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STRESS-STRENGTH BASED ON m-GENERALIZED ORDER STATISTICS AND CONCOMITANT FOR DEPENDENT FAMILIES

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Abstract. The stress-strength model is proposed based on the m-generalized order statistics and the corresponding concomitant. For the dependency between m-generalized order statistics and its concomitant, a bivariate copula expansion is considered and the stress-strength model is obtained for two special cases of order statistics and upper record values. In the particular case of copula function, the generalized Farlie-Gumbel-Morgenstern bivariate distribution function is considered with proportional reversed hazard functions as marginal functions. Based on the order statistics and record values, two estimators of stress-strength are presented using a procedure similar to the inference functions for margins. Finally, a simulation study is carried out which shows the good performance of the proposed estimators for a finite sample.

Keywords: copula function; Dagum distribution; generalized order statistics; Farlie-Gumbel-Morgenstern distribution; proportional reversed hazard family; record values

MSC 2010: 62G30, 62N05

1. Introduction

In the stress-strength reliability, one usually considers two random variables such that one of them is denoted as stress (X) and the other is called strength (Y) of a component or a system. If the stress exceeds the strength the component fails, while the component works whenever the stress does not exceed the strength. The reliability is defined as the probability that the component works, i.e. $R = P(Y > X)$.

Using different specifications of the model and alternative estimation methods, the stress-strength problem has been widely discussed in the literature by many authors. Most of the literature has developed under the assumption of independence between

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stress and strength, by focusing on different specifications of the distribution functions of the random variables X and Y and using different types of data. The book [26] has provided an excellent review of the studies of the stress-strength model until 2003. Given the vast amount of papers published after 2003, we can only mention a few of the most recent contributions: [36], [38], [42], [2] and [40], to name but a few.

Certainly, a much smaller number of papers addressed the stress-strength problem when the random variables are dependent. The evaluation and the estimation of R were discussed in the literature when (X, Y) follows the bivariate normal distribution by Gupta and Subramanian [20], the bivariate Pareto distribution by Hanagal [21], the bivariate beta distribution by Nadarajah [30]. Recently, the stress-strength model in the case of dependence using the copula approach was considered by Domma and Giordano [15], [16].

Special attention paid by many authors to the stress-strength model is due to the wide applicability in various fields of science. In fact, being introduced in the reliability context, it has been adapted and applied in engineering, medicine, economics, biology, and psychology. For example, in a clinical study, $P(Y > X)$ measures the effectiveness of the treatment when X and Y are the responses of a control group and treatment group, respectively. Other known applications in this context concern the evaluation of the area under the ROC curve for diagnostic tests with continuous outcomes [1]. In economics, it has been used to evaluate the distance between the income distributions [12] and, more recently, as a measure of a household financial fragility which occurs whenever expenses exceed the household yearly income for dependence case [15].

In recent years, much attention has been turned to the estimation of R based on various types of incomplete and ordered data, such as censored samples, order statistics, and record values. In the case when the samples are progressively Type-II censored, we point out the papers by Raqab and Madi [35] when stress and strength are two independent generalized Rayleigh distributions, Asgharzadeh et al. [3] and Valiollahi et al. [41] when X and Y are two independent Weibull random variables, Saraçoğlu et al. [37] when X and Y are independent exponential random variables, Basirat et al. [7] in the case when stress and strength follow proportional hazard rate families. The estimation problem of the stress-strength model based on record values has been addressed by various authors in the case of independence between stress and strength and with different specifications of the distribution functions of X and Y. For example, a different formulation of generalized exponential distribution has been studied by Jaheen [22], and Baklizi [5]. The Weibull specifications for X and Y have been used by Baklizi [6]. Nadar and Kızılaslan [29] used the Kumaraswamy distribution. The case when stress and strength are distributed according to a proportional reversed hazard family (PRHF) has been proposed by Condino et al. [11].

For the purposes of this paper, it is important to highlight that the above-mentioned papers using the ordered data apply in the estimation phase of $R = P(Y > X)$. A different situation is a case dealt with in the paper by Pakdaman and Ahmadi [34] in which the main aim is to specify the stress-strength model based on the rth order stress component, $X_{r:n_1}$, and the kth order strength component, $Y_{k:n_2}$, i.e. $P(X_{r:n_1} < Y_{k:n_2})$, when stress and strength are independent exponential random variables.

In many contexts, being concerned with either order statistic or record value, we observe the value of another variable, indicated in the literature by the term of concomitant. The general theory of concomitants of order statistics was originally studied by Yang [43]. An exhaustive review on concomitants of order statistics is given in [13]. Let (X_i, Y_i), $i = 1, \ldots, n$, be n pairs of independent random variables from a bivariate cumulative distribution function (cdf) $F(x, y)$. Let $X_{r:n}$ be the rth order statistic, then the Y value associated with $X_{r:n}$ is the concomitant of the rth order statistic and is denoted by $Y_{[r:n]}$. The works [4] and [44] have presented only some recent contributions to the concomitant of order statistics. *The most important use of concomitants arises in selection procedures when k individuals are chosen on the basis of their X-values. Then the corresponding Y-values represent performance on an associated characteristic.* For a recent utilization in an economic context see the application in [17]. Less attention has been paid in the literature to studies related to concomitants of record value. We recall the works by Chacko and Thomas [10] and Bose and Gangopadhyay [9]. We refer to the paper [33] for a possible application in biosciences. It is of great importance to emphasize that the values of the concomitant variable, $Y_{[r:n]}$, are not necessarily in ascending order as X; furthermore, a priori it is not possible to establish whether the concomitant value is greater or less than the value of the order statistic or the record value.

Recently, some authors have placed their attention on concomitants of generalized order statistics (GOSs) proposed by Kamps [25] as a unified approach including several models of ordered random variables as special cases, e.g. order statistics, the kth record values and Pfeifer's record model, progressively type-II censored order statistics, among others. Beg and Ahsanullah [8] have examined concomitants of GOSs from Farlie-Gumbel-Morgenstern (FGM) distribution, Tahmasebi et al. [39] have studied the concomitants of Dual-GOSs from Morgenstern type bivariate generalized exponential distribution, Domma and Giordano [17] have considered the concomitants of m-GOSs from Generalized Farlie-Gumbel-Morgenstern (GFGM) distribution. It should be noted that since between ordered data and the corresponding concomitant there exists a dependence structure, it is necessary to use a dependence structure such as the copula function.

The main objective of this work is to study the probability that the kth order statistic (or record statistic) is less than the concomitant of the kth order statistic (or record statistic). To the best of our knowledge, this work is the first paper that tackles this issue which can be applied in different fields of science. For example, in an athletic competition, we can observe the athlete's time, say A, in correspondence with the personal record of the athlete B. Evidently, the time of athlete A is the concomitant value observed in correspondence of the record time of B. But we do not know whether the personal record of B is enough to win the competition. In other words, the concomitant time of athlete A may be shorter than the time records of athlete B. Following this scheme, many other real-life problems can be described.

In this paper, we introduce the above problem from a general viewpoint of studying the stress-strength model in the case of the concomitant based on the GOSs. In addition, we use the PRHF as marginal distributions and the copula function to model the dependence structure. Relevant special cases of concomitant from order statistics and record values are studied in depth by specifying a particular copula function and selecting a particular member of the PRHF as marginal distributions.

The paper is organized as follows. Section 2 is devoted to illustrating the concomitants of m-GOSs and we are presenting the general formulation of the stress-strength model. To account for dependence we use the copula function and the expansion proposed by Nadarajah [31], in Section 3. Moreover, the probability that the nth m-GOSs is smaller than its concomitant, in the special cases of m-GOSs, namely the order statistics and the upper record values, are reported in Section 3. In Section 4, we study the relevant case of the stress-strength model with GFGM bivariate distribution. The inference problem is studied in Section 5 and the performance of the estimators of stress-strength is evaluated with the simulation studies in Section 6.

2. Concomitant of m-GOSs and stress-strength model

In this section, we introduce the m-GOSs and determine the probability density function (pdf) of the concomitant of m-GOSs. In the final part of the section, we provide a general formulation of the stress-strength model in the case of the concomitant of m-GOSs.

Suppose that F is an absolutely continuous cdf with survival function $\overline{F} = 1 - F$, and pdf f. Assume that $n \in \mathbb{N}$, $\widetilde{m} = (m_1, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$, and $k > 0$ are parameters such that $\gamma_i = k+n-i+M_i > 0$ for $i = 1, 2, \ldots, n-1$, where $M_i = \sum_{j=i}^{n-1} m_i$. We denote the random variables of GOSs by $X_{(r,n,\widetilde{m},k)}$, $r = 1, \ldots, n$, and their *joint* *pdf* is given by

$$f_{X_{(1,n,\widetilde{m},k)},\ldots,X_{(n,n,\widetilde{m},k)}}(x_1,\ldots,x_n) = k\prod_{i=1}^{n}\gamma_i\bigg(\prod_{j=1}^{n-1}\overline{F}^{m_j}(x_j)f(x_j)\bigg)\overline{F}^{k-1}(x_n)f(x_n) \tag{2.1}$$

on the cone $F^{-1}(0) < x_1 \leqslant \ldots \leqslant x_n < F^{-1}(1)$, where $F^{-1}(\cdot)$ is the inverse function of $u = F(x)$, i.e. a quantile function. In the particular case, when $m_1 = m_2 = \ldots = m_{n-1} = m$, and $\gamma_i = k + (n-i)(m+1)$ for $i = 1,\ldots,n-1$, the random variable $X_{(r,n,\widetilde{m},k)}$, is called m-GOSs and is denoted by $X_{(r,n,m,k)}, r = 1,\ldots,n$. Using (2.1), the marginal pdf of the rth m-GOSs is

$$f_{(r,n,m,k)}(x) = \frac{1}{(r-1)!}\bigg(\prod_{i=1}^{r}\gamma_i\bigg)\overline{F}^{\gamma_r-1}(x)f(x)t_m^{r-1}(F(x)), \tag{2.2}$$

where

$$t_m(F(x)) = \begin{cases} \dfrac{1}{m+1}[1-\overline{F}^{m+1}(x)], & m \neq -1, \\ -\log \overline{F}(x), & m = -1,\ F(x) \in [0,1). \end{cases}$$

Special cases of m-GOSs can be obtained by appropriate choices of the parameters m and k. For example, it is easy to verify that for $m = 0$ and $k = 1$, the m-GOSs becomes the rth order statistic, whereas for $m = 1$ and $k = 1$, $X_{(r,n,m,k)}$ denotes the rth upper record.

Consider observations (X_i, Y_i), $i = 1,2,\ldots,n$, drawn from a bivariate joint pdf $h_{X,Y}(x,y)$ with marginal pdfs $f(x)$ and $g(y)$. We can order these pairs based on one of the random variables X or Y. In this context, if we order the X_i's involved in (X_i, Y_i), $i = 1,2,\ldots,n$, the realization of Y associated with the rth m-GOS of X is defined as the concomitant of $X_{(r,n,m,k)}$, and is denoted by $Y_{[r,n,m,k]}$. By [8] and using (2.2), the pdf of $Y_{[r,n,m,k]}$, $r = 1,\ldots,n$, is obtained by

$$g_{[r,n,m,k]}(y) = \int_{-\infty}^{\infty} g(y|x)f_{(r,n,m,k)}(x)\,\mathrm{d}x,$$

where $g(y|x)$ is the conditional pdf of Y given X.

In order to obtain the stress-strength based on the nth m-GOSs and its concomitant, it is necessary to derive the joint pdf between $X_{(n,n,m,k)}$ and $Y_{[n,n,m,k]}$. By expressing $g_{[r,n,m,k]}(y)$ using (2.2), it is possible to deduce that the joint pdf is given by

$$h_{X_{(n,n,m,k)},Y_{[n,n,m,k]}}(x,y) = h_{X,Y}(x,y)\frac{\prod_{i=1}^{n}\gamma_i}{(n-1)!}\overline{F}^{\gamma_n-1}(x)t_m^{n-1}(F(x)). \tag{2.3}$$

Now, using (2.3) we can determine the probability that the nth m-GOS, $X_{(n,n,m,k)}$, is smaller than its concomitant, $Y_{[n,n,m,k]}$, i.e.

$$
\begin{aligned}
(2.4) \qquad R &:= P(Y_{[n,n,m,k]} > X_{(n,n,m,k)}) \\
&= \iint_{y>x} g(y|x) f(x) \frac{\prod_{i=1}^{n} \gamma_i}{(n-1)!} \overline{F}^{\gamma_n - 1}(x) t_m^{n-1}(F(x))\, \mathrm{d}y\, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{\prod_{i=1}^{n} \gamma_i}{(n-1)!} \overline{F}^{\gamma_n - 1}(x) t_m^{n-1}(F(x)) f(x) \int_{x}^{\infty} g(y|x)\, \mathrm{d}y\, \mathrm{d}x \\
&= E_F \left[\frac{\prod_{i=1}^{n} \gamma_i}{(n-1)!} \overline{F}^{\gamma_n - 1}(X) t_m^{n-1}(F(X)) \{1 - G(X|X)\} \right],
\end{aligned}
$$

where $G(\cdot|\cdot)$ is the conditional cdf of Y given X, and $E_F(\cdot)$ is the expectation with respect to $F(x)$. It is clear that to use the general formulation shown in equation (2.4), it is necessary to specify the joint pdf between X and Y, the function $t(\cdot)$, i.e. the special cases of m-GOS and the marginal distributions $G(\cdot)$ and $F(\cdot)$. To this end, in the next sections, we use the copula function that allows us to specify the dependence structure and the marginal distributions separately. We investigate the stress-strength defined in (2.4) for the special cases of order statistics and record values based on the PRHF as marginal distributions.

3. A COPULA-BASED APPROACH TO ACCOUNT FOR DEPENDENCE

In the literature, it is well-known that a copula function is a joint distribution with uniform marginal distributions, the books [23] and [32] are exhaustive references for more details. The popularity of the copula as a tool to model the dependence stems from the fact that in a joint distribution the dependence structure, defined by copula function, and marginal distributions can be specified separately. Moreover, it is important to highlight that the marginal distributions not necessarily belong to the same family of distribution.

Let X and Y be two continuous random variables with joint distribution function $H_{X,Y}(x,y)$ and marginal distribution functions $F(x)$ and $G(y)$, respectively. Sklar's theorem states that any bivariate distribution function can be written as $H_{X,Y}(x,y) = C(F(x), G(y))$, where $C(\cdot,\cdot)$ is a unique copula function. It is easy to verify that the joint pdf is given by $h_{X,Y}(x,y) = c(F(x), G(y)) f(x) g(y)$, where

$$c(F(x), G(y)) = \frac{\partial^2 C(F(x), G(y))}{\partial F(x) \partial G(y)}$$

is the copula density.

Using the copula function, we can write the stress-strength of m-GOSs and its concomitant as

$$R = \iint_{y>x} c(F(x), G(y)) g(y) f(x) \frac{\prod_{i=1}^{n} \gamma_i}{(n-1)!} \overline{F}^{\gamma_n - 1}(x) t_m^{n-1}(F(x)) \, \mathrm{d}y \, \mathrm{d}x. \tag{3.1}$$

The equation shows that using the copula function, we can consider various cases for X and Y. For example, X and Y are either independent, $C(F(x), G(y)) = F(x)G(y)$, or dependent random variables. Also, we can assume that they have either the same distribution, $X \overset{d}{=} Y$, or different distributions.

In order to model the dependence between X and Y, we use a general form of the bivariate copula introduced by Nadarajah [31] which covers many known copula functions in the literature. The Nadarajah's expansion for the bivariate copula is

$$C(F(x), G(y)) = \sum_{j=1}^{d} \alpha_j F(x)^{a_j} G(y)^{b_j},$$

with the bivariate copula density

$$c(F(x), G(y)) = \sum_{j=1}^{d} \alpha_j a_j b_j F(x)^{a_j - 1} G(y)^{b_j - 1}, \tag{3.2}$$

where $d \geqslant 1$ is an integer and $\{(\alpha_j, a_j, b_j) \colon j \geqslant 1\}$ are some real numbers.

In this paper, we use the Nadarajah's expansion in order to model the dependence between m-GOSs and the concomitant. So, by substituting (3.2) in (3.1) one obtains

$$\begin{aligned} R = \sum_{j=1}^{d} \alpha_j a_j b_j \int_{-\infty}^{\infty} \frac{\prod_{i=1}^{n} \gamma_i}{(n-1)!} \overline{F}^{\gamma_n - 1}(x) t_m^{n-1}(F(x)) F^{a_j - 1}(x) f(x) \\ \times \left\{ \int_{x}^{\infty} G^{b_j - 1}(y) g(y) \, \mathrm{d}y \right\} \mathrm{d}x \\ = \sum_{j=1}^{d} \alpha_j a_j \frac{\prod_{i=1}^{n} \gamma_i}{(n-1)!} E_F[\overline{F}^{\gamma_n - 1}(X) t_m^{n-1}(F(X)) F^{a_j - 1}(X) \{1 - G^{b_j}(X)\}]. \end{aligned} \tag{3.3}$$

In the next subsection, we consider two special cases of m-GOSs and attain the stress-strength in these cases.

3.1. Specification of the function $t(\cdot)$: the special cases of m-GOSs. Two special cases of m-GOSs are considered and the stress-strength based on them are obtained in this section.

The best-known cases of m-GOSs are the order statistics and the upper record values. The stress-strength of order statistics and upper record values using Nadarajah's expansion copula is as follows.

(1) *Order statistics.*

In order to obtain the stress-strength in the case of ordered statistics, it is necessary to highlight that if we put $m = 0$ and $k = 1$ in m-GOSs, then $\gamma_i = n - i + 1$ and it can be shown that

$$\frac{\prod_{i=1}^{n} \gamma_i}{(n-1)!} \overline{F}^{\gamma_n - 1}(x) t_m^{n-1}(F(x)) = nF^{n-1}(x). \tag{3.4}$$

Therefore, substituting equation (3.4) in (3.3), after simple algebra, we derive

$$\begin{aligned} R_1 &:= P(Y_{[n,n,0,1]} > X_{(n,n,0,1)}) \\ &= n \sum_{j=1}^{d} \alpha_j a_j \int_{-\infty}^{\infty} [(F(x))^{n+a_j-2} f(x) - (F(x))^{n+a_j-2} f(x) (G(x))^{b_j}] \,\mathrm{d}x \\ &= n \sum_{j=1}^{d} \alpha_j a_j \left[\frac{1}{n + a_j - 1} - \int_0^1 u^{n+a_j-2} (G(F^{-1}(u)))^{b_j} \,\mathrm{d}u \right]. \end{aligned} \tag{3.5}$$

Unfortunately, there is no general solution to solve the last integral in (3.5) equation. However, it may be solved for some distributions, as we show in the next section.

(2) *Upper record.*

If we set $m = -1$ and $k = 1$ in m-GOSs, then

$$\frac{\prod_{i=1}^{n} \gamma_i}{(n-1)!} \overline{F}^{\gamma_n - 1}(x) t_m^{n-1}(F(x)) = \frac{(-\ln \overline{F}(x))^{n-1}}{(n-1)!}. \tag{3.6}$$

In order to calculate the stress-strength model, we need the following remarks.

Remark 3.1 ([19], p. 53). For $-1 \leqslant z < 1$, we have $-\ln(1-z) = \sum_{k=1}^{\infty} z^k/k$.

R e m a r k 3.2 ([19], p. 17). Power series raised to powers:

$$\left(\sum_{k=0}^{\infty} a_k z^k\right)^n = \sum_{k=0}^{\infty} c_k z^k,$$

where

$$c_0 = a_0^n, c_m = \frac{1}{ma_0} \sum_{k=1}^{m} (kn - m + k) a_k c_{m-k}$$

for $m \geqslant 1$ and n is a natural number.

Using equation (3.6), Remarks 3.1 and 3.2 in equation (3.3), we have

$$\begin{aligned} R_2 &:= P(Y_{[n,n,-1,1]} > X_{(n,n,-1,1)}) \\ &= \frac{n}{(n-1)!} \sum_{j=1}^{d} \alpha_j a_j \Big[\int_{-\infty}^{\infty} (F(x))^{a_j-1} (-\ln \overline{F}(x))^{n-1} f(x)\,\mathrm{d}x \\ &\quad - \int_{-\infty}^{\infty} (F(x))^{a_j-1} (-\ln \overline{F}(x))^{n-1} f(x) (G(x))^{b_j}\,\mathrm{d}x\Big]. \end{aligned}$$

Now, let

$$I_1 = \int_{-\infty}^{\infty} (F(x))^{a_j-1} (-\ln \overline{F}(x))^{n-1} f(x)\,\mathrm{d}x$$

and

$$I_2 = \int_{-\infty}^{\infty} (F(x))^{a_j-1} (-\ln \overline{F}(x))^{n-1} f(x) (G(x))^{b_j}\,\mathrm{d}x.$$

By choosing $u = -\ln \overline{F}(x)$, it is easy to verify that

$$\begin{aligned} I_1 &= \int_{0}^{\infty} (1 - \mathrm{e}^{-u})^{a_j-1} u^{n-1} \mathrm{e}^{-u}\,\mathrm{d}u \\ &= \sum_{k=0}^{a_j-1} \binom{a_j-1}{k} (-1)^k \int_0^{\infty} u^{n-1} \mathrm{e}^{-u(k+1)}\,\mathrm{d}u \\ &= \sum_{k=0}^{a_j-1} \binom{a_j-1}{k} \frac{(-1)^k \Gamma(n)}{(k+1)^n}, \end{aligned}$$

and by Remarks 3.1 and 3.2 it can be shown that

$$\begin{aligned} I_2 &= \int_{-\infty}^{\infty} (F(x))^{a_j-1} \left(\sum_{k=1}^{\infty} \frac{(F(x))^k}{k}\right)^{n-1} (G(x))^{b_j} f(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} (F(x))^{a_j-1} \left(\sum_{i=0}^{\infty} \frac{(F(x))^{i+1}}{i+1}\right)^{n-1} (G(x))^{b_j} f(x)\,\mathrm{d}x \\ &= \sum_{i=0}^{\infty} c_i \int_{-\infty}^{\infty} (F(x))^{a_j+n+i-2} (G(x))^{b_j} f(x)\,\mathrm{d}x, \end{aligned}$$

where $c_0 = 1$, and

$$c_m = \frac{1}{m} \sum_{k=1}^{m} \frac{(kn - m)}{k+1} c_{m-k}$$

for $m \geqslant 1$.

Ultimately, using the integrals I_1 and I_2, we determine the expression of R_2,

$$R_2 = \frac{1}{(n-1)!} \sum_{j=1}^{d} \alpha_j a_j \Bigg[\sum_{k=0}^{a_j - 1} \binom{a_j - 1}{k} (-1)^k \frac{\Gamma(n)}{(k+1)^n} - \sum_{i=0}^{\infty} c_i \int_{-\infty}^{\infty} (F(x))^{a_j + n + i - 2} (G(x))^{b_j} f(x) \, \mathrm{d}x \Bigg]. \tag{3.7}$$

To solve the last integral we need to specify the form of the cdfs $F(\cdot)$ and $G(\cdot)$.

3.2. The PRHF as marginal distributions for Nadarajah's expansion copula. In this section, we use the PRHF as marginal distributions for the copula function and then we obtain the expressions of the stress-strength model in the special cases of order statistics (R_1) and upper record values (R_2).

In a general setting, the cdf of a PRHF is defined as $F(z) = [F_1(z)]^{\alpha}$, with $-\infty \leqslant a \leqslant z \leqslant b \leqslant \infty$, and $\alpha > 0$, where $F_1(\cdot)$ is an arbitrary continuous cdf and $F_1(a) = 0$, $F_1(b) = 1$ (see, for example, [27], p. 234). The cdf $F_1(\cdot)$ is called the *baseline distribution* and α is a *resilience parameter*. The PRHF is a very flexible family of distributions and it includes several well-known distributions as special cases, such as the generalized exponential, the Burr X, the Topp-Leone, the Dagum, and Type I generalized logistic, just to name a few.

Let $F_1(\cdot)$ be the cdf of the baseline distribution with the pdf $f_1(\cdot)$. We assume that

$$X \sim F(x; \alpha_x) = [F_1(x)]^{\alpha_x} \quad \text{and} \quad Y \sim G(y; \alpha_y) = [F_1(y)]^{\alpha_y}, \tag{3.8}$$

with the corresponding pdfs

$$f(x; \alpha_x) = \alpha_x f_1(x) [F_1(x)]^{\alpha_x - 1} \quad \text{and} \quad g(y; \alpha_y) = \alpha_y f_1(y) [F_1(y)]^{\alpha_y - 1}. \tag{3.9}$$

Now, we obtain the stress-strength based on the m-GOSs and its concomitant in the special cases of order statistics and upper record values using the PRHF. So, for order statistics by using (3.5), (3.8), and (3.9) we have

$$\begin{aligned} R_1 &= n \sum_{j=1}^{d} \alpha_j a_j \Bigg[\frac{1}{n + a_j - 1} - \alpha_x \int_{-\infty}^{\infty} [F_1(x)]^{\alpha_x (n + a_j - 1) + \alpha_y b_j - 1} f_1(x) \, \mathrm{d}x \Bigg] \\ &= n \sum_{j=1}^{d} \alpha_j a_j \Bigg[\frac{1}{n + a_j - 1} - \frac{\alpha_x}{\alpha_x (n + a_j - 1) + \alpha_y b_j} \Bigg], \end{aligned}$$

and according to (3.7), (3.8), and (3.9) for upper record values it can be shown that

$$R_2 = \frac{1}{(n-1)!}\sum_{j=1}^{d}\alpha_j a_j\Bigg[\sum_{k=0}^{a_j-1}\binom{a_j-1}{k}(-1)^k\frac{\Gamma(n)}{(k+1)^n} - \sum_{i=0}^{\infty}c_i\frac{\alpha_x}{\alpha_x(n+a_j+i-1)+\alpha_y b_j}\Bigg].$$

It is important to notice that the stress-strength model for order statistics and upper record based on the PRHF does not depend on the common baseline distribution for the cdfs of the random stress and of the random strength.

4. Stress-strength model for GFGM distribution family

In this section, we consider the GFGM bivariate distribution function to derive the stress-strength model. In particular, we use the specification proposed by Bairamov et al. [4] of the GFGM distribution, which is an extension of the original FGM distribution obtained by introducing additional parameters in the FGM distribution to increase the range of dependence measures.

The GFGM distribution family introduced by Farlie [18] is the most general form of the FGM family, defined by

$$H(x,y) = F(x)G(y)\{1+\theta A(F(x))B(G(y))\}, \tag{4.1}$$

where $A(\cdot)$ and $B(\cdot)$ are differentiable functions on the unit interval and $A(t)\to 0$ and $B(t)\to 0$ as $t\to 1$, and θ is the dependence parameter. In a special case, when $\theta=0$, then X and Y are independent. The bivariate pdf of the GFGM is defined by

$$h(x,y) = \{1+\theta[A(F(x))+F(x)A'(F(x))][B(G(y))+G(y)B'(G(y))]\}F(x)G(y),$$

where

$$A'(F(x)) = \frac{\partial A(F(x))}{\partial F(x)} \quad \text{and} \quad B'(G(y)) = \frac{\partial B(G(y))}{\partial G(y)}.$$

Evidently, the FGM distribution is a special case of the GFGM distribution with dependence parameter θ, when $A(F(x)) = 1-F(x)$ and $B(G(y)) = 1-G(y)$, and the cdf of FGM is

$$H(x,y) = F(x)G(y)\{1+\theta(1-F(x))(1-G(y))\},$$

with the corresponding bivariate pdf

$$h(x,y) = f(x)g(y)\{1+\theta(2F(x)-1)(2G(y)-1)\}. \tag{4.2}$$

If we consider $A(F(x)) = [1 - F^{m_1}(x)]^{p_1}$ and $B(G(y)) = [1 - G^{m_2}(y)]^{p_2}$ in equation (4.1), then we obtain the following generalization of the FGM distribution as

$$H(x,y) = F(x)G(y)\{1 + \theta[1 - F^{m_1}(x)]^{p_1}[1 - G^{m_2}(y)]^{p_2}\}, \tag{4.3}$$

with $m_1, m_2, p_1, p_2 > 0$ (see [4]). It should be noted that θ has an admissible range $\theta_l \leqslant \theta \leqslant \theta_u$ where

$$\begin{aligned}\theta_l &= -\min\left\{1, \frac{1}{m_1 m_2}\left[\frac{1+m_1p_1}{m_1(p_1-1)}\right]^{p_1-1}\left[\frac{1+m_2p_2}{m_2(p_2-1)}\right]^{p_2-1}\right\}, \\ \theta_u &= \min\left\{\frac{1}{m_1}\left[\frac{1+m_1p_1}{m_1(p_1-1)}\right]^{p_1-1}, \frac{1}{m_2}\left[\frac{1+m_2p_2}{m_2(p_2-1)}\right]^{p_2-1}\right\}.\end{aligned} \tag{4.4}$$

It is easy to show that the corresponding pdf of (4.3) is given by

$$\begin{aligned} h(x,y) = f(x)g(y)\Bigg\{1 + \theta \sum_{i=0}^{p_1-1}\sum_{j=0}^{p_2-1} \xi_{p_1-1,i}\xi_{p_2-1,j}F^{m_1 i}(x)G^{m_2 j}(y) \\ \times [1-(1+m_1p_1)F^{m_1}(x)][1-(1+m_2p_2)G^{m_2}(y)]\Bigg\},\end{aligned} \tag{4.5}$$

where $\xi_{s,t}$ is $\binom{s}{t}(-1)^t$.

Now, using equation (3.1) and equation (4.5), it can be shown that the stress-strength model of m-GOS and its concomitant in this case is

$$\begin{aligned} R^{(\mathrm{GFGM})} &= P(Y_{[n,n,m,k]} > X_{(n,n,m,k)}) \\ &= \frac{\prod_{i=1}^{n}\gamma_i}{(n-1)!} E_F[\overline{F}^{\gamma_n-1}(X)t_m^{n-1}(F(X))\{1-G(X)\}] \\ &\quad + \theta \sum_{r=0}^{p_1-1}\sum_{t=0}^{p_2-1} \xi_{p_1-1,r}\xi_{p_2-1,t}\frac{\prod_{i=1}^{n}\gamma_i}{(n-1)!} \\ &\quad \times E_F\big[\overline{F}^{\gamma_n-1}(X)t_m^{n-1}(F(X))F^{m_1 r}(X)\{1-\eta_1F^{m_1}(X)\} \\ &\quad \times \{\eta_{2,t}(1-G^{m_2t+1}(X)) - \eta_{3,t}(1-G^{m_2(t+1)+1}(X))\}\big],\end{aligned} \tag{4.6}$$

where

$$\eta_1 = 1 + m_1p_1, \quad \eta_{2,t} = \frac{1}{m_2t+1}, \quad \text{and} \quad \eta_{3,t} = \frac{1+m_2p_2}{m_2(t+1)+1}.$$

As was done in the previous section, to use the equation (4.6), we have to specify the function $t(\cdot)$ and the marginal distribution functions $F(\cdot)$ and $G(\cdot)$.

Below we consider the stress-strength model on the basis of the m-GOS and its concomitant, in the simple but relevant case of the FGM bivariate distribution, a particular case of the GFGM bivariate distribution. Let the density copula of FGM be as (4.2), then we have

$$
\begin{aligned}
R^{(\mathrm{FGM})} &= \frac{\prod_{i=1}^{n}\gamma_i}{(n-1)!} E_F[\overline{F}^{\gamma_n-1}(X) t_m^{n-1}(F(X))\{1-G(X)\}] \\
&\quad + \theta \frac{\prod_{i=1}^{n}\gamma_i}{(n-1)!} E_F[\overline{F}^{\gamma_n-1}(X) t_m^{n-1}(F(X))(2F(X)-1)(G(X)-G^2(X))].
\end{aligned}
$$

4.1. Specification of $t(\cdot)$ function: the special cases of m-GOS. In what follows, the stress-strength model of order statistics and upper record values, special cases of the m-GOS is presented using the GFGM bivariate distribution.

(1) *Order statistics.*

In the case of order statistics, we can use the result obtained in the previous section; in particular, by substituting equation (3.4) in equation (4.6), we derive

$$
\begin{aligned}
\text{(4.7)}\quad R_1^{(\mathrm{GFGM})} &= nE_F[F^{n-1}(X)(1-G(X))] \\
&\quad + n\theta \sum_{r=0}^{p_1-1}\sum_{t=0}^{p_2-1} \xi_{p_1-1,r}\xi_{p_2-1,t} E_F[F^{n+m_1 r-1}(X)\{1-\eta_1 F^{m_1}(X)\} \\
&\quad \times \{\eta_{2,t}(1-G^{m_2 t+1}(X)) - \eta_{3,t}(1-G^{m_2(t+1)+1}(X))\}].
\end{aligned}
$$

(2) *Upper record.*

According to (3.6), it can be written

$$
\begin{aligned}
R_2^{(\mathrm{GFGM})} &= \int_0^\infty f(x)\frac{[-\ln(1-F(x))]^{n-1}}{(n-1)!}\int_x^\infty g(y)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad + \theta \sum_{r=0}^{p_1-1}\sum_{t=0}^{p_2-1} \xi_{p_1-1,r}\xi_{p_2-1,t}\int_0^\infty F(x)F^{m_1 r}(x)[1-(1+m_1p_1)F^{m_1}(x)] \\
&\quad \times \frac{[-\ln(1-F(x))]^{n-1}}{(n-1)!}\int_x^\infty g(y)G^{m_2 t}(y)[1-(1+m_2p_2)G^{m_2}(y)]\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}
$$

Now, using Remarks 3.1 and 3.2 and after some simple algebra, we get

$$
\begin{aligned}
(4.8)\qquad R_2^{(\mathrm{GFGM})} &= 1 - \frac{1}{(n-1)!}\sum_{i=0}^{\infty} c_i \int_0^{\infty} (F(x))^{n+i-1} G(x) f(x)\,\mathrm{d}x \\
&+ \theta \sum_{r=0}^{p_1-1}\sum_{t=0}^{p_2-1} \xi_{p_1-1,r}\xi_{p_2-1,t}\bigg\{\frac{\eta_{2,t}-\eta_{3,t}}{(n-1)!}\sum_{i=0}^{\infty}\frac{c_i}{m_1 r+n+i} \\
&+ \frac{\eta_1(\eta_{3,t}-\eta_{2,t})}{(n-1)!}\sum_{i=0}^{\infty}\frac{c_i}{m_1(r+1)+n+i} \\
&- \frac{1}{(n-1)!}\sum_{i=0}^{\infty} c_i \int_0^{\infty} \big[\eta_{2,t}(F(x))^{m_1 r+n+i-1}(G(x))^{m_2 t+1} \\
&- \eta_{3,t}(F(x))^{m_1 r+n+i-1}(G(x))^{m_2(t+1)+1} \\
&- \eta_1\eta_{2,t}(F(x))^{m_1(r+1)+n+i-1}(G(x))^{m_2 t+1} \\
&+ \eta_1\eta_{3,t}(F(x))^{m_1(r+1)+n+i-1}(G(x))^{m_2(t+1)+1}\big] f(x)\,\mathrm{d}x\bigg\}.
\end{aligned}
$$

In the next subsection, we consider the PRHF as marginal distributions in order to calculate the stress-strength model.

4.2. The PRHF as marginal distributions for GFGM copula. According to the results obtained in the previous sections, we assume the PRHF as marginal distributions for GFGM copula and obtain the stress-strength in special cases.

(1) *Order statistics.*

Using (3.8), (3.9), and (4.7), we can write

$$
\begin{aligned}
(4.9)\quad R_1^{(\mathrm{GFGM})} &= 1 - \frac{n\alpha_x}{n\alpha_x+\alpha_y} \\
&+ n\theta \sum_{r=0}^{p_1-1}\sum_{t=0}^{p_2-1} \xi_{p_1-1,r}\xi_{p_2-1,t}\bigg\{\frac{\eta_{2,t}}{n+m_1 r} - \frac{\eta_{2,t}\alpha_x}{\alpha_x(n+m_1 r)+\alpha_y(m_2 t+1)} \\
&- \frac{\eta_{3,t}}{n+m_1 r} + \frac{\eta_{3,t}\alpha_x}{\alpha_x(n+m_1 r)+\alpha_y(m_2(t+1)+1)} \\
&+ \frac{\eta_1\eta_{2,t}\alpha_x}{\alpha_x(n+m_1(r+1))+\alpha_y(m_2 t+1)} - \frac{\eta_1\eta_{2,t}}{n+m_1(r+1)} + \frac{\eta_1\eta_{3,t}}{n+m_1(r+1)} \\
&- \frac{\eta_1\eta_{3,t}\alpha_x}{\alpha_x(n+m_1(r+1))+\alpha_y(m_2(t+1)+1)}\bigg\}.
\end{aligned}
$$

(2) *Upper record.*

According to (3.8), (3.9), and (4.8) and after simple algebra we have

$$
\begin{aligned}
(4.10) \quad R_2^{(\mathrm{GFGM})} = {} & 1 - \frac{\alpha_x}{(n-1)!} \sum_{i=0}^{\infty} \frac{c_i}{\alpha_x(n+i)+\alpha_y} \\
& + \theta \sum_{r=0}^{p_1-1} \sum_{t=0}^{p_2-1} \xi_{p_1-1,r}\xi_{p_2-1,t} \Bigg\{ \frac{(\eta_{2,t}-\eta_{3,t})}{(n-1)!} \sum_{i=0}^{\infty} \frac{c_i}{m_1 r+n+i} \\
& + \frac{\eta_1(\eta_{3,t}-\eta_{2,t})}{(n-1)!} \sum_{i=0}^{\infty} \frac{c_i}{m_1(r+1)+n+i} \\
& - \frac{\alpha_x \eta_{2,t}}{(n-1)!} \sum_{i=0}^{\infty} \frac{c_i}{\alpha_x(m_1 r+n+i)+\alpha_y(m_2 t+1)} \\
& + \frac{\alpha_x \eta_{3,t}}{(n-1)!} \sum_{i=0}^{\infty} \frac{c_i}{\alpha_x(m_1 r+n+i)+\alpha_y(m_2(t+1)+1)} \\
& + \frac{\alpha_x \eta_1 \eta_{2,t}}{(n-1)!} \sum_{i=0}^{\infty} \frac{c_i}{\alpha_x(m_1(r+1)+n+i)+\alpha_y(m_2 t+1)} \\
& - \frac{\alpha_x \eta_1 \eta_{3,t}}{(n-1)!} \sum_{i=0}^{\infty} \frac{c_i}{\alpha_x(m_1(r+1)+n+i)+\alpha_y(m_2(t+1)+1)} \Bigg\}.
\end{aligned}
$$

It is observed that the stress-strength models obtained in equations (4.9) and (4.10) are free distributions with respect to baseline distributions.

5. Stress-strength estimation using inference functions for margins method

In this section, we propose a procedure to estimate the stress-strength model by considering the GFGM distribution as the bivariate distribution between random stress and random strength. In this case, it is well-known in the literature that the maximum likelihood estimation (MLE) of the dependence parameter θ does not exist. Therefore, we suggest using a procedure similar to the inference functions for the margins (IFM) method ([23] and [24]). First of all, we apply the maximum likelihood method to estimate the shape parameters, separately. It is easy to show that the MLEs of α_x and α_y based on a random sample with size n are given by

$$
(5.1) \qquad \widehat{\alpha}_x = -\frac{n}{\sum_{i=1}^{n} \ln F_1(X_i)}, \quad \widehat{\alpha}_y = -\frac{n}{\sum_{i=1}^{n} \ln F_1(Y_i)}.
$$

Subsequently, we use a nonparametric estimator of θ. Domma and Giordano [17] showed that an unbiased estimator of θ based on the GFGM copula is given by

$$\hat{\theta} = \frac{(2+m_1p_1)(2+m_2p_2)\widehat{\tau}}{8p_1p_2B(2/m_1,p_1)B(2/m_2,p_2)},$$

with $B(\cdot,\cdot)$ being the Beta function, where $\widehat{\tau}$ is the unbiased estimator of Kendall's τ

$$\widehat{\tau} = \binom{n}{2}^{-1} \sum_{1\leqslant i\leqslant j\leqslant n} \operatorname{sgn}(X_i - X_j)\operatorname{sgn}(Y_i - Y_j). \tag{5.2}$$

Now, replacing the estimators of $\alpha_x, \alpha_y, \theta$, in (4.9) and (4.10) we derive the estimators of stress-strength for order statistics and upper record. In so doing, the estimators of $R_1^{(\mathrm{GFGM})}$ and $R_2^{(\mathrm{GFGM})}$ are given by

$$\begin{aligned}\widehat{R}_1^{(\mathrm{GFGM})} = 1 &- \frac{n\widehat{\alpha}_x}{n\widehat{\alpha}_x+\widehat{\alpha}_y} + n\hat{\theta}\sum_{r=0}^{p_1-1}\sum_{t=0}^{p_2-1}\xi_{p_1-1,r}\xi_{p_2-1,t} \\ &\times\left\{\frac{\eta_{2,t}}{n+m_1r} - \frac{\eta_{2,t}\widehat{\alpha}_x}{\widehat{\alpha}_x(n+m_1r)+\widehat{\alpha}_y(m_2t+1)}\right. \\ &\quad - \frac{\eta_{3,t}}{n+m_1r} + \frac{\eta_{3,t}\widehat{\alpha}_x}{\widehat{\alpha}_x(n+m_1r)+\widehat{\alpha}_y(m_2(t+1)+1)} \\ &\quad + \frac{\eta_1\eta_{2,t}\widehat{\alpha}_x}{\widehat{\alpha}_x(n+m_1(r+1))+\widehat{\alpha}_y(m_2t+1)} \\ &\quad - \frac{\eta_1\eta_{2,t}}{n+m_1(r+1)} + \frac{\eta_1\eta_{3,t}}{n+m_1(r+1)} \\ &\quad \left. - \frac{\eta_1\eta_{3,t}\widehat{\alpha}_x}{\widehat{\alpha}_x(n+m_1(r+1))+\widehat{\alpha}_y(m_2(t+1)+1)}\right\},\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}\widehat{R}_2^{(\mathrm{GFGM})} &= 1 - \frac{\widehat{\alpha}_x}{(n-1)!}\sum_{i=0}^{\infty}\frac{c_i}{\widehat{\alpha}_x(n+i)+\widehat{\alpha}_y} \\ &+ \hat{\theta}\sum_{r=0}^{p_1-1}\sum_{t=0}^{p_2-1}\xi_{p_1-1,r}\xi_{p_2-1,t}\left\{\frac{\eta_{2,t}-\eta_{3,t}}{(n-1)!}\sum_{i=0}^{\infty}\frac{c_i}{m_1r+n+i}\right. \\ &+ \frac{\eta_1(\eta_{3,t}-\eta_{2,t})}{(n-1)!}\sum_{i=0}^{\infty}\frac{c_i}{m_1(r+1)+n+i}\end{aligned} \tag{5.4}$$

$$
\begin{aligned}
&- \frac{\widehat{\alpha}_x \eta_{2,t}}{(n-1)!} \sum_{i=0}^{\infty} \frac{c_i}{\widehat{\alpha}_x(m_1 r + n + i) + \widehat{\alpha}_y(m_2 t + 1)} \\
&+ \frac{\widehat{\alpha}_x \eta_{3,t}}{(n-1)!} \sum_{i=0}^{\infty} \frac{c_i}{\widehat{\alpha}_x(m_1 r + n + i) + \widehat{\alpha}_y(m_2(t+1) + 1)} \\
&+ \frac{\widehat{\alpha}_x \eta_1 \eta_{2,t}}{(n-1)!} \sum_{i=0}^{\infty} \frac{c_i}{\widehat{\alpha}_x(m_1(r+1) + n + i) + \widehat{\alpha}_y(m_2 t + 1)} \\
&- \frac{\widehat{\alpha}_x \eta_1 \eta_{3,t}}{(n-1)!} \sum_{i=0}^{\infty} \frac{c_i}{\widehat{\alpha}_x(m_1(r+1) + n + i) + \widehat{\alpha}_y(m_2(t+1) + 1)} \Bigg\}.
\end{aligned}
$$

Although $\widehat{R}_1^{(\mathrm{GFGM})}$ and $\widehat{R}_2^{(\mathrm{GFGM})}$ are non-linear functions with respect to (X_i, Y_i), $i = 1, \ldots, n$, some mathematical properties are investigated such as asymptotic unbiasedness and asymptotic distribution of the $\widehat{R}_1^{(\mathrm{GFGM})}$ in the following theorem.

Theorem 5.1. *The function $\widehat{R}_1^{(\mathrm{GFGM})}$ has an asymptotic normal distribution as*

$$
\frac{\sqrt{n}(\widehat{R}_1^{(\mathrm{GFGM})} - R_1^{(\mathrm{GFGM})})}{\sqrt{V^*}} \xrightarrow{d} N(0,1),
$$

where

$$
V^* = \left[\frac{\partial R_1^{(\mathrm{GFGM})}}{\partial \alpha_x}, \frac{\partial R_1^{(\mathrm{GFGM})}}{\partial \alpha_y}, \frac{\partial R_1^{(\mathrm{GFGM})}}{\partial \theta} \right] \begin{bmatrix} \alpha_x^2 & 0 & 0 \\ 0 & \alpha_y^2 & 0 \\ 0 & 0 & 4\overline{K}^2 \sigma_\tau^2 \end{bmatrix} \begin{bmatrix} \dfrac{\partial R_1^{(\mathrm{GFGM})}}{\partial \alpha_x} \\ \dfrac{\partial R_1^{(\mathrm{GFGM})}}{\partial \alpha_y} \\ \dfrac{\partial R_1^{(\mathrm{GFGM})}}{\partial \theta} \end{bmatrix},
$$

with

$$
\overline{K} = \frac{(2 + m_1 p_1)(2 + m_2 p_2)}{8 p_1 p_2 B(2/m_1, p_1) B(2/m_2, p_2)},
$$

and $\sigma_\tau^2 = \mathrm{Var}(\mathrm{E}[\mathrm{sgn}(X - \widetilde{X})\,\mathrm{sgn}(Y - \widetilde{Y}) | X, Y])$, where $(\widetilde{X}, \widetilde{Y})$ is an independent copy of (X, Y).

Proof. The proof of Theorem 5.1 is given in Appendix. □

In the next section, the efficiency of the proposed estimators is investigated using simulation and numerical computation.

6. Simulation study

The simulation study helps us to evaluate the performance of the introduced estimators of the stress-strength model for order statistics and record statistics. Toward this end, a simulation study is done in three stages as follows:

(1) To generate a data set from the GFGM(m_1, m_2, p_1, p_2) copula with dependence parameter θ, we first consider a fixed θ, say θ_0, based on the admissible range in (4.4) and marginal Dagum distribution for X and Y, i.e., $X \sim \mathrm{Da}(\alpha_x, \lambda, \delta)$ and $Y \sim \mathrm{Da}(\alpha_y, \lambda, \delta)$, where $\alpha_x, \alpha_y, \delta > 0$ are shape parameters and $\lambda > 0$ is a scale parameter. It is obvious that the Dagum distribution belongs to PRHF; in fact, the cdf of Dagum distribution can be written as $F(w; \alpha_w \lambda, \delta) = [F_1(w; \lambda, \delta)]^{\alpha_w}$, where $F_1(w; \lambda, \delta) = (1 + \lambda w^{-\delta})^{-1}$. Next, we generate a random pairs of size n from GFGM copula using the following algorithm (see [32], p. 41), for fixed value of the sample size n, the marginal parameters α_x, α_y, λ and δ and the copula parameters m_1, m_2, p_1, p_2 and θ:

- ▷ Generate two independent random samples of numbers with size n from the $U(0,1)$ distribution, i.e. u_i, t_i for $i = 1, 2, \ldots, n$, where u_i and t_i are independent observations from $U(0,1)$ distribution.
- ▷ Compute v_i using numerical methods, where v_i is the numerical value from the equation $C(v_i|u_i) = t_i$ and where $C(v_i|u_i)$ stands for the conditional copula of GFGM. By repeating this process B times, we have B data sets of size n from GFGM copula.
- ▷ Obtain the n simulated pairs of data, say (x_i, y_i) for $i = 1, 2, \ldots, n$, by the following quantiles of Dagum distributions:

$$x_i = \left(\frac{\lambda}{u_i^{-1/\alpha_x} - 1} \right)^{1/\delta}, \quad y_i = \left(\frac{\lambda}{v_i^{-1/\alpha_y} - 1} \right)^{1/\delta}.$$

(2) Now, using the simulated data from the previous stage, we compute the values of estimators $\hat{\theta}$, $\widehat{\alpha}_x$, $\widehat{\alpha}_y$, $\widehat{R}_1^{(\mathrm{GFGM})}$ and $\widehat{R}_2^{(\mathrm{GFGM})}$. Note that, to estimate θ, we should first obtain the value of Kendall's τ of (x_i, y_i) using (5.2); also, to obtain $\widehat{\alpha}_x$, $\widehat{\alpha}_y$, we should use Dagum distributions as baseline distributions in (5.1).

(3) We apply two criteria: the mean squared error (MSE) and the average of the relative estimates (AVRE) for evaluating the performance of $\hat{\theta}$, $\widehat{R}_1^{(\mathrm{GFGM})}$ and $\widehat{R}_2^{(\mathrm{GFGM})}$, in which they are defined on the basis of B iterations as

$$\mathrm{MSE}_n(\widehat{T}) = \frac{1}{B} \sum_{j=1}^{B} (\widehat{T}_j - T_0)^2, \quad \mathrm{AVRE}_n(\widehat{T}) = \frac{1}{B} \sum_{j=1}^{B} \frac{\widehat{T}_j}{T_0},$$

where $\widehat{T}_j$ and T_0 are the value of the estimator at the jth iteration and the fixed value of the parameter, respectively.

Based on the above stages, we performed the simulation for some combinations of (m_1, m_2, p_1, p_2) with $B = 10000$ iterations for the sample size, i.e. $n = 5, 6, 7, 8, 9, 10$. Since the range of θ is a function of (m_1, m_2, p_1, p_2), we consider two arbitrary different values of θ, one positive and one negative for each combination of (m_1, m_2, p_1, p_2). Also, we have used Dagum marginal distributions $\mathrm{Da}(1.5, 1, 3)$ and $\mathrm{Da}(0.5, 1, 3)$ to generate the data according to the aforementioned stages. It should be noted that the obtained results do not change remarkably in comparison with other choices of the Dagum parameters. The AVRE criterion indicates the behavior of each estimator with respect to the chosen fixed value of that parameter.

Table 1 contains the simulation results of MSE and AVRE for $\widehat{R}_1^{(\mathrm{GFGM})}$, $\widehat{R}_2^{(\mathrm{GFGM})}$ and $\hat{\theta}$. There exist 3 different combinations of (m_1, m_2, p_1, p_2) in Table 1 and it is easy to provide quite similar outcomes for other combinations.

On the basis of the obtained results in Table 1, the estimators $\widehat{R}_1^{(\mathrm{GFGM})}$ and $\widehat{R}_2^{(\mathrm{GFGM})}$ carry out quite well. Their estimated values are really near to the chosen fixed ones of them because the AVRE of two estimators are very close to 1 and this is desirable. As expected, the bias is decreasing when the sample size increases and the MSE tends decreasingly to zero as the sample size increases. This trend can be observed for two estimators $\widehat{R}_1^{(\mathrm{GFGM})}$ and $\widehat{R}_2^{(\mathrm{GFGM})}$ and also $\hat{\theta}$ in all cases such as different combinations of (m_1, m_2, p_1, p_2) and various values of θ_0. On the other hand, the values of AVREs and MSEs for estimators are not remarkably sensitive with respect to changes of (m_1, m_2, p_1, p_2), because by changing the combination (m_1, m_2, p_1, p_2), the values of AVREs and MSEs have not really changed.

By comparing the estimators $\widehat{R}_1^{(\mathrm{GFGM})}$ and $\widehat{R}_2^{(\mathrm{GFGM})}$, it can be found that both estimators decrease with respect to n, but the MSE of $\widehat{R}_2^{(\mathrm{GFGM})}$ is always less than that of $\widehat{R}_1^{(\mathrm{GFGM})}$ in each row of Table 1. Also, the AVRE of $\widehat{R}_2^{(\mathrm{GFGM})}$ is closer to 1 than the AVRE of $\widehat{R}_2^{(\mathrm{GFGM})}$. These evidences show that the estimator $\widehat{R}_2^{(\mathrm{GFGM})}$ has better performance than the estimator $\widehat{R}_1^{(\mathrm{GFGM})}$ to estimate the stress-strength parameter. In other words, based on the results of Table 1, record statistics and their concomitants may estimate the stress-strength parameter better than order statistics and their concomitants in view of both the AVRE and MSE criteria.

m_1	m_2	p_1	p_2	θ_0	n	AVRE($\widehat{R}_1^{(\mathrm{GFGM})}$)	MSE($\widehat{R}_1^{(\mathrm{GFGM})}$)	AVRE($\widehat{R}_2^{(\mathrm{GFGM})}$)	MSE($\widehat{R}_2^{(\mathrm{GFGM})}$)	AVRE($\hat{\theta}$)	MSE($\hat{\theta}$)
3	2	3	2	−.855	5	1.17581	0.02284	1.05045	0.00094	1.00474	2.50868
					6	1.15644	0.01421	1.01267	0.00017	1.01030	1.87495
					7	1.15710	0.00943	1.00374	3.59283e−05	0.99856	1.54660
					8	1.13096	0.00575	1.00115	7.52363e−06	0.99860	1.23249
					9	1.11392	0.00387	1.00038	1.42016e−06	1.00665	1.04358
					10	1.10856	0.00268	1.00014	2.85406e−07	0.99039	0.91220
				0.425	5	1.08260	0.02318	1.03635	0.00075	0.99100	2.73333
					6	1.08083	0.01478	1.01016	0.00013	0.99388	2.08866
					7	1.08276	0.00968	1.00285	2.57740e−05	1.01174	1.66481
					8	1.08038	0.00638	1.00084	4.90315e−06	1.03810	1.35129
					9	1.07746	0.00453	1.00028	9.33792e−07	1.00408	1.17564
					10	1.06709	0.00316	1.00009	1.76982e−07	0.98681	1.01576
	5	4	4	−0.217	5	1.13623	0.03132	1.04259	0.00107	0.96907	0.78532
					6	1.11247	0.01954	1.01082	0.00016	1.02122	0.59966
					7	1.09818	0.01300	1.00307	2.98309e−05	1.05114	0.47672
					8	1.10185	0.00844	1.00094	5.84757e−06	0.97144	0.39710
					9	1.11311	0.00634	1.00037	1.29379e−06	0.97798	0.33637
					10	1.08576	0.00403	1.00012	2.30822e−07	0.97487	0.29640
				0.415	5	1.08693	0.03429	1.04089	0.00102	0.99240	0.76323
					6	1.07540	0.02220	1.01048	0.00015	1.00555	0.57971
					7	1.11148	0.01293	1.00319	3.12329e−05	1.00234	0.46797
					8	1.07948	0.01051	1.00076	4.58857e−06	1.02411	0.37184
					9	1.07465	0.00734	1.00027	9.00991e−07	0.98244	0.32326
					10	1.06330	0.00504	1.00008	1.55180e−07	0.99077	0.27938
3	4	5	4	−0.634	5	1.15737	0.02130	1.05024	0.00081	0.99292	3.17357
					6	1.15911	0.01243	1.01401	0.00016	0.96648	2.35162
					7	1.12484	0.00705	1.00351	3.17829e−05	0.98135	1.84516
					8	1.12564	0.00471	1.00120	7.46519e−06	0.97523	1.51453
					9	1.10579	0.00305	1.00037	1.37963e−06	0.94758	1.29789
					10	1.08691	0.00204	1.00013	2.51885e−07	0.99947	1.09205
				0.473	5	1.08505	0.02251	1.03610	0.00057	0.99494	3.27339
					6	1.08922	0.01306	1.00979	0.00010	1.02428	2.47461
					7	1.07828	0.00807	1.00251	2.18929e−05	0.99739	2.00666
					8	1.07132	0.00503	1.00077	4.56301e−06	0.99401	1.64287
					9	1.07272	0.00335	1.00025	8.60785e−07	1.00408	1.42765
					10	1.06833	0.00231	1.00009	1.66416e−07	0.97998	1.20997

Table 1. The AVRE and MSE of $\widehat{R}_1^{(\mathrm{GFGM})}$, $\widehat{R}_2^{(\mathrm{GFGM})}$ and $\hat{\theta}$ for different combinations of (m_1, m_2, p_1, p_2).

Appendix

For the proof of Theorem 5.1, we first prove that $\widehat{R}_1^{(\mathrm{GFGM})}$ is an asymptotically unbiased estimator. For this purpose, let

$$\begin{aligned}\widehat{R}_1^{(\mathrm{GFGM})} &= 1 - RA_1 \\ &\quad + n \sum_{r=0}^{p_1-1} \sum_{t=0}^{p_2-1} \xi_{p_1-1,r} \xi_{p_2-1,t} \{A\hat{\theta} - \hat{\theta} RA_2 + \hat{\theta} RA_3 + \hat{\theta} RA_4 - \hat{\theta} RA_5\},\end{aligned}$$

where

$$A = \frac{\eta_{2,t}}{n + m_1 r} - \frac{\eta_{3,t}}{n + m_1 r} - \frac{\eta_1 \eta_{2,t}}{n + m_1(r+1)} + \frac{\eta_1 \eta_{3,t}}{n + m_1(r+1)},$$

and for $j = 1, \ldots, 5$,

$$RA_j = \frac{k_{j1} \widehat{\alpha}_x}{k_{j2} \widehat{\alpha}_x + k_{j3} \widehat{\alpha}_y},$$

with

$$\begin{aligned}&k_{11} = k_{12} = n \text{ and } k_{13} = 1, \\ &k_{21} = \eta_{2,t},\ k_{22} = n + m_1 r \text{ and } k_{23} = m_2 t + 1, \\ &k_{31} = \eta_{3,t},\ k_{32} = n + m_1 r \text{ and } k_{33} = m_2(t+1) + 1, \\ &k_{41} = \eta_1 \eta_{2,t},\ k_{42} = n + m_1(r+1) \text{ and } k_{43} = m_2 t + 1, \\ &k_{51} = \eta_1 \eta_{3,t},\ k_{52} = n + m_1(r+1) \text{ and } k_{53} = m_2(t+1) + 1.\end{aligned}$$

Remark 6.1. For construction, the estimators $\hat{\theta}$, $\widehat{\alpha}_x$, and $\widehat{\alpha}_y$ are independent of each other.

Now, we verify if the estimator $\widehat{R}_1^{(\mathrm{GFGM})}$ is unbiased,

$$\begin{aligned}\mathrm{E}[\widehat{R}_1^{(\mathrm{GFGM})}] &= 1 - \mathrm{E}[RA_1] \\ &\quad + n \sum_{r=0}^{p_1-1} \sum_{t=0}^{p_2-1} \xi_{p_1-1,r} \xi_{p_2-1,t} \{\mathrm{E}[A\hat{\theta}] - \mathrm{E}[\hat{\theta}]\mathrm{E}[RA_2] + \mathrm{E}[\hat{\theta}]\mathrm{E}[RA_3] \\ &\quad + \mathrm{E}[\hat{\theta}]\mathrm{E}[RA_4] - \mathrm{E}[\hat{\theta}]\mathrm{E}[RA_5]\}.\end{aligned}$$

The general problem is to calculate the expectation of RA_j. From known results in the literature, we know that for two random variables (see [28], p. 181)

$$\mathrm{E}\Big[\frac{X}{Y}\Big] \approx \frac{\mathrm{E}(X)}{\mathrm{E}(Y)} - \frac{\mathrm{Cov}(X,Y)}{\mathrm{E}^2(Y)} + \frac{\mathrm{E}(X)}{\mathrm{E}^3(Y)} \mathrm{Var}(Y), \tag{6.1}$$

and

$$\operatorname{Var}\Big[\frac{X}{Y}\Big] \approx \Big[\frac{\mathrm{E}(X)}{\mathrm{E}(Y)}\Big]^2\Big\{\frac{\operatorname{Var}(X)}{\mathrm{E}^2(X)} + \frac{\operatorname{Var}(Y)}{\mathrm{E}^2(Y)} - \frac{2\operatorname{Cov}(X,Y)}{\mathrm{E}(X)\mathrm{E}(Y)}\Big\}. \tag{6.2}$$

Using (6.1), we can write

$$\begin{aligned}
\mathrm{E}[RA_j] &= \mathrm{E}\Big[\frac{k_{j1}\widehat{\alpha}_x}{k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y}\Big] \\
&\approx \frac{\mathrm{E}[k_{j1}\widehat{\alpha}_x]}{\mathrm{E}[k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y]} - \frac{\operatorname{Cov}(k_{j1}\widehat{\alpha}_x, k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y)}{(\mathrm{E}[k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y])^2} \\
&\quad + \frac{\mathrm{E}[k_{j1}\widehat{\alpha}_x]\operatorname{Var}(k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y)}{(\mathrm{E}[k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y])^3}.
\end{aligned}$$

Remark 6.2. (i) We have

$$\begin{aligned}
&\operatorname{Cov}(k_{j1}\widehat{\alpha}_x, k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y) \\
&\quad = \mathrm{E}\{k_{j1}\widehat{\alpha}_x[k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y]\} - \mathrm{E}[k_{j1}\widehat{\alpha}_x]\mathrm{E}[k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y] \\
&\quad = \mathrm{E}[k_{j1}k_{j2}\widehat{\alpha}_x^2 + k_{j1}k_{j3}\widehat{\alpha}_x\widehat{\alpha}_y] - \mathrm{E}[k_{j1}\widehat{\alpha}_x]\{\mathrm{E}[k_{j2}\widehat{\alpha}_x] + \mathrm{E}[k_{j3}\widehat{\alpha}_y]\} \\
&\quad = k_{j1}k_{j2}\mathrm{E}[\widehat{\alpha}_x^2] + k_{j1}k_{j3}\mathrm{E}[\widehat{\alpha}_x\widehat{\alpha}_y] - k_{j1}k_{j2}\{\mathrm{E}[\widehat{\alpha}_x]\}^2 - k_{j1}k_{j3}\mathrm{E}[\widehat{\alpha}_x]\mathrm{E}[\widehat{\alpha}_y] \\
&\quad = k_{j1}k_{j2}\{\mathrm{E}[\widehat{\alpha}_x^2] - (\mathrm{E}[\widehat{\alpha}_x])^2\} + k_{j1}k_{j3}\{\mathrm{E}[\widehat{\alpha}_x\widehat{\alpha}_y] - \mathrm{E}[\widehat{\alpha}_x]\mathrm{E}[\widehat{\alpha}_y]\} \\
&\quad = k_{j1}k_{j2}\operatorname{Var}(\widehat{\alpha}_x) + k_{j1}k_{j3}\underbrace{\operatorname{Cov}(\widehat{\alpha}_x, \widehat{\alpha}_y)}_{0} = k_{j1}k_{j2}\operatorname{Var}(\widehat{\alpha}_x),
\end{aligned} \tag{6.3}$$

where the last equality follows from Remark 6.1.

(ii) We get $\mathrm{E}[k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y] = k_{j2}\mathrm{E}[\widehat{\alpha}_x] + k_{j3}\mathrm{E}[\widehat{\alpha}_y]$.

(iii) It is easy to see

$$\begin{aligned}
\operatorname{Var}(k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y) &= k_{j2}^2\operatorname{Var}(\widehat{\alpha}_x) + k_{j3}^2\operatorname{Var}(\widehat{\alpha}_y) + 2k_{j2}k_{j3}\underbrace{\operatorname{Cov}(\widehat{\alpha}_x, \widehat{\alpha}_y)}_{0} \\
&= k_{j2}^2\operatorname{Var}(\widehat{\alpha}_x) + k_{j3}^2\operatorname{Var}(\widehat{\alpha}_y).
\end{aligned}$$

By Remark 6.2, we derive

$$\begin{aligned}
\mathrm{E}[RA_j] \approx \frac{k_{j1}\mathrm{E}[\widehat{\alpha}_x]}{k_{j2}\mathrm{E}[\widehat{\alpha}_x] + k_{j3}\mathrm{E}[\widehat{\alpha}_y]} &- \frac{k_{j1}k_{j2}\operatorname{Var}(\widehat{\alpha}_x)}{(k_{j2}\mathrm{E}[\widehat{\alpha}_x] + k_{j3}\mathrm{E}[\widehat{\alpha}_y])^2} \\
&+ \frac{k_{j1}\mathrm{E}[\widehat{\alpha}_x][k_{j2}^2\operatorname{Var}(\widehat{\alpha}_x) + k_{j3}^2\operatorname{Var}(\widehat{\alpha}_y)]}{(k_{j2}\mathrm{E}[\widehat{\alpha}_x] + k_{j3}\mathrm{E}[\widehat{\alpha}_y])^3}.
\end{aligned}$$

Remark 6.3. By asymptotic properties of MLE, we can write $\sqrt{n}(\widehat{\alpha}_z - \alpha_z) \xrightarrow{d} N(0, (I(\alpha_z))^{-1})$ for $z = x, y$, where $I(\alpha_z)$ is the Fisher information, i.e. $\widehat{\alpha}_z$ is asymptotically unbiased for α_z, with variance equal to $\operatorname{Var}(\widehat{\alpha}_z) = (I(\alpha_z))^{-1}/n = \alpha_z^2/n$.

By Remark 6.3, the asymptotic $\mathrm{E}[RA_j]$ is given by

$$\mathrm{E}[RA_j] \approx \frac{k_{j1}\alpha_x}{k_{j2}\alpha_x + k_{j3}\alpha_y} - \frac{k_{j1}k_{j2}\alpha_x^2/n}{(k_{j2}\alpha_x + k_{j3}\alpha_y)^2} + \frac{k_{j1}\alpha_x[k_{j2}^2\alpha_x^2/n + k_{j3}^2\alpha_y^2/n]}{(k_{j2}\alpha_x + k_{j3}\alpha_y)^3}, \quad j = 1, \ldots, 5. \tag{6.4}$$

So:

▷ For $j = 1$, we have

$$\mathrm{E}[RA_1] \approx \frac{n\alpha_x}{n\alpha_x + \alpha_y} + B_1,$$

where

$$B_1 = -\frac{n\alpha_x^2}{(n\alpha_x + \alpha_y)^2} + \frac{\alpha_x[n^2\alpha_x^2 + \alpha_y^2]}{(n\alpha_x + \alpha_y)^3}.$$

▷ For $j = 2$, we get

$$\mathrm{E}[RA_2] \approx \frac{\eta_{2,t}\alpha_x}{(n + m_1 r)\alpha_x + (m_2 t + 1)\alpha_y} + B_{2,r,t},$$

where

$$B_{2,r,t} = -\frac{\eta_{2,t}(n + m_1 r)\alpha_x^2/n}{[(n + m_1 r)\alpha_x + (m_2 t + 1)\alpha_y]^2} + \frac{\eta_{2,t}\alpha_x[(n + m_1 r)^2\alpha_x^2/n + (m_2 t + 1)^2\alpha_y^2/n]}{[(n + m_1 r)\alpha_x + (m_2 t + 1)\alpha_y]^3}.$$

▷ For $j = 3$, we derive

$$\mathrm{E}[RA_3] \approx \frac{\eta_{3,t}\alpha_x}{(n + m_1 r)\alpha_x + (m_2(t+1) + 1)\alpha_y} + B_{3,r,t},$$

where

$$B_{3,r,t} = -\frac{\eta_{3,t}(n + m_1 r)\alpha_x^2/n}{[(n + m_1 r)\alpha_x + (m_2(t+1) + 1)\alpha_y]^2} + \frac{\eta_{3,t}\alpha_x[(n + m_1 r)^2\alpha_x^2/n + (m_2(t+1) + 1)^2\alpha_y^2/n]}{[(n + m_1 r)\alpha_x + (m_2(t+1) + 1)\alpha_y]^3}.$$

▷ For $j = 4$, we have

$$\mathrm{E}[RA_4] \approx \frac{\eta_1\eta_{2,t}\alpha_x}{(n + m_1(r+1))\alpha_x + (m_2 t + 1)\alpha_y} + B_{4,r,t},$$

where

$$B_{4,r,t} = -\frac{\eta_1\eta_{2,t}(n + m_1(r+1))\alpha_x^2/n}{[(n + m_1(r+1))\alpha_x + (m_2 t + 1)\alpha_y]^2} + \frac{\eta_1\eta_{2,t}\alpha_x[(n + m_1(r+1))^2\alpha_x^2/n + (m_2 t + 1)^2\alpha_y^2/n]}{[(n + m_1(r+1))\alpha_x + (m_2 t + 1)\alpha_y]^3}.$$

▷ For $j = 5$, we get

$$\mathrm{E}[RA_5] \approx \frac{\eta_1\eta_{3,t}\alpha_x}{(n+m_1(r+1))\alpha_x + (m_2(t+1)+1)\alpha_y} + B_{5,r,t},$$

where

$$\begin{aligned} B_{5,r,t} = & -\frac{\eta_1\eta_{3,t}(n+m_1(r+1))\alpha_x^2/n}{[(n+m_1(r+1))\alpha_x + (m_2(t+1)+1)\alpha_y]^2} \\ & + \frac{\eta_1\eta_{3,t}\alpha_x[(n+m_1(r+1))^2\alpha_x^2/n + (m_2(t+1)+1)^2\alpha_y^2/n]}{[(n+m_1(r+1))\alpha_x + (m_2(t+1)+1)\alpha_y]^3}. \end{aligned}$$

Remark 6.4. Note that $\hat{\theta}$ is the unbiased estimator of θ (see [17]).

Therefore, using Remark 6.4 we can write

$$\begin{aligned} \mathrm{E}[\widehat{R}_1^{(\mathrm{GFGM})}] &\approx 1 - \frac{n\alpha_x}{n\alpha_x + \alpha_y} \\ &+ n\sum_{r=0}^{p_1-1}\sum_{t=0}^{p_2-1} \xi_{p_1-1,r}\xi_{p_2-1,t}\Big\{A\theta - \theta\Big[\frac{\eta_{2,t}\alpha_x}{(n+m_1r)\alpha_x + (m_2t+1)\alpha_y} + B_{2,r,t}\Big] \\ &+ \theta\Big[\frac{\eta_{3,t}\alpha_x}{(n+m_1r)\alpha_x + (m_2(t+1)+1)\alpha_y} + B_{3,r,t}\Big] \\ &+ \theta\Big[\frac{\eta_1\eta_{2,t}\alpha_x}{(n+m_1(r+1))\alpha_x + (m_2t+1)\alpha_y} + B_{4,r,t}\Big] \\ &- \theta\Big[\frac{\eta_1\eta_{3,t}\alpha_x}{(n+m_1(r+1))\alpha_x + (m_2(t+1)+1)\alpha_y} + B_{5,r,t}\Big]\Big\} + B_1 \\ &= 1 - \frac{n\alpha_x}{n\alpha_x+\alpha_y} + n\theta\sum_{r=0}^{p_1-1}\sum_{t=0}^{p_2-1}\xi_{p_1-1,r}\xi_{p_2-1,t}\Big\{A - \frac{\eta_{2,t}\alpha_x}{(n+m_1r)\alpha_x + (m_2t+1)\alpha_y} \\ &+ \frac{\eta_{3,t}\alpha_x}{(n+m_1r)\alpha_x + (m_2(t+1)+1)\alpha_y} + \frac{\eta_1\eta_{2,t}\alpha_x}{(n+m_1(r+1))\alpha_x + (m_2t+1)\alpha_y} \\ &- \frac{\eta_1\eta_{3,t}\alpha_x}{(n+m_1(r+1))\alpha_x + (m_2(t+1)+1)\alpha_y}\Big\} + B_1 \\ &+ n\theta\sum_{r=0}^{p_1-1}\sum_{t=0}^{p_2-1}\xi_{p_1-1,r}\xi_{p_2-1,t}\{-B_{2,r,t} + B_{3,r,t} + B_{4,r,t} - B_{5,r,t}\} \\ &= R_1^{(\mathrm{GFGM})} + B_1 + n\theta\sum_{r=0}^{p_1-1}\sum_{t=0}^{p_2-1}\xi_{p_1-1,r}\xi_{p_2-1,t}\{-B_{2,r,t} + B_{3,r,t} + B_{4,r,t} - B_{5,r,t}\}. \end{aligned}$$

On the other hand, it can be shown that $\lim\limits_{n\to\infty} B_1 = 0$ and $\lim\limits_{n\to\infty} nB_{j,r,t} = 0$, for $j = 2, 3, 4, 5$. Hence, we can conclude that $\widehat{R}_1^{(\mathrm{GFGM})}$ is the asymptotically unbiased estimator of $R_1^{(\mathrm{GFGM})}$.

Now, we calculate the variance of estimator $\widehat{R}_1^{(\mathrm{GFGM})}$.

$$
\begin{aligned}
(6.5)\quad \mathrm{Var}(\widehat{R}_1^{(\mathrm{GFGM})}) &= \mathrm{Var}(RA_1) + n^2 \sum_{r=0}^{p_1-1}\sum_{t=0}^{p_2-1} \xi_{p_1-1,r}^2 \xi_{p_2-1,t}^2 \\
&\quad \times \{A^2\mathrm{Var}(\hat{\theta}) + \mathrm{Var}(\hat{\theta}RA_2) + \mathrm{Var}(\hat{\theta}RA_3) + \mathrm{Var}(\hat{\theta}RA_4) + \mathrm{Var}(\hat{\theta}RA_5)\} \\
&\quad + 2n \sum_{r=0}^{p_1-1}\sum_{t=0}^{p_2-1} \xi_{p_1-1,r}\xi_{p_2-1,t}\big\{-\mathrm{Cov}(RA_1, A\hat{\theta}) - \mathrm{Cov}(RA_1, \hat{\theta}RA_2) \\
&\quad - \mathrm{Cov}(RA_1, \hat{\theta}RA_3) - \mathrm{Cov}(RA_1, \hat{\theta}RA_4) + \mathrm{Cov}(RA_1, \hat{\theta}RA_5) \\
&\quad - \mathrm{Cov}(A\hat{\theta}, \hat{\theta}RA_2) + \mathrm{Cov}(A\hat{\theta}, \hat{\theta}RA_3) + \mathrm{Cov}(A\hat{\theta}, \hat{\theta}RA_4) + \mathrm{Cov}(A\hat{\theta}, \hat{\theta}RA_5) \\
&\quad - \mathrm{Cov}(\hat{\theta}RA_2, \hat{\theta}RA_3) - \mathrm{Cov}(\hat{\theta}RA_2, \hat{\theta}RA_4) + \mathrm{Cov}(\hat{\theta}RA_2, \hat{\theta}RA_5) \\
&\quad + \mathrm{Cov}(\hat{\theta}RA_3, \hat{\theta}RA_4) - \mathrm{Cov}(\hat{\theta}RA_3, \hat{\theta}RA_5) - \mathrm{Cov}(\hat{\theta}RA_4, \hat{\theta}RA_5)\big\}.
\end{aligned}
$$

By [17] we have

$$
(6.6)\qquad \mathrm{Var}(\hat{\theta}) = \Big[\frac{(2+m_1p_1)(2+m_2p_2)}{8p_1p_2B(2/m_1,p_1)B(2/m_2,p_2)}\Big]^2 \mathrm{Var}(\widehat{\tau}),
$$

where $\mathrm{Var}(\widehat{\tau})$ is reported in Appendix of [17].

For $j = 2, 3, 4, 5$, we can write

$$
\mathrm{Var}(\hat{\theta}RA_j) = \mathrm{Var}\Big(\hat{\theta}\frac{k_{j1}\widehat{\alpha}_x}{k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y}\Big),
$$

with $\hat{\theta}$ and RA_j being independent, so

$$
(6.7)\qquad \mathrm{Var}(\hat{\theta}RA_j) = \mathrm{Var}(\hat{\theta})\mathrm{Var}(RA_j) + \mathrm{Var}(\hat{\theta})[\mathrm{E}(RA_j)]^2 + \mathrm{Var}(RA_j)[\mathrm{E}(\hat{\theta})]^2.
$$

Note that $\mathrm{E}(RA_j)$ is reported in equation (6.4) and $\mathrm{Var}(\hat{\theta})$ is given in (6.6). Moreover, $\hat{\theta}$ is the unbiased estimator of θ and using (6.2), we get

$$
\begin{aligned}
\mathrm{Var}(RA_j) &= \mathrm{Var}\Big(\frac{k_{j1}\widehat{\alpha}_x}{k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y}\Big) \\
&\approx \Big\{\frac{\mathrm{E}[k_{j1}\widehat{\alpha}_x]}{\mathrm{E}[k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y]}\Big\}^2 \Big\{\frac{\mathrm{Var}(k_{j1}\widehat{\alpha}_x)}{\mathrm{E}[k_{j1}\widehat{\alpha}_x]^2} + \frac{\mathrm{Var}(k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y)}{\mathrm{E}[k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y]^2} \\
&\quad - 2\frac{\mathrm{Cov}(k_{j1}\widehat{\alpha}_x, k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y)}{\mathrm{E}[k_{j1}\widehat{\alpha}_x]\mathrm{E}[k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y]}\Big\}.
\end{aligned}
$$

By Remarks 6.2 and 6.3, we have

$$
\begin{aligned}
(6.8)\quad &\mathrm{Var}(RA_j) \\
&\approx \Big\{\frac{k_{j1}\alpha_x}{k_{j2}\alpha_x + k_{j3}\alpha_y}\Big\}^2 \Big\{\frac{1}{n} + \frac{k_{j2}^2\alpha_x^2 + k_{j3}^2\alpha_y^2}{n[k_{j2}\alpha_x + k_{j3}\alpha_y]^2} - \frac{2k_{j2}\alpha_x}{n[k_{j2}\alpha_x + k_{j3}\alpha_y]}\Big\}, \\
&\quad j = 1, \ldots, 5.
\end{aligned}
$$

Now we calculate the covariances in equation (6.5). First note that $\mathrm{Cov}(RA_1, A\hat{\theta}) = 0$, because RA_1 and $\hat{\theta}$ are independent. Furthermore,

$$
\begin{aligned}
(6.9)\quad \mathrm{Cov}(A\hat{\theta}, \hat{\theta}RA_j) &= \mathrm{E}[A\hat{\theta}^2 RA_j] - \mathrm{E}[A\hat{\theta}]\mathrm{E}[\hat{\theta}RA_j] \\
&= A\mathrm{E}[\hat{\theta}^2]\mathrm{E}[RA_j] - A(\mathrm{E}[\hat{\theta}])^2\mathrm{E}[RA_j] = A\mathrm{Var}(\hat{\theta})\mathrm{E}[RA_j] \\
&\approx A\Big[\frac{(2+m_1p_1)(2+m_2p_2)}{8p_1p_2B(2/m_1,p_1)B(2/m_2,p_2)}\Big]^2 \mathrm{Var}(\widehat{\tau})\mathrm{E}[RA_j], \quad j = 2, \ldots, 5.
\end{aligned}
$$

In addition, for $j = 2, \ldots, 5$, we have

$$
(6.10)\quad \mathrm{Cov}(RA_1, \hat{\theta}RA_j) = \mathrm{E}[RA_1\hat{\theta}RA_j] - \mathrm{E}[RA_1]\mathrm{E}[\hat{\theta}RA_j] = \mathrm{E}[\hat{\theta}]\mathrm{Cov}(RA_1, RA_j),
$$

and

$$
\begin{aligned}
(6.11)\quad \mathrm{Cov}(\hat{\theta}RA_i, \hat{\theta}RA_j) &= \mathrm{E}[\hat{\theta}^2 RA_iRA_j] - \mathrm{E}[\hat{\theta}RA_i]\mathrm{E}[\hat{\theta}RA_j] \\
&= \mathrm{E}[\hat{\theta}^2]\mathrm{E}[RA_iRA_j] - \mathrm{E}[\hat{\theta}]^2\mathrm{E}[RA_i]\mathrm{E}[RA_j] \\
&= (\mathrm{Var}(\hat{\theta}) + \mathrm{E}[\hat{\theta}]^2)\mathrm{E}[RA_iRA_j] - \mathrm{E}[\hat{\theta}]^2\mathrm{E}[RA_i]\mathrm{E}[RA_j] \\
&= \mathrm{Var}(\hat{\theta})\mathrm{E}[RA_iRA_j] + \mathrm{E}[\hat{\theta}]^2\mathrm{E}[RA_iRA_j] \\
&\quad\; - \mathrm{E}[\hat{\theta}]^2\mathrm{E}[RA_i]\mathrm{E}[RA_j] \\
&= \mathrm{Var}(\hat{\theta})\mathrm{E}[RA_iRA_j] + \mathrm{E}[\hat{\theta}]^2\mathrm{Cov}(RA_i, RA_j).
\end{aligned}
$$

Now, we need to compute $\mathrm{Cov}(RA_i, RA_j)$. To this end,

$$
\begin{aligned}
(6.12)\quad \mathrm{Cov}(RA_i, RA_j) &= \mathrm{E}[RA_iRA_j] - \mathrm{E}[RA_i]\mathrm{E}[RA_j] \\
&= \mathrm{E}\Big[\frac{k_{i1}\widehat{\alpha}_x}{k_{i2}\widehat{\alpha}_x + k_{i3}\widehat{\alpha}_y}\frac{k_{j1}\widehat{\alpha}_x}{k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y}\Big] \\
&\quad - \mathrm{E}\Big[\frac{k_{i1}\widehat{\alpha}_x}{k_{i2}\widehat{\alpha}_x + k_{i3}\widehat{\alpha}_y}\Big]\mathrm{E}\Big[\frac{k_{j1}\widehat{\alpha}_x}{k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y}\Big] \\
&= \mathrm{E}\Big[\frac{k_{i1}k_{j1}\widehat{\alpha}_x^2}{(k_{i2}\widehat{\alpha}_x + k_{i3}\widehat{\alpha}_y)(k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y)}\Big] - \mathrm{E}[RA_i]\mathrm{E}[RA_j].
\end{aligned}
$$

By (6.1), we get

$$
\begin{aligned}
\mathrm{E}\Big[\frac{k_{i1}k_{j1}\widehat{\alpha}_x^2}{[k_{i2}\widehat{\alpha}_x + k_{i3}\widehat{\alpha}_y][k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y]}\Big] &\approx \frac{k_{i1}k_{j1}\mathrm{E}[\widehat{\alpha}_x^2]}{\mathrm{E}[(k_{i2}\widehat{\alpha}_x + k_{i3}\widehat{\alpha}_y)(k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y)]} \\
&- \frac{\mathrm{Cov}(k_{i1}k_{j1}\widehat{\alpha}_x^2, [k_{i2}\widehat{\alpha}_x + k_{i3}\widehat{\alpha}_y][k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y])}{\{\mathrm{E}[(k_{i2}\widehat{\alpha}_x + k_{i3}\widehat{\alpha}_y)(k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y)]\}^2} \\
&+ \frac{k_{i1}k_{j1}\mathrm{E}[\widehat{\alpha}_x^2]}{\{\mathrm{E}[(k_{i2}\widehat{\alpha}_x + k_{i3}\widehat{\alpha}_y)(k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y)]\}^3}\mathrm{Var}([k_{i2}\widehat{\alpha}_x + k_{i3}\widehat{\alpha}_y][k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y]).
\end{aligned}
$$

To calculate the last equation, we need to compute

$$I_1 := \mathrm{E}[(k_{i2}\widehat{\alpha}_x + k_{i3}\widehat{\alpha}_y)(k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y)],$$
$$I_2 := \mathrm{Cov}(k_{i1}k_{j1}\widehat{\alpha}_x^2, [k_{i2}\widehat{\alpha}_x + k_{i3}\widehat{\alpha}_y][k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y])$$

and

$$I_3 := \mathrm{Var}([k_{i2}\widehat{\alpha}_x + k_{i3}\widehat{\alpha}_y][k_{j2}\widehat{\alpha}_x + k_{j3}\widehat{\alpha}_y]).$$

For this purpose, we can write

$$\begin{aligned} I_1 &= \mathrm{E}[k_{i2}k_{j2}\widehat{\alpha}_x^2 + k_{i2}k_{j3}\widehat{\alpha}_x\widehat{\alpha}_y + k_{i3}k_{j2}\widehat{\alpha}_x\widehat{\alpha}_y + k_{i3}k_{j3}\widehat{\alpha}_y^2] \\ &= k_{i2}k_{j2}\mathrm{E}[\widehat{\alpha}_x^2] + (k_{i2}k_{j3} + k_{i3}k_{j2})\mathrm{E}[\widehat{\alpha}_x\widehat{\alpha}_y] + k_{i3}k_{j3}\mathrm{E}[\widehat{\alpha}_y^2] \\ &= k_{i2}k_{j2}(\mathrm{Var}(\widehat{\alpha}_x) + \mathrm{E}[\widehat{\alpha}_x]^2) + (k_{i2}k_{j3} + k_{i3}k_{j2})\mathrm{E}[\widehat{\alpha}_x]\mathrm{E}[\widehat{\alpha}_y] \\ &\quad + k_{i3}k_{j3}(\mathrm{Var}(\widehat{\alpha}_y) + \mathrm{E}[\widehat{\alpha}_y]^2) \\ &= k_{i2}k_{j2}\Big[\frac{\alpha_x^2}{n} + \alpha_x^2\Big] + (k_{i2}k_{j3} + k_{i3}k_{j2})\alpha_x\alpha_y + k_{i3}k_{j3}\Big[\frac{\alpha_y^2}{n} + \alpha_y^2\Big]. \end{aligned} \tag{6.13}$$

To obtain I_2, since $\sqrt{n}(\widehat{\alpha}_z - \alpha_z) \xrightarrow{d} N(0, \alpha_z^2)$ for $z = x, y$, we notice that

$$\mathrm{E}[\widehat{\alpha}_z^3] = (\mathrm{E}[\widehat{\alpha}_z])^3 + 3\mathrm{Var}(\widehat{\alpha}_z)\mathrm{E}[\widehat{\alpha}_z] = \alpha_z^3 + 3\frac{\alpha_z^3}{n},$$

and

$$\mathrm{E}[\widehat{\alpha}_z^4] = (\mathrm{E}[\widehat{\alpha}_z])^4 + 6\mathrm{Var}(\widehat{\alpha}_z)(\mathrm{E}[\widehat{\alpha}_z])^2 + 3(\mathrm{Var}(\widehat{\alpha}_z))^2 = \alpha_z^4 + 6\frac{\alpha_z^4}{n} + 3\frac{\alpha_z^4}{n^2}.$$

Hence,

$$\begin{aligned} I_2 &= \mathrm{E}[k_{i1}k_{j1}\widehat{\alpha}_x^2\{k_{i2}k_{j2}\widehat{\alpha}_x^2 + (k_{i2}k_{j3} + k_{i3}k_{j2})\widehat{\alpha}_x\widehat{\alpha}_y + k_{i3}k_{j3}\widehat{\alpha}_y^2\}] \\ &\quad - k_{i1}k_{j1}\mathrm{E}[\widehat{\alpha}_x^2]I_1 \\ &= \mathrm{E}[k_{i1}k_{j1}k_{i2}k_{j2}\widehat{\alpha}_x^4 + k_{i1}k_{j1}(k_{i2}k_{j3} + k_{i3}k_{j2})\widehat{\alpha}_x^3\widehat{\alpha}_y \\ &\quad + k_{i1}k_{j1}k_{i3}k_{j3}\widehat{\alpha}_x^2\widehat{\alpha}_y^2] - k_{i1}k_{j1}\mathrm{E}[\widehat{\alpha}_x^2]I_1 \\ &= k_{i1}k_{j1}k_{i2}k_{j2}\Big[\alpha_x^4 + 6\frac{\alpha_x^4}{n} + 3\frac{\alpha_x^4}{n^2}\Big] \\ &\quad + k_{i1}k_{j1}(k_{i2}k_{j3} + k_{i3}k_{j2})\Big[\alpha_x^3 + 3\frac{\alpha_x^3}{n}\Big]\alpha_y \\ &\quad + k_{i1}k_{j1}k_{i3}k_{j3}\Big[\frac{\alpha_x^2}{n} + \alpha_x^2\Big]\Big[\frac{\alpha_y^2}{n} + \alpha_y^2\Big] - k_{i1}k_{j1}\Big(\alpha_x^2 + \frac{\alpha_x^2}{n}\Big) \\ &\qquad \times \Big\{k_{i2}k_{j2}\Big[\frac{\alpha_x^2}{n} + \alpha_x^2\Big] + (k_{i2}k_{j3} + k_{i3}k_{j2})\alpha_x\alpha_y + k_{i3}k_{j3}\Big[\frac{\alpha_y^2}{n} + \alpha_y^2\Big]\Big\} \\ &= k_{i1}k_{j1}k_{i2}k_{j2}\Big[2 + \frac{1}{n}\Big]\frac{2\alpha_x^4}{n} + k_{i1}k_{j1}(k_{i2}k_{j3} + k_{i3}k_{j2})\frac{2\alpha_x^3\alpha_y}{n}, \end{aligned} \tag{6.14}$$

and

$$
\begin{aligned}
(6.15)\qquad I_3 &= \mathrm{Var}(k_{i2}k_{j2}\widehat{\alpha}_x^2 + [k_{i2}k_{j3} + k_{i3}k_{j2}]\widehat{\alpha}_x\widehat{\alpha}_y + k_{i3}k_{j3}\widehat{\alpha}_y^2) \\
&= k_{i2}^2k_{j2}^2\mathrm{Var}(\widehat{\alpha}_x^2) + [k_{i2}k_{j3} + k_{i3}k_{j2}]^2\mathrm{Var}(\widehat{\alpha}_x\widehat{\alpha}_y) + k_{i3}^2k_{j3}^2\mathrm{Var}(\widehat{\alpha}_y^2) \\
&= 2\frac{k_{i2}^2k_{j2}^2}{n}\Big(2 + \frac{1}{n}\Big)\alpha_x^4 + [k_{i2}k_{j3} + k_{i3}k_{j2}]^2\Big(\frac{1}{n} + 2\Big)\frac{\alpha_x^2\alpha_y^2}{n} \\
&\quad + 2\frac{k_{i3}^2k_{j3}^2}{n}\Big(2 + \frac{1}{n}\Big)\alpha_y^4.
\end{aligned}
$$

Furthermore, the estimator of Kendall's τ is a U-statistic, i.e.

$$
\widehat{\tau} = \binom{n}{2}^{-1} \sum_{1\leqslant i<j\leqslant n} \mathrm{sgn}(X_i - X_j)\,\mathrm{sgn}(Y_i - Y_j).
$$

From the theory of U-statistics we know that the τ-estimator is an unbiased and strongly consistent estimator and asymptotically normal as $\sqrt{n}(\widehat{\tau} - \tau) \xrightarrow{d} N(0, 4\sigma_\tau^2)$, as $n \to \infty$, where $\sigma_\tau^2 = \mathrm{Var}(\mathrm{E}[\mathrm{sgn}(X - \widetilde{X})\,\mathrm{sgn}(Y - \widetilde{Y})|X, Y])$ and where $(\widetilde{X}, \widetilde{Y})$ is an independent copy of (X, Y) (see [14]). Hence, using $\mathrm{Var}(\widehat{\tau}) = 4\sigma_\tau^2/n$, and by substituting equations (6.6)–(6.15) in (6.5), we obtain the variance of $\widehat{R}_1^{(\mathrm{GFGM})}$.

Consequently, $\widehat{R}_1^{(\mathrm{GFGM})}$ is a consistent estimator, because it is a function of consistent estimates. Moreover, since $\sqrt{n}(\widehat{\tau} - \tau) \xrightarrow{d} N(0, 4\sigma_\tau^2)$, we can conclude that $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, 4\overline{K}^2\sigma_\tau^2)$, as $n \to \infty$, where $\hat{\theta} = \overline{K}\widehat{\tau}$ and $\theta = \overline{K}\tau$. Now, using the asymptotic distributions of $\widehat{\alpha}_x$ and $\widehat{\alpha}_y$ and since $\widehat{\alpha}_x$, $\widehat{\alpha}_y$ and $\hat{\theta}$ are independent, therefore,

$$
\sqrt{n}\left(\begin{pmatrix} \widehat{\alpha}_x \\ \widehat{\alpha}_y \\ \hat{\theta} \end{pmatrix} - \begin{pmatrix} \alpha_x \\ \alpha_y \\ \theta \end{pmatrix}\right) \xrightarrow{d} N_{3\times 3}\left(\mathbf{0}, \begin{pmatrix} \alpha_x^2 & 0 & 0 \\ 0 & \alpha_y^2 & 0 \\ 0 & 0 & 4\overline{K}^2\sigma_\tau^2 \end{pmatrix}\right).
$$

Hence, by the Delta method we can conclude that

$$
\frac{\sqrt{n}(\widehat{R}_1^{(\mathrm{GFGM})} - R_1^{(\mathrm{GFGM})})}{\sqrt{V^*}} \xrightarrow{d} N(0, 1),
$$

where

$$
V^* = \left[\frac{\partial R_1^{(\mathrm{GFGM})}}{\partial \alpha_x}, \frac{\partial R_1^{(\mathrm{GFGM})}}{\partial \alpha_y}, \frac{\partial R_1^{(\mathrm{GFGM})}}{\partial \theta}\right] \begin{bmatrix} \alpha_x^2 & 0 & 0 \\ 0 & \alpha_y^2 & 0 \\ 0 & 0 & 4\overline{K}^2\sigma_\tau^2 \end{bmatrix} \begin{bmatrix} \dfrac{\partial R_1^{(\mathrm{GFGM})}}{\partial \alpha_x} \\ \dfrac{\partial R_1^{(\mathrm{GFGM})}}{\partial \alpha_y} \\ \dfrac{\partial R_1^{(\mathrm{GFGM})}}{\partial \theta} \end{bmatrix}.
$$

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