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Archivum Mathematicum, Vol. 55 (2019), No. 4, 261–264

Persistent URL: <http://dml.cz/dmlcz/147879>

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ON THE γ -EQUIVALENCE OF SEMIHOLONOMIC JETS

MIROSLAV DOUPOVEC AND IVAN KOLÁŘ

ABSTRACT. It is well known that the concept of holonomic r -jet can be geometrically characterized in terms of the contact of individual curves. However, this is not true for the semiholonomic r -jets, [5], [8]. In the present paper, we discuss systematically the semiholonomic case.

In [5], the second author introduced the concept of equivalence with respect to curves, or γ -equivalence, of semiholonomic r -jets, when he studied the contact of spaces with higher order connection according to C. Ehresmann, [4]. In the present paper, we study the general form of this problem. In Section 5, we describe how to discuss the γ -equivalence of two arbitrary semiholonomic r -jets. We use an original concept of k -sesquiholonomic r -jet, that generalizes an idea by P. Libermann, [8].

Unless otherwise specified, we use the terminology and notation from the book [6].

1. SEMIHOLONOMIC r -JETS

The r -th semiholonomic prolongation $\bar{J}^r Y \rightarrow M$ of a fibered manifold $p: Y \rightarrow M$ is defined as follows. By induction, we have constructed a projection $\pi_{r-1}^{r-2}: \bar{J}^{r-1} Y \rightarrow \bar{J}^{r-2} Y$. The elements of $\bar{J}^r Y$ are 1-jets $j_x^1 s$ of the local sections $s: M \rightarrow \bar{J}^{r-1} Y$ satisfying

$$(1) \quad s(x) = j_x^1(\pi_{r-1}^{r-2} \circ s).$$

If x^i, y^p are some local fiber coordinates on Y and $y_i^p, \dots, y_{i_1 \dots i_{r-1}}^p$ are the induced local coordinates on $\bar{J}^{r-1} Y$ arbitrary in all subscripts, then the induced local coordinates on $\bar{J}^r Y \rightarrow \bar{J}^{r-1} Y$ are

$$(2) \quad y_{i_1 \dots i_r}^p(j_x^1 s) = \frac{\partial y_{i_1 \dots i_{r-1}}^p(s)}{\partial x^{i_r}}(x).$$

Hence even $y_{i_1 \dots i_r}^p$ are arbitrary in all subscripts. The r -th holonomic prolongation $J^r Y$ is a subbundle of $\bar{J}^r Y$, whose all coordinates are symmetric in all subscripts. The space $\bar{J}^r(M, N)$ of semiholonomic r -jets of M into N is defined as the r -th

2010 *Mathematics Subject Classification*: primary 58A20; secondary 58A32.

Key words and phrases: jet bundle, semiholonomic r -jet.

The second author was supported by GA ČR under the grant GA17-01171F.

Received April 2, 2019. Editor J. Slovák.

DOI: 10.5817/AM2019-4-261

semiholonomic prolongation of the product fibered manifold $Y = M \times N \rightarrow M$. We write α or β for the source or target projection.

The semiholonomic r -jets are endowed with the restriction of the composition of nonholonomic r -jets by Ehresmann, [3]. If $X = j_x^1 f \in \overline{J}^r(M, N)$ and $Y = j_y^1 g \in \overline{J}^r(N, Q)$, $y = \beta X$, then

$$(3) \quad Y \circ X = j_x^1(g(\beta f(u)) \circ f(u)) \in \overline{J}^r(M, Q), \quad u \in M,$$

with the composition of semiholonomic $(r - 1)$ -jets on the right hand side. The composition of holonomic r -jets coincides with the classical one.

2. THE EQUIVALENCE WITH RESPECT TO CURVES

In [5], the second author introduced a special version of this idea, when he investigated the contact of spaces with higher order connection in the sense of Ehresmann, [4]. In the general situation, we define

Definition 1. Two r -jets $B, C \in \overline{J}_x^r(M, N)_y$ are equivalent by curves, or γ -equivalent, if

$$(4) \quad B \circ A = C \circ A \quad \text{for all} \quad A = j_0^r \gamma, \quad \gamma: \mathbb{R} \rightarrow M, \quad \gamma(0) = x.$$

We shall write $B \sim_\gamma C$.

It is well known that two holonomic r -jets $B, C \in J_x^r(M, N)_y$ are equivalent by curves, if and only if $B = C$. But in the semiholonomic case, the situation is different.

In the simplest case $B, C \in \overline{J}_0^2(\mathbb{R}^m, \mathbb{R}^n)_0$, $B = (y_i^p, y_{ij}^p)$, $C = (z_i^p, z_{ij}^p)$ and $A = (a_1^i, a_2^i) \in J_0^2(\mathbb{R}, \mathbb{R}^m)_0$, $B \circ A = C \circ A$ for all A means

$$(5) \quad (y_i^p a_1^i, y_{ij}^p a_1^i a_2^j + y_i^p a_2^i) = (z_i^p a_1^i, z_{ij}^p a_1^i a_2^j + z_i^p a_2^i),$$

i.e. $y_i^p = z_i^p$ and $y_{ij}^p = z_{ij}^p$. This proves that $B, C \in \overline{J}_x^2(M, N)_y$ are γ -equivalent if and only if the symmetrizations of B and C in $J_x^2(M, N)_y$ coincide.

3. THE k -SESQUIHOLONOMIC r -JETS

According to P. Libermann, [8], $\overline{J}^r(M, N)$ is a pullback of $TN \otimes \otimes^r T^*M$ over $\overline{J}^{r-1}(M, N)$. (We remark that the fact $J^r(M, N)$ is a pullback of $TN \otimes S^r T^*M$ over $J^{r-1}(M, N)$ was deduced in [6].) Further, she defined the r -th sesquiholonomic prolongation $\check{J}^r(M, N) \subset \overline{J}^r(M, N)$ by

$$(6) \quad X \in \check{J}^r(M, N) \quad \text{means} \quad \pi_r^{r-1} X \in J^{r-1}(M, N).$$

So, $\check{J}^r(M, N)$ is the pullback of $TN \otimes \otimes^r T^*M$ over $J^{r-1}(M, N)$. Further, the tensor symmetrization $TN \otimes \otimes^r T^*M \rightarrow TN \otimes S^r T^*M$ induces a map $\rho_r : \check{J}^r(M, N) \rightarrow J^r(M, N)$, see also [2]. Analogously to (5), one verifies that $B, C \in \check{J}_x^r(M, N)_y$ are γ -equivalent, if and only if $\pi_r^{r-1} B = \pi_r^{r-1} C$ and

$$(7) \quad \rho_r(B) = \rho_r(C) \in J_x^r(M, N)_y.$$

We generalize the concept of sesquiholonomic r -jet as follows.

Definition 2. A jet $X \in \overline{J}^r(M, N)$ is called k -sesquiholonomic, $k < r$, if $\pi_r^k X \in J^k(M, N)$.

We shall write $X \in \check{J}^{r,k}(M, N)$. So sesquiholonomic in the sense of Libermann means $(r - 1)$ -sesquiholonomic under our approach.

Proposition 1. $B, C \in \check{J}_x^r(M, N)_y$ are γ -equivalent, if and only if $\pi_r^{r-1} B = \pi_r^{r-1} C$ and $\rho_r(B) = \rho_r(C)$.

Proof. The highest order coordinates of B or C are $y_{i_1 \dots i_r}^p$ or $z_{i_1 \dots i_r}^p$, respectively. One finds easily that $B \sim_\gamma C$ means

$$(8) \quad z_{i_1 \dots i_r}^p a_1^{i_1} \dots a_1^{i_r} = y_{i_1 \dots i_r}^p a_1^{i_1} \dots a_1^{i_r}.$$

This is the coordinate form of $\rho_r(B) = \rho_r(C)$. □

4. THE CASE $r = 3$

It is useful to discuss this special case separately.

Proposition 2. $B, C \in \overline{J}_0^3(M, N)_0$ are γ -equivalent, if and only if $\pi_3^2 B = \pi_3^2 C \in J_0^2(M, N)_0$ and $\rho_3(B) = \rho_3(C) \in J_0^3(M, N)_0$.

Proof. Let $B = (y_i^p, y_{ij}^p, y_{ijk}^p), C = (z_i^p, z_{ij}^p, z_{ijk}^p)$ and $A = (a_1^i, a_2^i, a_3^i) \in J_0^3(\mathbb{R}, \mathbb{R}^m)_0$. From (3), we deduce for $C \circ A = B \circ A$

$$(9) \quad z_i^p a_1^i = y_i^p a_1^i,$$

$$(10) \quad z_{ij}^p a_1^i a_1^j + z_i^p a_2^i = y_{ij}^p a_1^i a_1^j + y_i^p a_2^i,$$

$$(11) \quad z_{ijk}^p a_1^i a_1^j a_1^k + z_{ij}^p (a_2^i a_1^j + a_1^i a_2^j) + z_{ij}^p a_2^i a_2^j + z_i^p a_3^i = \{y\},$$

where $\{y\}$ in (11) means that all z 's on the left hand side are replaced by the corresponding y 's. Since a_1^i are arbitrary quantities, (9)–(11) imply

$$(12) \quad z_i^p = y_i^p, \quad z_{(ij)}^p = y_{(ij)}^p, \quad z_{(ijk)}^p = y_{(ijk)}^p.$$

Further, for $a_2^i = 1, a_1^j = 1$ and other a 's equal to zero, we obtain from (11) the additional conditions

$$(13) \quad z_{(ij)}^p + z_{ij}^p = y_{(ij)}^p + y_{ij}^p,$$

what implies $z_{ij}^p = y_{ij}^p$. This proves our assertion. □

We remark that the coordinate formula for the composition $Y \circ X$ of two arbitrary semiholonomic r -jets is deduced in [1]. However, in our case the coordinate form of $X = A$ is very special. So we find more suitable the direct use of (3) than the specialization of the general formula from [1].

5. THE GENERAL SITUATION

Consider two k -sesquiholonomic r -jets $B, C \in \check{J}^{r,k}(M, N)$. If $B \sim_\gamma C$, then $\pi_r^s B \sim_\gamma \pi_r^s C$ for all $s \geq k$. In the case $s = k$, these jets are holonomic and $\pi_r^k B \sim_\gamma \pi_r^k C$ is equivalent to $\pi_r^k B = \pi_r^k C$. Then $\pi_r^{k+1} B, \pi_r^{k+1} C \in \check{J}^{k+1}(M, N)$ and $\pi_r^{k+1} B \sim_\gamma \pi_r^{k+1} C$ is equivalent to $\rho_{k+1}(\pi_r^{k+1} B) = \rho_{k+1}(\pi_r^{k+1} C)$. On the other hand, if the last equation is not satisfied, we do not have $\pi_r^{k+1} B \sim_\gamma \pi_r^{k+1} C$. Hence even $B \sim_\gamma C$ cannot be true.

The situation $r = 3$ is specific in that sense, that we can deduce $\pi_3^2 B = \pi_3^2 C$ directly from (11) with no additional conditions on B and C .

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