Amir Hossein Salehi Shayegan; Reza Bayat Tajvar; Alireza Ghanbari; Ali Safaie

Inverse source problem in a space fractional diffusion equation from the final overdetermination

Applications of Mathematics, Vol. 64 (2019), No. 4, 469–484

Persistent URL: http://dml.cz/dmlcz/147800

INVERSE SOURCE PROBLEM IN A SPACE FRACTIONAL DIFFUSION EQUATION FROM THE FINAL OVERDETERMINATION

Amir Hossein Salehi Shayegan, Reza Bayat Tajvar, Alireza Ghanbari, Ali Safaie, Tehran

Received September 19, 2018. Published online June 3, 2019.

Abstract. We consider the problem of determining the unknown source term $f = f(x,t)$ in a space fractional diffusion equation from the measured data at the final time $u(x,T) = \psi(x)$. In this way, a methodology involving minimization of the cost functional $J(f) = \int_0^l (u(x,t;f)|_{t=T} - \psi(x))^2 \,\mathrm{d}x$ is applied and shown that this cost functional is Fréchet differentiable and its derivative can be formulated via the solution of an adjoint problem. In addition, Lipschitz continuity of the gradient is proved. These results help us to prove the monotonicity and convergence of the sequence $\{J'(f^{(n)})\}$, where $f^{(n)}$ is the nth iteration of a gradient like method. At the end, the convexity of the Fréchet derivative is given.

Keywords: inverse source problem; space fractional diffusion equation; weak solution theory; adjoint problem; Lipschitz continuity

MSC 2010: 65N21, 65N20

1. Introduction

We study the inverse problem associated with the space fractional diffusion problem

$$u_t(x,t) - \frac{1}{2}\,{}^{R}D_x^{\alpha} u(x,t) - \frac{1}{2}\,{}_x^{R}D^{\alpha} u(x,t) = f(x,t), \quad (x,t) \in Q_T, \tag{1.1}$$

$$u(0,t) = u(l,t) = 0, \quad t \in (0,T), \tag{1.2}$$

$$u(x,0) = \varphi(x), \quad x \in \Lambda, \tag{1.3}$$

where $u_t := \partial u/\partial t$, $\Lambda = (0,l)$, $Q_T = \Lambda \times (0,T)$ and $1 < \alpha < 2$. Here ${}^{R}D_x^{\alpha} u(x,t)$ and ${}_x^{R}D^{\alpha} u(x,t)$ denote the left and right Riemann-Liouville fractional derivatives, respectively, which are defined for $x \in (0, l)$ by

$$
\begin{aligned}
{}^{R}D_x^{\alpha} u(x,t) &= \frac{1}{\Gamma(2-\alpha)} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \int_0^x \frac{u(\xi,t)}{(x-\xi)^{\alpha-1}} \,\mathrm{d}\xi, \\
{}_{x}^{R}D^{\alpha} u(x,t) &= \frac{1}{\Gamma(2-\alpha)} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \int_x^l \frac{u(\xi,t)}{(\xi-x)^{\alpha-1}} \,\mathrm{d}\xi.
\end{aligned}
$$

The inverse problem here consists of determining the source term $f = f(x,t)$ from the measured data at the final time

$$
u(x,T) = \psi(x). \tag{1.4}
$$

The function $\psi(x)$ is assumed to be the measured output data and also the functions f and φ are the inputs data. In this context, the inverse source problem (1.1)–(1.4) and the problem (1.1)–(1.3) for a given f will be referred to as the problem (ISP) and the direct problem, respectively.

It is worth pointing out that for $\alpha = 1$ and $\alpha = 2$, the ISP (1.1)–(1.4) is a classical ISP and has been studied by some researchers such as ISP for linear parabolic equations with final overdetermination, see [3], [8], nonlinear source term given by $f(x,t,u) = p(x)u^r$, see [1], ISP for the parabolic equation $u_t = \Delta u + p(x)u + f(u)$, see [2], determination of the unknown function $p(x)$ in the source term $F = p(x)f(u)$, see [16] and determination of the unknown source term $F(x,t)$ in $u_t = (k(x)u_x)_x + F(x,t)$, see [6], [7], [9]. But to our knowledge, there are few works on inverse source space fractional diffusion equations.

In this paper, we apply the weak solution theory and the adjoint problem approach to ISP (1.1)–(1.4). To this end, the auxiliary functional

$$
J(f) = \int_0^l (u(x,t;f)|_{t=T} - \psi(x))^2 \,\mathrm{d}x \tag{1.5}
$$

is introduced and ISP is reformulated as a minimization problem for this functional. It is shown that the gradient J' of the cost functional (1.5) is Lipschitz continuous. Then, an explicit formula for this gradient is obtained by the solution of the corresponding adjoint problem. Based on these results, monotonicity of the sequence $\{J(f^{(n)})\}$ is proved where $\{f^{(n)}\}$ is the sequence of iterations obtained by the gradient method.

Note that, as a consequence of the physical model, the inputs $\varphi = \varphi(x)$ and $f = f(x,t)$ may not be smooth and, especially in real problems, the input $f = f(x,t)$ belongs to $L_2(Q_T)$. This circumstance requires use of weak solution theory. So, the main motivation of the proposed approach and also the substantial difference of the provided results in this paper in comparison to the available scientific results, is to apply the weak solution theory which reduces the order of regularity. To be exact, the approach proposed here is based on weak solution theory and the adjoint method. This combination leads to an applicable method which is more consistent with the physical model in real-world problem.

The paper is organized as follows. In Section 2, we introduce a quasi solution of ISP (1.1)–(1.4), based on the weak solution of the direct problem (1.1)–(1.3). In Section 3, we introduce an adjoint space fractional diffusion problem and obtain an explicit relationship between the weak solution of this problem and the gradient of the cost functional (1.5). The Lipschitz continuity of the gradient is obtained in Section 4. These results help one to construct a gradient like iteration process for the sequence of approximate solutions $\{f^{(n)}\} \subset \chi$ of the inverse problem and prove monotonicity of the sequence of functionals $\{J(f^{(n)})\}$. In Section 5, convexity of the Fréchet derivative is studied and finally, in Section 6, an application of the considered problem is given.

2. Quasi solution of the inverse problem and the gradient

Let us denote by $\chi := L_2(Q_T)$ the set of admissible unknown source functions f. Evidently, the set χ is closed and convex. The weak solution of the direct problem (1.1)–(1.3) will be defined as the function $u \in B^{\alpha/2}(Q_T)$ satisfying the integral identity

$$\Pi(u,v) = F(v) \quad \forall\, v \in B^{\alpha/2}(Q_T), \tag{2.1}$$

where the bilinear form $\Pi(\cdot,\cdot)$ is defined by

$$\Pi(u,v) := (u_t, v)_{L_2(Q_T)} - \frac{1}{2}\big({}^R D_x^{\alpha/2} u, {}_x^R D^{\alpha/2} v\big)_{L_2(Q_T)} - \frac{1}{2}\big({}_x^R D^{\alpha/2} u, {}^R D_x^{\alpha/2} v\big)_{L_2(Q_T)},$$

and the functional $F(\cdot)$ is given by

$$F(v) := (f, v)_{L_2(Q_T)}.$$

Here

$$B^{\alpha}(Q_T) := L_\infty((0,T), L_2(\Lambda)) \cap L_2((0,T), H_0^{\alpha}(\Lambda))$$

is a Banach space with respect to the norm

$$\|v\|_{B^{\alpha}(Q_T)} = \Big(\max_{0\leqslant t\leqslant T} \|v(\cdot,t)\|^2_{L_2(\Lambda)} + \|v\|^2_{L_2((0,T),H_0^{\alpha}(\Lambda))}\Big)^{1/2},$$

where

$$L_2((0,T),H_0^\alpha(\Lambda)) = \{v;\ \|v(\cdot,t)\|_{H_0^\alpha(\Lambda)} \in L_2(0,T)\},$$

endowed with the norm

$$\|v\|_{L_2((0,T),H_0^\alpha(\Lambda))} = \|\|v(\cdot,t)\|_{H_0^\alpha(\Lambda)}\|_{L_2(0,T)}.$$

In the above definition $H_0^\alpha(\Lambda)$ denotes the usual fractional Sobolev space with respect to the norm $\|\cdot\|_{H_0^\alpha(\Lambda)}$ (for more details see [11]). Now suppose that the following assumptions hold:

(A1) $1 < \alpha < 2$,
(A2) $f \in \chi$,
(A3) $\varphi \in L_2(\Lambda)$,
(A4) $\psi \in L_2(\Lambda)$.

Then it is proved that the weak solution $u \in B^{\alpha/2}(Q_T)$ of the direct problem (1.1)–(1.3) exists and is unique, see [11], [5], [10]. We denote this weak solution by $u(x,t;f)$ corresponding to a given $f \in \chi$. If this function satisfies the additional condition (1.4), then it must satisfy the equation

$$u(x,t;f)|_{t=T} = \psi(x), \quad x \in \Lambda. \tag{2.2}$$

However, due to measurement errors in practice, the exact equality in the above equation is usually not achieved [7]. For this reason, we define a quasi solution of the inverse problem as a solution of the minimization problem for the cost functional J, given by (1.5). In doing so, find $f_* \in \chi$ such that

$$J(f_*) = \inf_{f\in\chi} J(f). \tag{2.3}$$

Clearly, if $J(f_*) = 0$, then the quasi solution $f_* \in \chi$ is a strict solution of the inverse problem (1.1)–(1.4) and also $f_* \in \chi$ satisfies the functional equation (2.2). In addition, in view of the weak solution theory for space fractional diffusion problems, from [11] we have

$$\|u(\cdot,T;f^{(n)}) - u(\cdot,T;f)\|_{L_2(\Lambda)} \lesssim \|f^{(n)} - f\|_{L_2(Q_T)}.$$

This, in particular, means that if the sequence $\{f^{(n)}\} \in \chi$ weakly converges to the function $f \in \chi$, then the sequence of traces $\{u(x,T;f^{(n)})\}$ of the corresponding solutions of the direct problem (1.1)–(1.3) converges in the L_2-norm to the solution $u(x,T;f)$, which implies $J(f^{(n)}) \to J(f)$, as $n \to \infty$ (see [10], [11]). This means that the functional J is continuous with respect to the weak convergence in χ, hence due to the Weierstrass existence theorem, see [6], [9], the set of solutions

$$\chi_* := \Big\{ f_* \in \chi \colon \ J(f_*) = \inf_{f\in\chi} J(f) \Big\},$$

of the minimization problem (2.3) is not an empty set.

Remark 2.1. It is worth pointing out that the solution of the problem (2.1) does not imply the boundary conditions imposed upon the strong solution of (1.1)–(1.3) for functions in $B^{\alpha/2}(Q_T)$ with $\alpha/2 < 1/2$. In fact it has no sense to define the trace at $x = 0$ (and also $x = l$) for functions in $B^{\alpha/2}(Q_T)$ with $0 < \alpha < 1$. But in our problem, according to [11], [10] the the trace at time $t = 0$ (and also $t = T$) and boundary conditions are well-defined.

3. Fréchet differentiability of the cost functional and its gradient

Let f and $f+\delta f \in \chi$ be source functions. We denote by $u(x,t;f)$ and $u(x,t;f+\delta f)$ the corresponding solutions of the problem (1.1)–(1.3). Then

$$\delta u(x,t;f) := u(x,t;f+\delta f) - u(x,t;f),$$

is the solution of the problem

$$\delta u_t(x,t) - \frac{1}{2}{}^R D_x^{\alpha} \delta u(x,t) - \frac{1}{2}{}_x^R D^{\alpha} \delta u(x,t) = \delta f(x,t), \quad (x,t) \in Q_T, \tag{3.1}$$

$$\delta u(0,t) = \delta u(l,t) = 0, \quad t \in (0,T), \tag{3.2}$$

$$\delta u(x,0) = 0, \quad x \in \Lambda. \tag{3.3}$$

The first variation ΔJ of the cost functional J is

$$\Delta J(f) := J(f+\delta f) - J(f) = 2\int_0^l (u(x,t;f)|_{t=T} - \psi(x))\delta u(x,t;f)|_{t=T} \, \mathrm{d}x + \int_0^l (\delta u(x,t;f)|_{t=T})^2 \, \mathrm{d}x, \tag{3.4}$$

where $\delta u(x,t;f)$ is the solution of (3.1)–(3.3).

Lemma 3.1. *Let $f, f+\delta f \in \chi$ be given source functions. If $u = u(x,t;f)$ is the solution of the direct problem (1.1)–(1.3) and $p = p(x,t)$ is the solution of the* *adjoint problem*

$$p_t(x,t)+\frac{1}{2}{}^R D_x^\alpha p(x,t)+\frac{1}{2}{}_x^R D^\alpha p(x,t)=0,\quad (x,t)\in Q_T, \tag{3.5}$$

$$p(0,t)=p(l,t)=0,\qquad t\in(0,T), \tag{3.6}$$

$$p(x,T)=q(x),\qquad x\in\Lambda, \tag{3.7}$$

with an arbitrary function $q=q(x)\in L_2(\Lambda)$, then the following integral identity holds:

$$\int_0^l q(x)\delta u(x,t;f)|_{t=T}\,\mathrm{d}x=\int_0^T\int_0^l \delta f(x,t)p(x,t)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.8}$$

P r o o f. Multiply (3.1) by p and integrate over Q_T to get

$$(\delta u_t,p)_{L_2(Q_T)}-\frac{1}{2}\big({}^R D_x^\alpha\delta u,p\big)_{L_2(Q_T)}-\frac{1}{2}\big({}_x^R D^\alpha\delta u,p\big)_{L_2(Q_T)}=(\delta f,p)_{L_2(Q_T)}. \tag{3.9}$$

According to [14], we have

$$\begin{aligned}\big({}^R D_x^\alpha\delta u,p\big)_{L_2(Q_T)}&=\big(\delta u,{}_x^R D^\alpha p\big)_{L_2(Q_T)},\\ \big({}_x^R D^\alpha\delta u,p\big)_{L_2(Q_T)}&=\big(\delta u,{}^R D_x^\alpha p\big)_{L_2(Q_T)}.\end{aligned} \tag{3.10}$$

Now, consider the first term on the left-hand side of (3.9). Applying integration by parts, we get

$$\begin{aligned}(\delta u_t,p)_{L_2(Q_T)}&=\int_0^l\int_0^T \delta u_t(x,t;f)p(x,t)\,\mathrm{d}t\,\mathrm{d}x\\ &=\int_0^l \delta u(x,t;f)|_{t=T}p(x,T)\,\mathrm{d}x-\int_0^l \delta u(x,t;f)|_{t=0}\,p(x,0)\,\mathrm{d}x\\ &\quad-(\delta u,p_t)_{L_2(Q_T)}.\end{aligned}$$

So, we obtain

$$(\delta u_t,p)_{L_2(Q_T)}=\int_0^l \delta u(x,t;f)|_{t=T}\,q(x)\,\mathrm{d}x-(\delta u,p_t)_{L_2(Q_T)}. \tag{3.11}$$

For the second and third terms on the left-hand side of (3.9), using (3.10) we have

$$\begin{aligned}&-\frac{1}{2}\big({}^R D_x^\alpha\delta u,p\big)_{L_2(Q_T)}-\frac{1}{2}\big({}_x^R D^\alpha\delta u,p\big)_{L_2(Q_T)}\\ &\quad=-\frac{1}{2}\big(\delta u,{}_x^R D^\alpha p\big)_{L_2(Q_T)}-\frac{1}{2}\big(\delta u,{}^R D_x^\alpha p\big)_{L_2(Q_T)}.\end{aligned} \tag{3.12}$$

Applying (3.11) and (3.12) in (3.9), we can obtain

$$\int_0^l \delta u(x,t;f)|_{t=T}\, q(x)\,\mathrm{d}x - (\delta u, p_t)_{L_2(Q_T)} - \frac{1}{2}\big(\delta u, {}^R_x D^\alpha p\big)_{L_2(Q_T)} - \frac{1}{2}\big(\delta u, {}^R D^\alpha_x p\big)_{L_2(Q_T)} = (\delta f, p)_{L_2(Q_T)},$$

and

$$\int_0^l \delta u(x,t;f)|_{t=T}\, q(x)\,\mathrm{d}x + \Big(\delta u, -p_t - \frac{1}{2}{}^R_x D^\alpha p - \frac{1}{2}{}^R D^\alpha_x p\Big)_{L_2(Q_T)} = (\delta f, p)_{L_2(Q_T)},$$

which leads to

$$\int_0^l \delta u(x,t;f)|_{t=T}\, q(x)\,\mathrm{d}x = \int_0^T\int_0^l \delta f(x,t)p(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

□

Corollary 3.1. *Let us choose an arbitrary control function* $q = q(x)$ *in* (3.8) *as*

$$q(x) := \frac{\delta u(x,t;f)|_{t=T}}{\|\delta u(x,t;f)|_{t=T}\|_{L_2(\Lambda)}}.$$

Then we obtain

$$\|\delta u(x,t;f)|_{t=T}\|_{L_2(\Lambda)} \leqslant \|p\|_{L_2(Q_T)}\|\delta f\|_{L_2(Q_T)},$$

where $\delta u = \delta u(x,t;f)$ *is the solution of* (3.1)–(3.3) *and* $p = p(x,t)$ *is defined in Lemma* 3.1. *We note that the existence and uniqueness of* (3.5)–(3.7) *are the straightforward results of* [11].

Corollary 3.2. *If in* (3.8) *we set* $q(x) := 2(u(x,t;f)|_{t=T} - \psi(x))$, *we obtain the useful identity*

$$2\int_0^l (u(x,t;f)|_{t=T} - \psi(x))\delta u(x,t;f)|_{t=T}\,\mathrm{d}x = \int_0^T\int_0^l \delta f(x,t)p(x,t)\,\mathrm{d}x\,\mathrm{d}t, \tag{3.13}$$

which will be used in the following.

Hereafter, in cases where no confusion could arise, the symbol p in (3.5)–(3.7) refers to the terminal condition q defined in Corollary 3.2.

The integral equality (3.13) yields that the first variation of the cost functional J may be written in the form

$$\begin{aligned}\Delta J(f) &= J(f+\delta f) - J(f) \\ &= \int_0^T\int_0^l \delta f(x,t)p(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^l (\delta u(x,t;f)|_{t=T})^2\,\mathrm{d}x.\end{aligned} \tag{3.14}$$

Now, we will show that the second term on the right-hand side of (3.14) is of the order $\mathrm{O}(\|\delta f\|^2_{L_2(Q_T)})$.

Lemma 3.2. *If $f\in\chi$ is a given source function and $u=u(x,t;f)\in B^{\alpha/2}(Q_T)$ is the corresponding solution of the direct problem* (1.1)–(1.3), *then we have the inequality*

$$\|\delta u(x,t;f)|_{t=T}\|_{L_2(\Lambda)} \leqslant \mathrm{e}^T\|\delta f\|_{L_2(Q_T)}. \tag{3.15}$$

Proof. Multiply both sides of (3.1) by δu and then integrate over Q_t, $t\in(0,T]$, to get

$$\begin{aligned}(\delta u_\tau,\delta u)_{L_2(Q_t)} &- \frac{1}{2}\big({}^R D_x^\alpha \delta u,\delta u\big)_{L_2(Q_t)} \\ &- \frac{1}{2}\big({}_x^R D^\alpha \delta u,\delta u\big)_{L_2(Q_t)} = (\delta f,\delta u)_{L_2(Q_t)}.\end{aligned} \tag{3.16}$$

Using an energy function

$$G(\tau) = \int_0^l \delta u^2(x,\tau;f)\,\mathrm{d}x,$$

we obtain $G'(\tau) = 2\int_0^l \delta u_\tau(x,\tau;f)\delta u(x,\tau;f)\,\mathrm{d}x$. Therefore, we get

$$\frac{1}{2}\int_0^t G'(\tau)\,\mathrm{d}\tau = \frac{1}{2}(G(t)-G(0)) = \frac{1}{2}\bigg(\int_0^l \delta u^2(x,t;f) - \int_0^l \delta u^2(x,0;f)\,\mathrm{d}x\bigg),$$

which means that

$$(\delta u_\tau,\delta u)_{L_2(Q_t)} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l \delta u^2(x,t;f)\,\mathrm{d}x. \tag{3.17}$$

On the other hand, we have

$$
\begin{aligned}
(3.18)\qquad & \frac{1}{2}\big({}^R D_x^{\alpha}\delta u,\delta u\big)_{L_2(Q_t)}+\frac{1}{2}\big({}_x^R D^{\alpha}\delta u,\delta u\big)_{L_2(Q_t)}\\
&=\frac{1}{2}\big({}^R D_x^{\alpha/2}\delta u,{}_x^R D^{\alpha/2}\delta u\big)_{L_2(Q_t)}+\frac{1}{2}\big({}_x^R D^{\alpha/2}\delta u,{}^R D_x^{\alpha/2}\delta u\big)_{L_2(Q_t)}\\
&=\frac{1}{2}\int_0^t\int_0^l {}^R D_x^{\alpha/2}\delta u(x,\tau;f)\,{}_x^R D^{\alpha/2}\delta u(x,\tau;f)\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+\frac{1}{2}\int_0^t\int_0^l {}_x^R D^{\alpha/2}\delta u(x,\tau;f)\,{}^R D_x^{\alpha/2}\delta u(x,\tau;f)\,\mathrm{d}x\,\mathrm{d}\tau\\
&=\big({}^R D_x^{\alpha/2}\delta u,{}_x^R D^{\alpha/2}\delta u\big)_{L_2(Q_t)}=\cos\Big(\frac{\pi\alpha}{2}\Big)\big\|{}^R D_x^{\alpha/2}\delta u\big\|^2_{L_2(Q_t)},
\end{aligned}
$$

where in the last equality we use Fourier transform property [4]. Consequently, applying the relations (3.17) and (3.18) in (3.16), we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l \delta u^2(x,t;f)\,\mathrm{d}x-2\cos\Big(\frac{\pi\alpha}{2}\Big)\big\|{}^R D_x^{\alpha/2}\delta u\big\|^2_{L_2(Q_t)}=2(f,\delta u)_{L_2(Q_t)},$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l \delta u^2(x,t;f)\,\mathrm{d}x\leqslant 2(f,\delta u)_{L_2(Q_t)}.$$

Now, using the Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l \delta u^2(x,t;f)\,\mathrm{d}x &\leqslant 2\int_0^l \delta u^2(x,t;f)\,\mathrm{d}x\int_0^l \delta f^2(x,t)\,\mathrm{d}x\\
&\leqslant \int_0^l \delta u^2(x,t;f)\,\mathrm{d}x+\int_0^l \delta f^2(x,t)\,\mathrm{d}x.
\end{aligned}
$$

Then applying Gronwall inequality, the desired result can be archived. □

By the definition of the Fréchet derivative, from (3.14) and (3.15) we conclude that the gradient of the cost functional J is the operator

$$J'(f)=p(x,t;f), \tag{3.19}$$

where p is the solution of (3.5)–(3.7).

By using Lemma 3.1 and Lemma 3.2, one can prove the following theorem.

Theorem 3.1. *Let the assumptions* (A1)–(A4) *hold. Then the cost functional J is Fréchet-differentiable, $J\in C^1(\chi)$. The Fréchet derivative at $f\in\chi$ of the cost functional J is defined via the solution of the adjoint problem* (3.5)–(3.7) *as*

$$J'(f)=p(x,t;f).$$

Corollary 3.3. *Let $J \in C^1(\chi)$ and let $\chi_* \subset \chi$ be the set of quasi solutions of the ISP* (1.1)–(1.4). *Then $f_* \in \chi_*$ is a strict solution of the ISP* (1.1)–(1.4) *if and only if $p(x,t;f_*) \equiv 0$ on Q_T.*

4. Lipschitz continuity of the gradient and the monotone iteration scheme

A gradient-type iteration algorithm for the minimization problem (2.3) has the form

$$f^{(n+1)} = f^{(n)} - \omega_n J'(f^{(n)}), \quad n = 0, 1, 2, \dots, \tag{4.1}$$

where $f^{(0)} \in \chi$ is a given initial iteration and $\omega_n > 0$ is a relaxation parameter. The choice of $\omega_n > 0$ defines different gradient methods. In many situations, especially when numerically solving nonlinear problems, the estimation of the relaxation parameter ω_n is a difficult problem. Hasanov et al. in [7] asserted that in the case of Lipschitz continuity of the Fréchet gradient, i.e., when J is of the Hölder class $C^{1,1}(\chi)$, the relaxation parameter can be estimated via the Lipschitz constant $L > 0$, as follows:

$$\omega_n \in (0, 2/L).$$

In fact, if $J \in C^{1,1}(\chi)$ and $\{f^{(n)}\} \subset \chi$ is a sequence of iterations defined by the above algorithm, then for $\omega_n \in (0, 2/L)$ the numerical sequence $\{J'(f^{(n)})\}$ is decreasing and $\lim_{n\to\infty} \|J'(f^{(n)})\| = 0$. Thus, the Lipschitz continuity of the gradient of the cost functional implies the monotonicity of the numerical sequence $\{J'(f^{(n)})\}$, where $f^{(n)}$ is the nth iteration of a gradient-type iteration.

As a result, now we will prove the Lipschitz continuity of the cost functional (1.5).

Lemma 4.1. *Let the assumptions* (A1)–(A4) *hold. Also, let the functions $p(x,t;f)$ and $p(x,t;f+\delta f)$ be the solutions of adjoint problem* (3.5)–(3.7) *with $q(x) = 2(u(x,t;f)|_{t=T} - \psi(x))$ and $q(x) = 2(u(x,t;f+\delta f)|_{t=T} - \psi(x))$, respectively. Then the functional J is of the Hölder class $C^{1,1}(\chi)$ and*

$$\|J'(f+\delta f) - J'(f)\|_{L_2(Q_T)} \leqslant L\|\delta f\|_{L_2(Q_T)}, \tag{4.2}$$

where $L := 2\mathrm{e}^T\sqrt{T}$ is a Lipschitz constant and

$$\|J'(f+\delta f) - J'(f)\|^2_{L_2(Q_T)} = \int_0^T\int_0^l (\delta p(x,t;f))^2\,\mathrm{d}x\,\mathrm{d}t, \tag{4.3}$$

in which $\delta p(x,t;f) = p(x,t;f+\delta f) - p(x,t;f)$ *is the solution of the problem*

$$\delta p_t(x,t) + \frac{1}{2}\,{}^R D_x^\alpha \delta p(x,t) + \frac{1}{2}\,{}_x^R D^\alpha \delta p(x,t) = 0, \quad (x,t) \in Q_T, \tag{4.4}$$

$$\delta p(0,T) = \delta p(l,t) = 0, \quad t \in (0,T), \tag{4.5}$$

$$\delta p(x,T) = 2\delta u(x,t;f)|_{t=T}, \quad x \in \Lambda. \tag{4.6}$$

P r o o f. Multiply (4.4) by $\delta p(x,t;f)$ and integrate over $(0,l)$ to get

$$(\delta p_t, \delta p)_{L_2(\Lambda)} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l (\delta p(x,t;f))^2\,\mathrm{d}x,$$

and

$$\frac{1}{2}\big({}^R D_x^\alpha\, \delta p, \delta p\big)_{L_2(\Lambda)} + \frac{1}{2}\big({}_x^R D^\alpha \delta p, \delta p\big)_{L_2(\Lambda)} = \cos\Big(\frac{\pi\alpha}{2}\Big)\big\|{}^R D_x^{\alpha/2}\delta p\big\|_{L_2(\Lambda)}^2.$$

Then we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l (\delta p(x,t;f))^2\,\mathrm{d}x = -\cos\Big(\frac{\pi\alpha}{2}\Big)\big\|{}^R D_x^{\alpha/2}\delta p\big\|_{L_2(\Lambda)}^2.$$

Now, we define

$$\Phi(t) := \int_0^l (\delta p(x,t;f))^2\,\mathrm{d}x.$$

Since $\Phi'(t) \geqslant 0$, $\Phi(t)$ is increasing on $(0,T]$, hence

$$\Phi(t) \leqslant \Phi(T), \quad t \in (0,T].$$

Consequently, we obtain

$$\begin{aligned}\int_0^l (\delta p(x,t;f))^2\,\mathrm{d}x &\leqslant \int_0^l (\delta p(x,T;f))^2\,\mathrm{d}x \\ &= 4\int_0^l (\delta u(x,t;f)|_{t=T})^2\,\mathrm{d}x \leqslant 4\mathrm{e}^{2T}\|\delta f\|_{L_2(Q_T)}^2,\end{aligned}$$

which concludes the proof. □

We will prove the monotonicity and convergence of the sequence $J(f^{(n)})$, where $f^{(n)}$, $n = 0,1,2,\ldots$, are defined by (4.1).

Lemma 4.2. *Let* $J \in C^{1,1}(\chi)$. *Then*

$$|J(f_1) - J(f_2) - (J'(f_2), f_2 - f_1)_{L_2(Q_T)}| \leqslant \frac{1}{2}L\|f_1 - f_2\|_{L_2(Q_T)}^2, \quad f_1, f_2 \in \chi, \tag{4.7}$$

where L *is defined in Lemma* 4.1.

P r o o f. According to [7], Lemma 3.4.3, page 112, one can easily prove the lemma. □

Lemma 4.3. *Let $f^{(n)}$, $n = 0, 1, 2, \ldots$, be iterations defined by* (4.1) *and the conditions of Lemmas* 4.1 *and* 4.2 *hold. Then $\{J(f^{(n)})\}$ is a decreasing convergent sequence and*

$$\lim_{n\to\infty} \|J'(f^{(n)})\|_{L_2(Q_T)} = 0. \tag{4.8}$$

P r o o f. Apply inequality (4.7) and take $f_1 = f^{(n)} - \omega_n J'(f^{(n)})$ and $f_2 = f^{(n)}$, $\omega_n > 0$. Then we get

$$J(f^{(n)} - \omega_n J'(f^{(n)})) - J(f^{(n)}) + \omega_n \|J'(f^{(n)})\|^2_{L_2(Q_T)} \leqslant \frac{1}{2} L\omega_n^2 \|J'(f^{(n)})\|^2_{L_2(Q_T)},$$

and

$$J(f^{(n)}) - J(f^{(n+1)}) \geqslant \omega_n \Big(1 - \frac{1}{2} L\omega_n\Big) \|J'(f^{(n)})\|^2_{L_2(Q_T)}.$$

The function $\omega_n(1 - \frac{1}{2}L\omega_n)$, $\omega_n > 0$ reaches its minimum value at $w_* := 1/L$, i.e., $w_* := 1/2\mathrm{e}^T\sqrt{T}$. Hence,

$$J(f^{(n)}) - J(f^{(n+1)}) \geqslant \frac{1}{4\mathrm{e}^T\sqrt{T}} \|J'(f^{(n)})\|^2_{L_2(Q_T)} \quad \forall\, f^{(n)}, f^{(n+1)} \in \chi.$$

The right-hand side is positive, which means that the sequence $\{J(f^{(n)})\}$ is decreasing. Since this sequence is bounded from below, this result also implies convergence of the numerical sequence $\{J(f^{(n)})\}$. In conclusion, passing to the limit in the above inequality, we obtain the second assertion (4.8) of the lemma. □

Notice that the optimal value $w_* := 1/L$ is in the range $(0, 2/L)$, which reveals the validation of the proposed gradient-type iteration algorithm.

Corollary 4.1. *If $\{f^{(n)}\} \subset \chi$ is the sequence of iterations defined by*

$$f^{(n+1)} = f^{(n)} - \omega_* J'(f^{(n)}), \quad \omega_* := \frac{1}{L} = \frac{1}{2\mathrm{e}^T\sqrt{T}}, \quad n = 0, 1, 2, \ldots,$$

then $\{J(f^{(n)})\}$ is a decreasing convergent sequence and

$$\lim_{n\to\infty} \|J'(f^{(n)})\|_{L_2(Q_T)} = 0.$$

5. Convexity of the Fréchet derivative

In this section, we will study the convexity of the cost functional J.

Lemma 5.1. *Let f and $f+\delta f \in \chi$. Then*

$$(J'(f+\delta f) - J'(f), \delta f)_{L_2(Q_T)} = 2\int_0^l \delta u(x,t;\varphi)|_{t=T})^2 \,\mathrm{d}x.$$

Proof. Using (3.14) and (3.19), we can write that

$$\begin{aligned}(J'(f+\delta f) - J'(f), \delta f)_{L_2(Q_T)} &= \int_0^T\int_0^l \delta f(x,t)\delta p(x,t;f)\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^l \delta u(x,t;f)|_{t=T}\,\delta q(x)\,\mathrm{d}x,\end{aligned}$$

where

$$\delta p(x,t;f) = p(x,t;f+\delta f) - p(x,t;f),$$

and

$$\begin{aligned}\delta q(x) &= 2(u(x,t;f+\delta f)|_{t=T} - \psi(x)) - 2(u(x,t;f)|_{t=T} - \psi(x))\\ &= 2\delta u(x,t;f)|_{t=T}.\end{aligned}$$

So we have

$$(J'(f+\delta f) - J'(f), \delta f)_{L_2(\Lambda)} = 2\int_0^l \delta u(x,t;f)|_{t=T}\delta u(x,t;f)|_{t=T}\,\mathrm{d}x.$$

□

Lemma 5.1 proves that J is convex. If for $f\in\chi$ we have

$$\int_0^l (\delta u(x,t;f)|_{t=T})^2\,\mathrm{d}x > 0, \tag{5.1}$$

then J is strictly convex. Using here the uniqueness theorem on minimal problems for strictly convex functionals defined on convex sets, we may derive the following unicity result.

Theorem 5.1. *Suppose that* (5.1) *hold. Then ISP* (1.1)–(1.4) *has at most one quasi solution.*

At the end, we give the convergence theorem of the sequence $J(f^{(n)})$.

Theorem 5.2. *Let the conditions of Lemmas* 4.1 *and* 4.2 *hold. Then for any initial source* $f^{(0)} \in \chi$ *the sequence of iterations* $\{f^{(n)}\}$, *given by* (4.1), *weakly converges in* $L_2(Q_T)$ *to a quasi solution* $f_* \in \chi_*$ *of the inverse problem* (1.1)–(1.4).

Proof. It is well known that a minimization problem for a continuous convex functional in a bounded closed and convex set has a solution. Therefore the minimizing sequence $\{f^{(n)}\} \subset \chi$ weakly converges to an element $f_* \in \chi_*$. Hence, for the sequence $\{f^{(n)}\} \subset \chi$ defined by (4.1) we have $f^{(n)} \rightharpoonup f_* \in \chi_*$ as $n \to \infty$. □

6. Application

In this section, we briefly present the physical background of the space fractional diffusion equation and introduce its concrete form. It is well known that the ordinary diffusion process is intimately related to the validity of the central limit theorem, which is characterized by the linear dependence of the mean square displacement $\langle x^2(t)\rangle \sim \kappa t$ on the diffusion coefficient κ. However, some diffusion processes, especially in various complex systems, no longer follow Gaussian behavior. This phenomenon is named anomalous diffusion which is described by the nonlinear growth of the mean square displacement $x(t)$ of a diffusion particle over time t: $\langle x^2(t)\rangle \sim \kappa_\alpha t^\alpha$, where κ_α is the diffusion coefficient, and α is the anomalous diffusion exponent. For different α, the anomalous diffusion is classified into subdiffusion ($0 < \alpha < 1$), normal diffusion ($\alpha = 1$), superdiffusion ($\alpha > 1$), and ballistic diffusion ($\alpha = 2$), see [12], [15], [13] and Fick's law is inevitable to be modified in order to precisely describe the anomalous diffusion behavior [12].

Following [12], [15], denote by $u = u(x,t)$ the probability distribution of the particles (or the concentration of solute) at point x and time t. For the pure diffusion process, the conservation of the mass equation can be expressed as

$$\frac{\partial u(x,t)}{\partial t} = -\frac{\partial H(x,t)}{\partial x} + f(x), \tag{6.1}$$

where H is the mass flux and $f(x)$ a stable source. Usually, the mass flux H refers to the Fick's first law

$$H(x,t) = -\kappa \frac{\partial u(x,t)}{\partial x}.$$

However, the anomalous diffusion is given by the generalized Fick's law

$$H(x,t) = -\kappa_\alpha \{p\, {}^R D_x^{\alpha-1} - q\, {}_x^R D^{\alpha-1}\} u, \tag{6.2}$$

where $\kappa_\alpha > 0$, and $p + q = 1$ for $p,\ q \geqslant 0$ (the case in [12] is $p = q = \frac{1}{2}$). Substituting (6.2) into (6.1) leads to

$$\frac{\partial u(x,t)}{\partial t} = \kappa_\alpha \nabla^\alpha_{p,q} + f(x), \tag{6.3}$$

where

$$\nabla^\alpha_{p,q} = \{p\,{}^R D^\alpha_x - q\,{}^R_x D^\alpha\},$$

still being meaningful when α tends to 2, and corresponding to the second order derivative.

In other words, the space fractional diffusion equation, we consider in this paper can describe the probability distribution of the particles having superdiffusion. Much progress has been made for numerically solving space fractional partial differential equations. Here instead of further pursuing research in this direction, we discuss the space fractional inverse diffusion equation, i.e., to determine an unknown source, which depends only on the spatial variable, in the one dimensional space fractional diffusion equation. Determination of the unknown source is to obtain information about a physical object or system by observed datum, and it is one of the most important and well-studied problems in many branches of engineering sciences. It is worth pointing out that, identifying the unknown source is an inverse and severely ill-posed problem [15].

Conclusion

In this paper, we considered an inverse source problem associated with a space fractional diffusion equation from the final overdetermination. Using the weak solution theory and the adjoint problem, we proved the existence and uniqueness of the quasi solution and constructed a monotone iteration scheme based on a gradient like method. To this end, we showed that the cost functional is Fréchet differentiable and its derivative can be formulated via the solution of the adjoint problem. In addition, Lipschitz continuity of the gradient and convergence of the iteration scheme were given.

References

[1] *I. Bushuyev*: Global uniqueness for inverse parabolic problems with final observation. Inverse Probl. *11* (1995), L11–L16. zbl MR doi
[2] *M. Choulli*: An inverse problem for a semilinear parabolic equation. Inverse Probl. *10* (1994), 1123–1132. zbl MR doi
[3] *M. Choulli, M. Yamamoto*: Generic well-posedness of an inverse parabolic problem—the Hölder-space approach. Inverse Probl. *12* (1996), 195–205. zbl MR doi

[4] *L. B. Feng, P. Zhuang, F. Liu, I. Turner, Y. T. Gu*: Finite element method for space-time fractional diffusion equation. Numer. Algorithms *72* (2016), 749–767. zbl MR doi
[5] *N. J. Ford, J. Xiao, Y. Yan*: A finite element method for time fractional partial differential equations. Fract. Calc. Appl. Anal. *14* (2011), 454–474. zbl MR doi
[6] *A. Hasanov*: Simultaneous determination of source terms in a linear parabolic problem from the final overdetermination: Weak solution approach. J. Math. Anal. Appl. *330* (2007), 766–779. zbl MR doi
[7] *A. Hasanov Hasanoğlu, V. G. Romanov*: Introduction to Inverse Problems for Differential Equations. Springer, Cham, 2017. zbl MR doi
[8] *V. Isakov*: Inverse parabolic problems with the final overdetermination. Commun. Pure Appl. Math. *54* (1991), 185–209. zbl MR doi
[9] *M. Kaya*: Determination of the unknown source term in a linear parabolic problem from the measured data at the final time. Appl. Math., Praha *59* (2014), 715–728. zbl MR doi
[10] *X. Li, C. Xu*: A space-time spectral method for the time fractional diffusion equation. SIAM J. Numer. Anal. *47* (2009), 2108–2131. zbl MR doi
[11] *X. Li, C. Xu*: Existence and uniqueness of the weak solution of the space-time fractional diffusion equation and a spectral method approximation. Commun. Comput. Phys. *8* (2010), 1016–1051. zbl MR doi
[12] *M. M. Meerschaert, C. Tadjeran*: Finite difference approximations for fractional advection-dispersion flow equations. J. Comput. Appl. Math. *172* (2004), 65–77. zbl MR doi
[13] *R. Metzler, J. Klafter*: The random walk's guide to anomalous diffusion: A fractional dynamics approach. Phys. Rep. *339* (2000), 1–77. zbl MR doi
[14] *A. H. Salehi Shayegan, A. Zakeri*: A numerical method for determining a quasi solution of a backward time-fractional diffusion equation. Inverse Probl. Sci. Eng. *26* (2018), 1130–1154. zbl MR doi
[15] *W. Y. Tian, C. Li, W. Deng, Y. Wu*: Regularization methods for unknown source in space fractional diffusion equation. Math. Comput. Simul. *85* (2012), 45–56. zbl MR doi
[16] *A. Zeghal*: Existence results for inverse problems associated with a nonlinear parabolic equation. J. Math. Anal. Appl. *272* (2002), 240–248. zbl MR doi

Authors' address: *Amir Hossein Salehi Shayegan* (corresponding author), *Reza Bayat Tajvar*, *Alireza Ghanbari*, *Ali Safaie*, Mathematics Department, Faculty of Basic Science, Khatam-ol-Anbia (PBU) University, Tehran, Iran, e-mail: `ahsalehi.kau@gmail.com`, `r.bayat.tajvar@gmail.com`, `ARGhanbari@gmail.com`, `alisafaie459@gmail.com`.