langle \sum_{i=1}^{n}a_{n+i}\lambda_iX_i - \sum_{i=1}^{n}a_i\lambda_iY_i, a_{2n+1}\lambda_jY_j\Big\rangle = \frac{1}{2}(n-1-2\lambda_j^2)a_ja_{2n+1}.
\end{aligned}$$

Case III: $X = -a_{2n+1}Y_j + a_{n+j}P$, $1 \leqslant j \leqslant n$. In this case we have

$$A_XX_i = \frac{1}{2}a_{n+j}Y_i + \frac{1}{2}\delta_{ij}a_{2n+1}P, \quad 1 \leqslant i \leqslant n, \quad A_XY_i = -\frac{1}{2}a_{n+j}X_i, \quad 1 \leqslant i \leqslant n,$$
$$A_XP = \frac{1}{2}a_{2n+1}X_j, \quad A_XQ = -a_{2n+1}\lambda_jX_j.$$

Similarly as above, we obtain

$$\mathrm{d}b_V(X) = \frac{1}{2}(n-1-2\lambda_j^2)a_{n+j}a_{2n+1}.$$

Case IV: $X = -a_{2n+2}X_j + a_jQ$, $1 \leqslant j \leqslant n$. Then we have

$$A_XX_i = 0, \quad A_XY_i = -\frac{1}{2}\delta_{ij}a_{2n+2}P, \quad 1 \leqslant i \leqslant n,$$
$$A_XP = -\frac{1}{2}a_{2n+2}Y_j, \quad A_XQ = a_{2n+2}\lambda_jY_j.$$

Thus

$$\mathrm{d}b_V(X) = -a_{2n+2}a_j\Big(\frac{1}{2} + \lambda_j^2\Big).$$

Case V: $X = -a_{2n+2}Y_j + a_{n+j}Q$, $1 \leqslant j \leqslant n$. Then we have

$$A_XX_i = \frac{1}{2}\delta_{ij}a_{2n+2}P, \quad 1 \leqslant i \leqslant n, \quad A_XY_i = 0,$$
$$A_XP = \frac{1}{2}a_{2n+2}X_j, \quad A_XQ = -a_{2n+2}\lambda_jX_j.$$

Thus

$$\mathrm{d}b_V(X) = -a_{2n+2}a_{n+j}\Big(\frac{1}{2} + \lambda_j^2\Big).$$

Case VI: $X = -a_{n+j}X_k + a_kY_j$, $1 \leqslant j,\, k \leqslant n$. Then we have

$$A_XX_i = -\frac{1}{2}\delta_{ij}a_kP, \quad 1 \leqslant i \leqslant n, \quad A_XY_i = -\frac{1}{2}\delta_{ik}a_{n+j}P, \quad 1 \leqslant i \leqslant n,$$
$$A_XP = -\frac{1}{2}\sum_{i=1}^{n}(\delta_{ij}a_kX_i + \delta_{ik}a_{n+j}Y_i), \quad A_XQ = \sum_{i=1}^{n}\lambda_i(\delta_{ik}a_{n+j}Y_i + \delta_{ij}a_kX_i)$$

and

$$\mathrm{d}b_V(X) = a_{n+j}a_k(\lambda_j^2 - \lambda_k^2).$$

Case VII: $X = -a_kX_j + a_jX_k$, $1 \leqslant j \neq k \leqslant n$. Then we have

$$A_XX_i = 0, \quad 1 \leqslant i \leqslant n, \quad A_XY_i = \frac{1}{2}(-\delta_{ij}a_k + \delta_{ik}a_j)P, \quad 1 \leqslant i \leqslant n,$$
$$A_XP = \frac{1}{2}\sum_{i=1}^{n}(-\delta_{ij}a_k + \delta_{ik}a_j)Y_i, \quad A_XQ = -\sum_{i=1}^{n}\lambda_i(-\delta_{ij}a_k + \delta_{ik}a_j)Y_i.$$

Thus

$$\mathrm{d}b_V(X) = a_ja_k(\lambda_k^2 - \lambda_j^2).$$

Case VIII: $X = -a_{n+k}Y_j + a_{n+j}Y_k$, $1 \leqslant j \neq k \leqslant n$. Then we have

$$A_XX_i = -\frac{1}{2}(-\delta_{ij}a_{n+k} + \delta_{ik}a_{n+j})P, \quad 1 \leqslant i \leqslant n, \quad A_XY_i = 0, \quad 1 \leqslant i \leqslant n,$$
$$A_XP = -\frac{1}{2}\sum_{i=1}^{n}(-\delta_{ij}a_{n+k} + \delta_{ik}a_{n+j})X_i, \quad A_XQ = \sum_{i=1}^{n}\lambda_i(-\delta_{ij}a_{n+k} + \delta_{ik}a_{n+j})X_i$$

and

$$\mathrm{d}b_V(X) = a_{n+j}a_{n+k}(\lambda_k^2 - \lambda_j^2).$$

From the above arguments, we conclude that V is a harmonic unit vector field if and only if the following system of equations holds:

$$\begin{cases} a_{2n+1}a_{2n+2} = 0, & \\ (n-1-2\lambda_j^2)a_ja_{2n+1} = 0, & 1 \leqslant j \leqslant n, \\ (n-1-2\lambda_j^2)a_{n+j}a_{2n+1} = 0, & 1 \leqslant j \leqslant n, \\ -a_{2n+2}a_j(\lambda_j^2 + \tfrac{1}{2}) = 0, & 1 \leqslant j \leqslant n, \\ -a_{2n+2}a_{n+j}(\lambda_j^2 + \tfrac{1}{2}) = 0, & 1 \leqslant j \leqslant n, \\ a_{n+j}a_k(\lambda_j^2 - \lambda_k^2) = 0, & 1 \leqslant j, k \leqslant n, \\ a_ja_k(\lambda_k^2 - \lambda_j^2) = 0, & 1 \leqslant j, k \leqslant n, \\ a_{n+j}a_{n+k}(\lambda_k^2 - \lambda_j^2) = 0, & 1 \leqslant j, k \leqslant n. \end{cases}$$

Now we return to the proof of the theorem. Firstly, if $a_{2n+2} \neq 0$, then $a_{2n+1} = a_j = a_{n+j} = 0$, $1 \leqslant j \leqslant n$, hence $\pm Q$ is a harmonic unit vector field.

Secondly, if $a_{2n+2} = 0$, and $a_{2n+1} \neq 0$ then from the second and the third equations we have

$$(n-1-2\lambda_j^2)a_j = 0, \quad (n-1-2\lambda_j^2)a_{n+j} = 0.$$

Denote $B = \{1, 2, \ldots, n\}$ and $A = \{j \in B \colon n - 1 - 2\lambda_j^2 = 0\}$. If j does not belong to the set A, then we have $a_j = a_{n+j} = 0$.

Thirdly, if $a_{2n+2} = 0$ and $a_{2n+1} = 0$, then the first five equations always hold.

Finally, if $\lambda_i^2 \neq \lambda_j^2$, $1 \leqslant i, j \leqslant n$, then by the last three equations we have $a_i = a_{n+i} = 0$ or $a_j = a_{n+j} = 0$, i.e., $(a_i^2 + a_{n+i}^2)(a_j^2 + a_{n+j}^2) = 0$. This completes the proof. $\square$

4. HARMONIC MAPS DETERMINED BY INVARIANT VECTOR FIELDS ON THE OSCILLATOR GROUP $G_n(\lambda)$

Using (3.2) and the Riemannian curvature formula $R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$, we can obtain the non-vanishing components of the curvature tensor field as follows:

$$\begin{aligned} &R(X_i,Y_j)X_s = -\frac{1}{2}\delta_{ij}Y_s - \frac{1}{4}\delta_{js}Y_i, && R(X_i,Y_j)Y_s = \frac{1}{2}\delta_{ij}X_s + \frac{1}{4}\delta_{is}X_j, \\ &R(X_i,X_j)Y_s = \frac{1}{4}(\delta_{js}Y_i - \delta_{is}Y_j), && R(Y_i,Y_j)X_s = \frac{1}{4}(\delta_{js}X_i - \delta_{is}X_j), \\ &R(X_i,P)X_j = \frac{1}{4}\delta_{ij}P, && R(X_i,P)P = -\frac{1}{4}X_i, \\ &R(Y_i,P)Y_j = \frac{1}{4}\delta_{ij}P, && R(Y_i,P)P = -\frac{1}{4}Y_i, \end{aligned} \tag{4.1}$$

where $1 \leqslant i, j, s \leqslant n$.

Proof of Theorem 1.2. If a left-invariant unit vector field V defines a harmonic map, then it is a harmonic vector field and satisfies the condition $\widetilde{\nu}_V(X) = \operatorname{tr}(Z \mapsto R(A_V Z, V)X) = 0$ for all $X \in \mathcal{S}$. Assume $V = \sum\limits_{s=1}^{n}(a_s X_s + a_{n+s}Y_s) + a_{2n+1}P + a_{2n+2}Q$. Then by the equations (3.4), (4.1) and the orthogonality of generators $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, P, Q\}$, we have the following:

Case 1: Set $X = X_j$, $1 \leqslant j \leqslant n$. Then we have

$$\begin{aligned} \sum_{i=1}^{n}\langle X_i, R(A_V X_i, V)X_j\rangle &= \sum_{i=1}^{n}\Big\langle X_i, R\Big(-\frac{1}{2}a_{n+i}P + \frac{1}{2}a_{2n+1}Y_i, V\Big)X_j\Big\rangle \\ &= \sum_{i=1}^{n}\Big\langle X_i, R\Big(\frac{1}{2}a_{2n+1}Y_i, \sum_{s=1}^{n} a_{n+s}Y_s\Big)X_j\Big\rangle \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\Big\langle X_i, \frac{1}{2}a_{2n+1}\sum_{s=1}^{n} a_{n+s}\Big(\frac{1}{4}\delta_{sj}X_i - \frac{1}{4}\delta_{ij}X_s\Big)\Big\rangle \\
&= \frac{n-1}{8}a_{2n+1}a_{n+j},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=1}^{n}\langle Y_i, R(A_VY_i, V)X_j\rangle &= \sum_{i=1}^{n}\Big\langle Y_i, R\Big(\frac{1}{2}a_iP - \frac{1}{2}a_{2n+1}X_i, V\Big)X_j\Big\rangle \\
&= \sum_{i=1}^{n}\Big\langle Y_i, R\Big(-\frac{1}{2}a_{2n+1}X_i, \sum_{s=1}^{n} a_{n+s}Y_s\Big)X_j\Big\rangle \\
&= \sum_{i=1}^{n}\Big\langle Y_i, \frac{1}{2}a_{2n+1}\sum_{s=1}^{n} a_{n+s}\Big(\frac{1}{2}\delta_{is}Y_j + \frac{1}{4}\delta_{sj}Y_i\Big)\Big\rangle \\
&= \frac{n+2}{8}a_{2n+1}a_{n+j}
\end{aligned}
$$

and

$$
\begin{aligned}
\langle P, R(A_VP, V)X_j\rangle &= \Big\langle P, R\Big(\frac{1}{2}\sum_{i=1}^{n}(-a_{n+i}X_i + a_iY_i), V\Big)X_j\Big\rangle \\
&= \Big\langle P, R\Big(-\frac{1}{2}\sum_{i=1}^{n} a_{n+i}X_i, a_{2n+1}P\Big)X_j\Big\rangle \\
&= \Big\langle P, -\frac{1}{8}a_{2n+1}\sum_{i=1}^{n} a_{n+i}\delta_{ij}P\Big\rangle = -\frac{1}{8}a_{2n+1}a_{n+j}.
\end{aligned}
$$

It is easy to see that $\langle Q, R(A_VQ, V)X_j\rangle = 0$. So we have

$$
\begin{aligned}
\widetilde{\nu}_V(X_j) &= \operatorname{tr}(Z \mapsto R(A_VZ, V)X_j) \\
&= \sum_{i=1}^{n}\langle X_i, R(A_VX_i, V)X_j\rangle + \sum_{i=1}^{n}\langle Y_i, R(A_VY_i, V)X_j\rangle \\
&\quad + \langle P, R(A_VP, V)X_j\rangle + \langle Q, R(A_VQ, V)X_j\rangle = \frac{n}{4}a_{2n+1}a_{n+j}.
\end{aligned}
$$

Case 2: Set $X = Y_j$, $1 \leqslant j \leqslant n$. Then we have

$$
\begin{aligned}
\sum_{i=1}^{n}\langle X_i, R(A_VX_i, V)Y_j\rangle &= \sum_{i=1}^{n}\langle X_i, R\Big(-\frac{1}{2}a_{n+i}P + \frac{1}{2}a_{2n+1}Y_i, V\Big)Y_j\rangle \\
&= \sum_{i=1}^{n}\Big\langle X_i, R\Big(\frac{1}{2}a_{2n+1}Y_i, \sum_{s=1}^{n} a_sX_s\Big)Y_j\Big\rangle \\
&= \sum_{i=1}^{n}\Big\langle X_i, -\frac{1}{2}a_{2n+1}\sum_{s=1}^{n} a_s\Big(\frac{1}{2}\delta_{si}X_j + \frac{1}{4}\delta_{sj}X_i\Big)\Big\rangle \\
&= -\frac{n+2}{8}a_{2n+1}a_j,
\end{aligned}
$$

$$\begin{aligned}\sum_{i=1}^{n}\langle Y_i, R(A_V Y_i, V)Y_j\rangle &= \sum_{i=1}^{n}\Big\langle Y_i, R\Big(\frac{1}{2}a_i P - \frac{1}{2}a_{2n+1}X_i, V\Big)Y_j\Big\rangle \\ &= \sum_{i=1}^{n}\Big\langle Y_i, R\Big(-\frac{1}{2}a_{2n+1}X_i, \sum_{s=1}^{n} a_s X_s\Big)Y_j\Big\rangle \\ &= \sum_{i=1}^{n}\Big\langle Y_i, -\frac{1}{8}a_{2n+1}\sum_{s=1}^{n} a_s(\delta_{sj}Y_i - \delta_{ij}Y_s)\Big\rangle \\ &= -\,\frac{n-1}{8}a_{2n+1}a_j\end{aligned}$$

and

$$\begin{aligned}\langle P, R(A_V P, V)X_j\rangle &= \Big\langle P, R\Big(\frac{1}{2}\sum_{i=1}^{n}(-a_{n+i}X_i + a_i Y_i), V\Big)Y_j\Big\rangle \\ &= \Big\langle P, R\Big(\frac{1}{2}\sum_{i=1}^{n} a_i Y_i, a_{2n+1}P\Big)Y_j\Big\rangle \\ &= \Big\langle P, \frac{1}{8}a_{2n+1}\sum_{i=1}^{n} a_i\delta_{ij}P\Big\rangle = \frac{1}{8}a_{2n+1}a_j.\end{aligned}$$

On the other hand, we also have $\langle Q, R(A_V Q, V)Y_j\rangle = 0$. Thus

$$\widetilde{\nu}_V(Y_j) = -\frac{n}{4}a_{2n+1}a_j.$$

Case 3: Set $X = P$. We get

$$\begin{aligned}\sum_{i=1}^{n}\langle X_i, R(A_V X_i, V)P\rangle &= \sum_{i=1}^{n}\Big\langle X_i, R\Big(-\frac{1}{2}a_{n+i}P + \frac{1}{2}a_{2n+1}Y_i, V\Big)P\Big\rangle \\ &= \sum_{i=1}^{n}\Big\langle X_i, R\Big(-\frac{1}{2}a_{n+i}P, \sum_{s=1}^{n} a_s X_s\Big)P\Big\rangle \\ &= \sum_{i=1}^{n}\Big\langle X_i, -\frac{1}{8}a_{n+i}\sum_{s=1}^{n} a_s X_s\Big\rangle = -\frac{1}{8}\sum_{i=1}^{n} a_{n+i}a_i\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{n}\langle Y_i, R(A_V Y_i, V)P\rangle &= \sum_{i=1}^{n}\Big\langle Y_i, R\Big(\frac{1}{2}a_i P - \frac{1}{2}a_{2n+1}X_i, V\Big)P\Big\rangle \\ &= \sum_{i=1}^{n}\Big\langle Y_i, R\Big(\frac{1}{2}a_i P, \sum_{s=1}^{n} a_{n+s}Y_s\Big)P\Big\rangle \\ &= \sum_{i=1}^{n}\Big\langle Y_i, \frac{1}{8}a_i\sum_{s=1}^{n} a_{n+s}Y_s\Big\rangle = \frac{1}{8}\sum_{i=1}^{n} a_{n+i}a_i.\end{aligned}$$

Moreover, $\langle P, R(A_V P, V)P\rangle = 0$ and $\langle Q, R(A_V Q, V)P\rangle = 0$. So we have

$$\widetilde{\nu}_V(P) = 0.$$

Case 4: Set $X = Q$. Then it is easily seen that $\widetilde{\nu}_V(Q) = 0$. Thus, we have

$$\widetilde{\nu}_V = \frac{1}{4}\sum_{j=1}^{n} n a_{2n+1}(a_{n+j} \otimes \alpha_j - a_j \otimes \beta_j).$$

On the other hand, it is easy to check that the vector field

$$V = \sum_{s=1}^{n}(a_s X_s + a_{n+s} Y_s) + a_{2n+1}P + a_{2n+2}Q$$

defines a harmonic map from $(G_n(\lambda), g)$ into its unit tangent bundle $(T_1 G_n(\lambda), g_S)$ if and only if V is a harmonic vector field satisfying the following equations:

$$\frac{n}{4} a_{2n+1} a_{n+j} = 0, \quad -\frac{n}{4} a_{2n+1} a_j = 0, \quad j = 1, \dots, n.$$

Consequently, if $\lambda_i^2 \neq \lambda_j^2$, then

$$V \in \{\pm P\} \cup \{\pm Q\} \cup \left(\mathcal{S} \cap \left\{\sum_{j=1}^{n}(a_j X_j + a_{n+j} Y_j)\right\}\right)$$

with $(a_i^2 + a_{n+i}^2)(a_j^2 + a_{n+j}^2) = 0$. This completes the proof. □

5. ENERGY ON COMPACT QUOTIENTS OF FOUR DIMENSIONAL OSCILLATOR GROUP $G_1(1)$

Since the action of any discrete subgroup Γ of a Lie group G by left translations is free and properly discontinuous, the set of orbits, namely the space of right cosets $\Gamma \setminus G$, is a C^∞ manifold and the natural projection $\pi : G \to \Gamma \setminus G$ is a C^∞ mapping (see [3]).

Furthermore, each left invariant vector field on G descends to $\Gamma \setminus G$, namely if X is left invariant, then $\pi_* X_{ba} = \pi_* X_a$ for all $a \in G$, $b \in \Gamma$ (see [11]). Similarly, each left invariant metric on G and all its left invariant tensors field can descend to the quotient space. And the projections of left invariant unit vector fields preserve the properties to be Killing, harmonic and to determine harmonic maps into the corresponding unit tangent bundles.

In Section 3, set $i = j = n = \lambda_j = 1$, $X_1 = X$, $Y_1 = Y$, we get a four dimensional oscillator group $G_1(1)$. It is a one dimensional solvable extension of three dimensional Heisenberg group H, and H admits a discrete subgroup Γ_1 such that $\Gamma_1 \setminus H$ is compact. Then there exists a discrete subgroup Γ of $G_1(1)$ such that $M = \Gamma \setminus G_1(1)$ is compact. We shall denote left invariant vector fields on $G_1(1)$ and their corresponding projections on $M = \Gamma \setminus G_1(1)$ by the same letter.

Now we calculate the energy of a smooth vector field $V\colon (M, g) \to (TM, g^s)$ on $M = \Gamma \setminus G_1(1)$.

Proposition 5.1. *Let $V = a_1X + a_2Y + a_3P + a_4Q$ be a smooth left invariant vector field on $M = \Gamma \setminus G_1(1)$. Then the energy of V is*

$$E(V) = \Big(2 + \frac{3}{4}\|V\|^2 - \frac{1}{2}a_3^2 - \frac{3}{4}a_4^2\Big)\operatorname{vol}(M).$$

Proof. By (3.3), we have

$$\begin{aligned} \nabla_X V &= \frac{1}{2}a_2P - \frac{1}{2}a_3Y, \quad & \nabla_Y V &= -\frac{1}{2}a_1P + \frac{1}{2}a_3X, \\ \nabla_P V &= \frac{1}{2}(a_2X - a_1Y), \quad & \nabla_Q V &= a_1Y - a_2X. \end{aligned}$$

Set $X = e_1$, $Y = e_2$, $P = e_3$, $Q = e_4$, then

$$\|\nabla V\|^2 = \sum_{i=1}^{4} g(\nabla_{e_i}V, \nabla_{e_i}V) = \frac{3}{2}a_1^2 + \frac{3}{2}a_2^2 + \frac{1}{2}a_3^2.$$

Considering $\|V\|^2 = a_1^2 + a_2^2 + a_3^2 + a_4^2$ in (2.1), we complete the proof. □

Let $G_1(1)$ be the four dimensional oscillator Lie group equipped with a left invariant Riemannian metric for which the generators $\{P, X, Y, Q\}$ of oscillator algebra $\mathfrak{g}_1(1)$ are orthonormal, and let Γ be a discrete subgroup such that $\Gamma \setminus G_1(1)$ is compact. By Theorem 1.1, we know V is a harmonic unit vector field on $\Gamma \setminus G_1(1)$ if and only if $V = \pm P$ or $V = \pm Q$ or $V = a_1X + a_2Y$ $(a_1^2 + a_2^2 = 1)$.

Proof of Theorem 1.3. If $V = \pm Q$, by Proposition 3.1, we know V is a parallel vector field. From (2.3), it is easy to see that V is stable. We have case (i) of Theorem 1.3.

If $V = \pm P$, let $X = l_1X + l_2Y + l_3Q \in \mathcal{H}^V$. Then by (3.2), we have

$$\|A_V\|^2 = \|\nabla P\|^2 = \frac{1}{2}.$$

From this and (3.3) we obtain

$$\|\nabla X\|^2 - \|A_V\|^2\|X\|^2 = l_1^2 + l_2^2 - \frac{1}{2}l_3^2.$$

So $(\mathrm{Hess}\, E)_V$ is negative on the subspace generated by Q. And we have case (ii) of Theorem 1.3.

If $V = a_1X + a_2Y$ $(a_1^2 + a_2^2 = 1)$, let $X = a_2X - a_1Y + a_3P + a_4Q \in \mathcal{H}^V$. Then by (3.3), we have

$$\|A_V\|^2 = \frac{3}{2}(a_1^2 + a_2^2) = \frac{3}{2}, \quad \|\nabla X\|^2 = \frac{3}{2} + \frac{1}{2}a_3^2.$$

So, we obtain

$$\|\nabla X\|^2 - \|A_V\|^2\|X\|^2 = -\frac{1}{2}a_3^2 - \frac{3}{2}a_4^2.$$

So $(\mathrm{Hess}\, E)_V$ is negative on the subspace generated by $\{P, Q\}$. And we have case (iii) of Theorem 1.3.

This completes the proof. ☐

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