Czechoslovak Mathematical Journal

Bo Zhang; Zhiqi Chen; Shaoqiang Deng

Pseudo-Riemannian weakly symmetric manifolds of low dimension

*Czechoslovak Mathematical Journal*, Vol. 69 (2019), No. 3, 811–835

Persistent URL: http://dml.cz/dmlcz/147791

**Terms of use:**

© Institute of Mathematics AS CR, 2019

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

**Czechoslovak Mathematical Journal, 69 (144) (2019), 811–835**

# PSEUDO-RIEMANNIAN WEAKLY SYMMETRIC MANIFOLDS OF LOW DIMENSION

Bo Zhang, Zhiqi Chen, Shaoqiang Deng, Tianjin

Received November 10, 2017. Published online March 26, 2019.

*Abstract.* We give a classification of pseudo-Riemannian weakly symmetric manifolds in dimensions 2 and 3, based on the algebraic approach of such spaces through the notion of a pseudo-Riemannian weakly symmetric Lie algebra. We also study the general symmetry of reductive 3-dimensional pseudo-Riemannian weakly symmetric spaces and particularly prove that a 3-dimensional reductive 2-fold symmetric pseudo-Riemannian manifold must be globally symmetric.

*Keywords*: pseudo-Riemannian manifold; pseudo-Riemannian weakly symmetric manifold; pseudo-Riemannian weakly symmetric Lie algebra; Lorentzian weakly symmetric manifold

*MSC 2010*: 53C30, 22E46

## 1. Introduction

The goal of this paper is to give a complete classification of 2- or 3-dimensional reductive weakly symmetric pseudo-Riemannian manifolds. Let us first explain our motivation to consider this problem. In the literature, there are several well-known important extensions of the theory of Riemannian symmetric spaces. Weakly symmetric spaces, introduced by Selberg in 1956 (see [8]), play key roles in number theory, Riemannian geometry and harmonic analysis. Pseudo-Riemannian symmetric spaces, including semisimple symmetric spaces, play central but complementary roles in number theory, differential geometry and relativity, Lie group representation theory and harmonic analysis. As the common extension of these two branches of symmetric space theory, pseudo-Riemannian weakly symmetric manifolds were introduced by the second author and Wolf in [2]. The study in [2], [3] shows that many

Supported by NSFC (no. 11301385, 11771331, 11671212, 51535008) of China and the Fundamental Research Funds for the Central Universities.

DOI: 10.21136/CMJ.2019.0515-17

results in the Riemannian case are no longer true for pseudo-Riemannian weakly symmetric manifolds, although some can actually be generalized to the pseudo case. On the other hand, as a special class of homogeneous pseudo-Riemannian manifolds, weakly symmetric pseudo-Riemannian manifolds will hopefully be very useful in the study of some physical problems. Therefore, a careful study of the geometric properties of such spaces will be of great interest.

In the positive definite case, the geometry of Riemannian weakly symmetric spaces has been studied rather extensively. As a central problem in this direction, the classification of Riemannian weakly symmetric spaces of certain special cases has been achieved [1], [10], [11]. However, all the classification results are obtained through a careful analysis of some other classes of homogeneous spaces and an algebraic approach was not available until the third author introduced the notion of weakly symmetric algebras in [4]. It turns out that the notion of a weakly symmetric Lie algebra is very useful in the study of weakly symmetric spaces. In particular, using this algebraic approach, the third author of this paper successfully classified weakly symmetric Finsler spaces of dimensions 2 and 3.

In this paper, we shall modify the notion of a weakly symmetric Lie algebra to include the pseudo case and use that to study pseudo-Riemannian weakly symmetric manifolds. We first show that any pseudo-Riemannian weakly symmetric manifold gives rise to a pseudo-Riemannian weakly symmetric Lie pair, and conversely, given any pseudo-Riemannian weakly symmetric Lie pair, one can construct a (not necessarily unique) pseudo-Riemannian weakly symmetric manifold. Using this intrinsic algebraic interpretation, we are able to classify reductive pseudo-Riemannian weakly symmetric manifolds in dimensions 2 and 3.

Our main results show that in the 2-dimensional case, no essentially new examples arise. However, in the 3-dimensional case, there appears several new types of weakly symmetric spaces, some of which are rather complicated. This phenomenon entails the fact that a further careful and serious study on pseudo-Riemannian weakly symmetric spaces would be rather interesting and may lead to more new phenomena. Moreover, we also study the general symmetry of 3-dimensional pseudo-Riemannian weakly symmetric spaces and particularly prove that a 2-fold reductive symmetric pseudo-Riemannian manifold must be globally symmetric. This result leads to the following open problem:

**Problem.** Is a 2-fold symmetric pseudo-Riemannian manifold globally symmetric?

We conjecture that in the Lorentian case the answer is positive. But in the general case, there may exist counterexamples.

The paper is organized as follows. In Section 2, we recall some basic definitions and results on pseudo-Riemannian weakly symmetric manifolds. In Section 3, we introduce pseudo-Riemannian weakly symmetric Lie pairs and reduce the study of pseudo-Riemannian weakly symmetric manifolds to that of pseudo-Riemannian weakly symmetric Lie algebras. In Sections 4 and 5, we give the classification of reductive Lorentzian weakly symmetric Lie algebras in dimensions 2 and 3, which implies the classification of pseudo-Riemannian weakly symmetric Lie algebras since the Riemannian cases in dimensions 2 and 3 are given in [4]. Finally, in Section 6, we study the symmetry of general 3-dimensional pseudo-Riemannian manifolds.

## 2. Prelimenaries

In this section, we recall some definitions and fundamental results.

**Definition 2.1.** A pseudo-Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$ is a smooth manifold $M$ with a non-degenerate inner product $\langle\cdot,\cdot\rangle$ on the fibers of its tangent bundle $TM$. Denote the signature of $\langle\cdot,\cdot\rangle$ as $(n_+, n_-)$, where $n_+ + n_- = \dim M$. The manifold $(M, \langle\cdot,\cdot\rangle)$ is called Riemannian if $\langle\cdot,\cdot\rangle$ has signature $(\dim M, 0)$, i.e., if the inner product is positive definite at any tangent space.

The following proposition is well known (see, e.g., [6]).

**Proposition 2.1.** *If a Lie group acts transitively on a connected manifold, then so does the identity component of the Lie group. In particular, if $(M, \langle\cdot,\cdot\rangle)$ is a connected homogeneous pseudo-Riemannian manifold, then the identity component $I(M, \langle\cdot,\cdot\rangle)^0$ of its isometry group is also transitive.*

There are a number of equivalent conditions that can be taken as the definition of weak symmetry for a Riemannian manifold. For the pseudo-Riemannian case, we use the one on reversing geodesics.

**Definition 2.2.** Let $(M, \langle\cdot,\cdot\rangle)$ be a pseudo-Riemannian manifold. Suppose that for every $m \in M$ and every nonzero tangent vector $X \in T_mM$ there is an isometry $\sigma = \sigma_{m,X}$ of $(M, \langle\cdot,\cdot\rangle)$ such that $\sigma(m) = m$ and $\mathrm{d}\sigma(X) = -X$. Then we say that $(M, \langle\cdot,\cdot\rangle)$ is a pseudo-Riemannian weakly symmetric manifold. In particular, if $(M, \langle\cdot,\cdot\rangle)$ is a Riemannian manifold, then we say that $(M, \langle\cdot,\cdot\rangle)$ is weakly symmetric.

For the pseudo-Riemannian case, we have:

**Proposition 2.2** ([2]). *Let $(M, \langle\cdot,\cdot\rangle)$ be a connected pseudo-Riemannian weakly symmetric manifold. Then $(M, \langle\cdot,\cdot\rangle)$ is a pseudo-Riemannian homogeneous space $G/H$, where $G = I(M, \langle\cdot,\cdot\rangle)^0$.*

The following result is very useful in the study of invariant weakly symmetric pseudo-Riemannian metrics on homogeneous manifolds.

**Proposition 2.3.** *Let $G$ be a Lie group and $H$ a closed subgroup of $G$. Suppose that the coset space $G/H$ is reductive, i.e., there exists a subspace $\mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of the Lie group $G$ such that $\mathfrak{g} = \mathfrak{h}+\mathfrak{p}$ is a direct sum such that $\mathrm{Ad}(h)(\mathfrak{p}) \subset \mathfrak{p}$ for any $h \in H$. If for any $X \in \mathfrak{p}$ there exists $h \in H$ such that $\mathrm{Ad}(h)(X) = -X$, then any $G$-invariant pseudo-Riemannian metric on $G/H$ is weakly symmetric.*

The proof is similar to the Riemmannian case in [12], so we omit it here.

## 3. Pseudo-Riemannian weakly symmetric Lie algebras

In this section, we will use the notion of a pseudo-Riemannian weakly symmetric Lie algebra to describe simply connected pseudo-Riemannian weakly symmetric manifolds. As usual, we only consider connected manifolds.

**Definition 3.1.** Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$. The pair $(\mathfrak{g}, \mathfrak{h})$ is called a weakly symmetric Lie algebra if there exists a series (finite or countable) of automorphisms $\{\sigma_0 = \mathrm{id}, \sigma_1, \sigma_2 \ldots,\}$ of $\mathfrak{g}$ satisfying the following conditions:

- ▷ (WSL1) Each $\sigma_i$, $i = 0, 1, 2, \ldots$, preserves the subspace $\mathfrak{h}$, i.e., $\sigma_i(\mathfrak{h}) = \mathfrak{h}$.
- ▷ (WSL2) For any pair $i, j \geqslant 0$ there exists $k \geqslant 0$ and a vector $X_{ij} \in \mathfrak{h}$ such that $\sigma_i\sigma_j = e^{\mathrm{ad}\, X_{ij}}\sigma_k$, where ad denotes the adjoint representation of $\mathfrak{g}$.
- ▷ (WSL3) For any $Y$ in the quotient spaces $\mathfrak{g}/\mathfrak{h}$ there exists $X_Y \in \mathfrak{h}$ and an index $m_Y$ such that $e^{\mathrm{ad}\, X_Y} \cdot \sigma_{m_Y}(Y) = -Y$ here we use the same notations ad and $\sigma_i$ to denote the induced actions on the quotient space $\mathfrak{g}/\mathfrak{h}$.

We usually say that the pair $(\mathfrak{g}, \mathfrak{h})$ is weakly symmetric with respect to $\{\sigma_0 = \mathrm{id}, \sigma_1, \sigma_2, \ldots\}$. If there exists an $\mathrm{ad}(\mathfrak{h})$-invariant non-degenerate inner product on the quotient space $\mathfrak{g}/\mathfrak{h}$ such that the induced action of any $\sigma_i$, $i = 0, 1, \ldots$, on $\mathfrak{g}/\mathfrak{h}$ is isometric, then the pair $(\mathfrak{g}, \mathfrak{h})$ is called a pseudo-Riemannian weakly symmetric Lie algebra. A weakly symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h})$ is called of finite type if the set of automorphisms $\{\sigma_0, \sigma_1, \ldots\}$ can be chosen to be a finite set.

Now we give an algebraic description of weakly symmetric pseudo-Riemannian spaces.

**Theorem 3.1.** *Let $(M, \langle\cdot,\cdot\rangle)$ be a pseudo-Riemannian weakly symmetric manifold. Then there exists a Lie group $G$ and a closed subgroup $H$ of $G$ such that $M = G/H$ and $\langle\cdot,\cdot\rangle$ is $G$-invariant. Furthermore, the Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{g}$*

*and $\mathfrak{h}$ are the Lie algebras of $G$ and $H$, respectively, is a pseudo-Riemannian weakly symmetric Lie algebra.*

Proof. Fix $x \in M$. By the definition, for any $v \in T_x(M)$ there exists an isometry $\tau$ of $(M, \langle\cdot,\cdot\rangle)$ such that $\tau(x) = x$ and $d\tau(v) = -v$. Let $\overline{G}$ be the full group of isometries of $(M, \langle\cdot,\cdot\rangle)$. Since a weakly symmetric pseudo-Riemannian manifold is homogeneous, $\overline{G}$ acts transitively on $M$. Then the unity component $G$ of $\overline{G}$ also acts transitively on $M$. Let $H$ be the isotropy subgroup of $G$ at $x$. Then $M$ is diffeomorphic to the coset space $G/H$ and the pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$ is $G$-invariant. Now we prove that the pair $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{g} = \operatorname{Lie} G$, $\mathfrak{h} = \operatorname{Lie} H$, is a pseudo-Riemannian weakly symmetric Lie algebra. Identify the quotient space $\mathfrak{g}/\mathfrak{h}$ with the tangent space $T_x(M)$. The isotropic action of $H$ on $T_x(M)$ corresponds to the action of $H$ on $\mathfrak{g}/\mathfrak{h}$ induced by the adjoint action of $H$ on $\mathfrak{g}$. Let $e$ be the unit element and $H_e$ be the identity component of $H$. Note that $H$ has at most countable components and $H_e$ is a normal subgroup of $H$. Now let $\{e, h_1, h_2, \ldots\}$ be a subset of $H$ such that $\{eH, h_1H, h_2H, \ldots\}$ are all the (distinct) elements of the quotient group. Let $\sigma_0$ be the identity transformation of $\mathfrak{g}$ and $\sigma_j = \operatorname{Ad}(h_j)$, $j = 1, 2, \ldots$ Then we assert the set $\{\sigma_0, \sigma_1, \sigma_2, \ldots\}$ satisfies conditions (WSL1), (WSL2), (WSL3). In fact, (WSL1) is obviously satisfied. Now for any pair $i, j$, suppose that in the quotient group $H/H_e$ we have $h_iH_eh_jH_e = h_kH_e$, then there exist $m_1, m_2, m_3 \in H_e$ such that $h_im_1h_jm_2 = h_km_3$, i.e., $h_ih_j = h_k(m_3m_2^{-1}(h_jm_1^{-1}h_j^{-1}))$. Since $h_jm_1^{-1}h_j^{-1} \in H_e$, we have $m = m_3m_2^{-1}(h_jm_1^{-1}h_j^{-1}) \in H_e$. Since $H_e$ is connected, it is generated (as a group) by the elements of the form $\exp(X)$, $X \in \mathfrak{h}$. Hence there exist $X_1, X_2, \ldots, X_l \in \mathfrak{h}$ such that $\exp(X_1)\exp(X_2)\ldots\exp(X_l) = m$. Then we have $\operatorname{Ad}(m) = e^{\operatorname{ad} X_1}e^{\operatorname{ad} X_2}\ldots e^{\operatorname{ad} X_l}$. Denote $X_{ij} = X_1 + X_2 \ldots + X_l$. Then we have $\sigma_i\sigma_j = \sigma_k e^{\operatorname{ad} X_{ij}} = \sigma_k e^{\operatorname{ad} X_{ij}}\sigma_k^{-1}\sigma_k = e^{\operatorname{ad}(\sigma_k(X_{ij}))} \cdot \sigma_k$, i.e., (WSL2) is satisfied. Now we prove (WSL3). By the definition, for any $Y \in \mathfrak{p}$ we can select $h \in H$ such that $\operatorname{Ad}(h)(Y) = -Y$. Suppose $h$ lies in the component $h_iH_e$. Then there exists $h_0 \in H_e$ such that $h = h_ih_0 = h_ih_0h_i^{-1}h_i$. Since $h_ih_0h_i^{-1} \in H_e$, we can write $h = h_i \exp Y_1 \exp Y_2 \ldots \exp Y_s$ for some $Y_1, Y_2, \ldots, Y_s \in \mathfrak{h}$. Then we have $\operatorname{Ad}(h) = e^{\operatorname{ad} Y_1}e^{\operatorname{ad} Y_2}\ldots e^{\operatorname{ad}(Y_s)}\operatorname{Ad}(h_i) = e^{\operatorname{ad}(Y_1+\ldots+Y_s)}\operatorname{Ad}(h_i)$. From this we easily see that (WSL3) is satisfied. This completes the proof of the theorem. □

Now we consider the converse statement of the above theorem. We will show that any pseudo-Riemannian weakly symmetric Lie algebra can give rise to a pseudo-Riemannian weakly symmetric manifold, although in general the spaces constructed from a pseudo-Riemannian weakly symmetric Lie algebra are not unique.

**Theorem 3.2.** *Let $(\mathfrak{g}, \mathfrak{h})$ be a pseudo-Riemannian weakly symmetric Lie algebra. Suppose that $G$ is a connected simply connected Lie group with the Lie algebra $\mathfrak{g}$*

*and $H$ is the unique connected Lie subgroup of $G$ with the Lie algebra $\mathfrak{h}$. If $H$ is closed in $G$, then there exists a $G$-invariant pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$ on the coset space $G/H$ such that $(G/H, Q)$ is a pseudo-Riemannian weakly symmetric space.*

Proof. The proof of this theorem is also very similar to the Riemannian case. However, for the completeness of the paper and for the convenience of the readers, we give a detailed proof here. Identify the tangent space $T_o(G/H)$ with $\mathfrak{g}/\mathfrak{h}$, where $o = eH$ is the origin of the coset space $G/H$. Since $(\mathfrak{g}, \mathfrak{h})$ is a pseudo-Riemannian weakly symmetric Lie algebra, there exists a non-degenerate inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}/\mathfrak{h}$ which is invariant under $\mathrm{ad}(\mathfrak{h})$ as well as the induced action of $\sigma_i$, $i = 1, 2, \ldots, s$, on $\mathfrak{g}/\mathfrak{h}$. We assert that $\langle\cdot,\cdot\rangle$ is also $\mathrm{Ad}(H)$-invariant. In fact, for any $h \in H$ and any index $j$, we have $\mathrm{Ad}\, h \cdot \tau_j = \tau_j \cdot (\tau_j^{-1} \cdot \mathrm{Ad}\, h \cdot \tau_j)$. Since $H$ is connected, it is generated by the elements of the form $\exp(X)$, $X \in \mathfrak{h}$. Hence, $h$ can be written as $\exp X_1 \exp X_2 \ldots \exp X_k$, where $X_i \in \mathfrak{h}$, $i = 1, 2, \ldots, k$. Then we have

$$\tau_j^{-1}\, \mathrm{Ad}\, h \cdot \tau_j = e^{\mathrm{ad}\, \tau_j X_1} e^{\mathrm{ad}\, \tau_j X_2} \ldots e^{\mathrm{ad}\, \tau_j X_k}.$$

From this, our assertion follows. Now using $\langle\cdot,\cdot\rangle$ we can construct a $G$-invariant pseudo-Riemannian metric $Q$ on the $G/H$ whose restriction to $T_o(G/H) = \mathfrak{p}$ is equal to $\langle\cdot,\cdot\rangle$ (see [7]). We assert that the homogeneous pseudo-Riemannian manifold constructed above is weakly symmetric. Note that $G$ is connected and simply connected, hence each automorphism $\tau_j$ of $\mathfrak{g}$ can be lifted to an automorphism of $G$ (see [6]), denoted by $\widetilde{\tau}_j$, $j = 0, 1, 2, \ldots$ Since $\tau_j(\mathfrak{h}) = \mathfrak{h}$, we easily see that $\widetilde{\tau}_j(H) \subset H$. Hence $\widetilde{\tau}_j$ induces a diffeomorphism of $G/H$, denoted by $\widehat{\tau}_j$, by sending $gH$ to $\widetilde{\tau}_j(g)H$. The diffeomorphism $\widehat{\tau}_j$ keeps the origin $o = eH$ invariant and its differential at $o$ is just the induced action of $\tau_j$ on $\mathfrak{g}/\mathfrak{h}$. From this we see that $\widehat{\tau}_j$ keeps the Riemannian metric $Q$ invariant, or in other words, $\widehat{\tau}_j$ lies in the isotropic subgroup (at $o$) of the full group of isometries of $(G/H, Q)$. By (WSL3), for any $Y \in \mathfrak{p} = T_o(G/H)$ we can choose $X_Y \in \mathfrak{h}$ and index $i_Y$ such that $e^{\mathrm{ad}\, X_Y} \tau_{i_Y}(Y) = -Y$. This means that the isometry $\tau_{\exp(X_Y)} \widehat{\tau}_{i_Y}$ of the pseduo-Riemannian manifold $(G/H, Q)$, where $\tau_h(gH) = hgH$, $h \in H$, reverses the tangent vector $Y$. Since $(G/H, Q)$ is homogeneous for any $m \in G/H$ and $Y \in T_m(G/H)$ there exists an isometry $f$ such that $f(m) = m$ and $\mathrm{d}f|_m(Y) = -Y$. Thus by Proposition 2.3, $(G/H, Q)$ is weakly symmetric. $\square$

## 4. Pseudo-Riemannian weakly symmetric manifolds of dimension 2

In this section we shall give a classification of reductive pseudo-Riemannian weakly symmetric manifolds of dimension 2. Since the classification of Riemannian weakly symmetric manifolds in dimension 2 has been given in [4], we only need to consider Lorentzian metrics.

The following result is well known.

**Lemma 4.1.** *The isotropic representation $\varrho$ of $\widetilde{H}$ on $T_x(M)$ is faithful, i.e., the mapping $\varrho\colon\ h \mapsto \mathrm{d}h|_X$, $h \in \widetilde{H}$, is one-to-one.*

Now we have:

**Theorem 4.1.** *Let $(G/H, \langle\cdot,\cdot\rangle)$ be a reductive Lorentzian weakly symmetric coset space of dimension 2, and $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{p}$ and $\varrho$ be the corresponding pseudo weakly symmetric Lie algebra as in Section 3. Then $(\mathfrak{g},\mathfrak{h})$ must be one of the following:*

(1) $\dim\mathfrak{g} = 2$, $\dim\mathfrak{h} = 0$, *and $\mathfrak{g}$ is an abelian Lie algebra. In this case, the set of automorphisms can be chosen as* $\{\mathrm{id}, -\mathrm{id}\}$.

(2) $\dim\mathfrak{g} = 3$, $\dim\mathfrak{h} = 1$, *and* $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ *(direct sum), where there exists a basis $X$ of $\mathfrak{h}$ and $\{\varepsilon_1, \varepsilon_2\}$ of $\mathfrak{p}$ such that*

$$[X,\varepsilon_1] = \varepsilon_2, \quad [X,\varepsilon_2] = \varepsilon_1, \quad [\varepsilon_1,\varepsilon_2] = aX.$$

*In this case, the Lorentzian metric $\langle\cdot,\cdot\rangle$ on $\mathfrak{p}$ can be defined by*

$$\langle\varepsilon_1,\varepsilon_1\rangle = -\langle\varepsilon_2,\varepsilon_2\rangle = 1$$

*and the set of automorphisms can be* $\{\mathrm{id},\sigma\}$, *where $\sigma$ is defined by*

$$\sigma(X) = X, \quad \sigma(\varepsilon_1) = -\varepsilon_1, \quad \sigma(\varepsilon_2) = -\varepsilon_2.$$

Proof. By Lemma 4.1, $\dim H = \dim\varrho(H)$. Since $\varrho(H)$ is a subgroup of $\mathrm{SO}(1,1)$, we have $\dim\varrho(H) \leqslant 1$. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ be a reductive decomposition of the Lie algebra and identify the tangent space $T_x(M)$ with $\mathfrak{p}$.

If $\dim\varrho(H) = 0$, then we have $H = \{e\}$. Hence $M$ is itself the Lie group $G$. By Theorem 3.1 and the assumption that $(M,\langle\cdot,\cdot\rangle)$ is the Lie algebra, $\mathfrak{g}$ admits finitely many automorphisms $\{\sigma_0 = \mathrm{id}, \sigma_1, \ldots, \sigma_s\}$ such that for any $Y \in \mathfrak{g} = \mathfrak{p}$ there exists an index $j_Y$ such that $\sigma_{j_Y}(Y) = -Y$. Let $V_j = \{Y \in \mathfrak{g}\colon\ \sigma_j(Y) = -Y\}$. Then $V_j$ are subspaces of $\mathfrak{g}$ and $\mathfrak{g} = \bigcup V_j$. Therefore there exists some $j_0$ such that $V_{j_0} = \mathfrak{g}$. Thus for any $X, Y \in \mathfrak{g}$,

$$-[X,Y] = \sigma_{j_0}[X,Y] = [\sigma_{j_0}(X), \sigma_{j_0}(Y)] = [-X,-Y] = [X,Y].$$

This implies that $\mathfrak{g}$ is abelian. That is, $G$ is a two-dimensional connected commutative Lie group.

If $\dim \varrho(H) = 1$, then we have $\dim H = 1$, since $\varrho$ is a faithful representation. So in the weakly symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h})$, $\dim \mathfrak{h} = 1$. Then there exists a Lorentzian inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{p}$ which is invariant under the actions of $\varrho(H)$. Thus, there exists a basis $\{\varepsilon_1, \varepsilon_2\}$ of $\mathfrak{p}$ satisfying

$$\langle \varepsilon_i, \varepsilon_j \rangle = (-1)^{i-1}\delta_{ij} \quad \forall\, i, j = 1, 2.$$

Moreover, there exists $X \neq 0$, $X \in \mathfrak{h}$, such that the matrix of $\operatorname{ad} X$ with respect to the above basis is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then we have

$$[X, [\varepsilon_1, \varepsilon_2]] = [[X, \varepsilon_1], \varepsilon_2] + [\varepsilon_1, [X, \varepsilon_2]] = 0.$$

Hence, there exists $a \in \mathbb{R}$ such that $[\varepsilon_1, \varepsilon_2] = aX$. Define an endomorphism $\sigma$ on $\mathfrak{g}$ by

$$\sigma(X) = X, \quad \sigma(\varepsilon_1) = -\varepsilon_1, \quad \sigma(\varepsilon_2) = -\varepsilon_2.$$

It is easy to check that $\sigma$ keeps the Lie brackets invariant. Thus, $\sigma$ is an automorphism of the Lie algebra. Moreover, we have $\sigma^2 = \mathrm{id}$. It follows that $(\mathfrak{g}, \mathfrak{h})$ is a Lorentzian weakly symmetric Lie algebra with respect to $(\mathrm{id}, \sigma)$. This completes the proof of the theorem. $\square$

## 5. Pseudo-Riemannian weakly symmetric manifolds in dimension 3

In this section we consider the classification of 3-dimensional reductive pseudo-Riemannian coset spaces. As in the 2-dimensional case, we only need to consider the Lorentzian spaces. The main result is the following theorem:

**Theorem 5.1.** *Let $(G/H, \langle \cdot, \cdot \rangle)$ be a reductive Lorentzian weakly symmetric coset space, and $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{p}$ and $\varrho$ be as in Section 3. Then $(\mathfrak{g}, \mathfrak{h})$ must be one of the following cases:*

(1) $\dim \mathfrak{g} = 3$, $\mathfrak{h} = \{0\}$, *and $\mathfrak{g}$ is an abelian Lie algebra. In this case, the set of automorphisms can be chosen as* $\{\mathrm{id}, -\mathrm{id}\}$.

(2) $\dim \mathfrak{g} = 4$, $\dim \mathfrak{h} = 1$, *and $\mathfrak{g}$ is an abelian Lie algebra. In this case, the set of automorphisms can be chosen as* $\{\mathrm{id}, -\mathrm{id}\}$.

(3) $\dim \mathfrak{g} = 4$, $\dim \mathfrak{h} = 1$, *and* $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ *(direct sum), where $\mathfrak{p}$ is an abelian ideal of $\mathfrak{g}$ and there exists a basis $X$ of $\mathfrak{h}$ and $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ of $\mathfrak{p}$ such that*

$$[X, \varepsilon_1] = -\varepsilon_2, \quad [X, \varepsilon_2] = \varepsilon_1, \quad [X, \varepsilon_3] = 0.$$

*In this case, the Lorentzian metric* $\langle \cdot, \cdot \rangle$ *on* $\mathfrak{p}$ *is defined by*

$$\langle \varepsilon_1, \varepsilon_1 \rangle = \langle \varepsilon_2, \varepsilon_2 \rangle = -\langle \varepsilon_3, \varepsilon_3 \rangle = 1,$$

*and the set of automorphisms can be chosen as* $\{\mathrm{id}, \sigma\}$, *where* $\sigma$ *is defined by*

$$\sigma(X) = -X, \quad \sigma(\varepsilon_1) = -\varepsilon_1, \quad \sigma(\varepsilon_2) = \varepsilon_2, \quad \sigma(\varepsilon_3) = -\varepsilon_3.$$

(4) $\dim \mathfrak{g} = 4$, $\dim \mathfrak{h} = 1$, *and* $\mathfrak{g} = \mathfrak{h} + \mathfrak{n}$ (*direct sum*), *where* $\mathfrak{n}$ *is an ideal of* $\mathfrak{g}$. *In this case, there exists a basis* $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ *of* $\mathfrak{n}$ *such that* $[\varepsilon_1, \varepsilon_2] = \varepsilon_3$, *i.e.,* $\mathfrak{n}$ *is the 3-dimensional Heisenberg Lie algebra. The action of a nonzero vector* $X$ *of* $\mathfrak{h}$ *on* $\mathfrak{n}$ *is the same as that in case* (3), *and the Lorentzian metric and the set of automorphisms can be chosen in the same way as those in case* (3).

(5) $\dim \mathfrak{g} = 4$, $\dim \mathfrak{h} = 1$, *and* $\mathfrak{g} = \mathfrak{h} + \mathfrak{su}(2)$ (*direct sum*), *where* $\mathfrak{su}(2)$ *is an ideal of* $\mathfrak{g}$. *In this case, there exists a basis* $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ *of* $\mathfrak{su}(2)$ *such that*

$$[\varepsilon_1, \varepsilon_2] = \varepsilon_3, \quad [\varepsilon_2, \varepsilon_3] = \varepsilon_1, \quad [\varepsilon_3, \varepsilon_1] = \varepsilon_2.$$

*The action of a nonzero vector* $X$ *of* $\mathfrak{h}$ *on* $\mathfrak{su}(2)$ *is the same as that in case* (3), *and the Lorentzian metric and the set of automorphisms can be chosen in the same way as those in case* (3).

(6) $\dim \mathfrak{g} = 4$, $\dim \mathfrak{h} = 1$, $\mathfrak{g} = \mathfrak{h} + \mathfrak{sl}(2)$ (*direct sum*), *where* $\mathfrak{sl}(2)$ *is an ideal of* $\mathfrak{g}$. *That is, there exists a basis* $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ *of* $\mathfrak{sl}(2)$ *such that*

$$[\varepsilon_1, \varepsilon_2] = \varepsilon_3, \quad [\varepsilon_2, \varepsilon_3] = -\varepsilon_1, \quad [\varepsilon_1, \varepsilon_3] = \varepsilon_2.$$

*The action of a nonzero vector* $X$ *of* $\mathfrak{h}$ *on* $\mathfrak{sl}(2)$ *is the same as that in case* (3), *and the Lorentzian metric and the set of automorphisms can be chosen in the same way as those in case* (3).

(7) $\dim \mathfrak{g} = 4$, $\dim \mathfrak{h} = 1$, $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ (*direct sum*), *where there exists a basis* $X$ *of* $\mathfrak{h}$ *and* $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ *of* $\mathfrak{p}$ *such that*

$$[X, \varepsilon_1] = 0, \quad [X, \varepsilon_2] = \varepsilon_3, \quad [X, \varepsilon_3] = \varepsilon_2, [\varepsilon_1, \varepsilon_2] = [\varepsilon_1, \varepsilon_3] = 0, \quad [\varepsilon_2, \varepsilon_3] = aX.$$

*In this case, the Lorentzian metric* $\langle \cdot, \cdot \rangle$ *on* $\mathfrak{p}$ *can be chosen in the same way as that in case* (3) *and the set of automorphisms can be chosen as* $\{\mathrm{id}, \sigma\}$, *where* $\sigma$ *is defined by*

$$\sigma(X) = X, \quad \sigma(\varepsilon_1) = -\varepsilon_1, \quad \sigma(\varepsilon_2) = -\varepsilon_2, \quad \sigma(\varepsilon_3) = -\varepsilon_3.$$

(8) $\dim \mathfrak{g} = 4$, $\dim \mathfrak{h} = 1$, *and* $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ (*direct sum*), *where there exists a basis* $X$ *of* $\mathfrak{h}$ *and* $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ *of* $\mathfrak{p}$ *such that*

$$[X, \varepsilon_1] = -\varepsilon_2, \quad [X, \varepsilon_2] = \varepsilon_1 + \varepsilon_3, \quad [X, \varepsilon_3] = \varepsilon_2,$$
$$[\varepsilon_1, \varepsilon_2] = [\varepsilon_2, \varepsilon_3] = aX, \quad [\varepsilon_1, \varepsilon_3] = 0.$$

*In this case, the Lorentzian metric and the set of automorphisms can be chosen in the same way as those in case* (7).

(9) $\dim \mathfrak{g} = 6$, $\dim \mathfrak{h} = 3$, *and* $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ (*direct sum*), *where there exists a basis* $\{A, B, C\}$ *of* $\mathfrak{h}$ *and* $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ *of* $\mathfrak{p}$ *such that*

$$\begin{aligned}
&[A, B] = -C, && [A, C] = B, && [B, C] = A,\\
&[A, \varepsilon_1] = -\varepsilon_2, && [A, \varepsilon_2] = \varepsilon_1, && [A, \varepsilon_3] = 0,\\
&[B, \varepsilon_1] = \varepsilon_3, && [B, \varepsilon_2] = 0, && [B, \varepsilon_3] = \varepsilon_1,\\
&[C, \varepsilon_1] = 0, && [C, \varepsilon_2] = \varepsilon_3, && [C, \varepsilon_3] = \varepsilon_2,\\
&[\varepsilon_1, \varepsilon_2] = -aA + b\varepsilon_3, && [\varepsilon_1, \varepsilon_3] = aB + b\varepsilon_2, && [\varepsilon_2, \varepsilon_3] = aC - b\varepsilon_1.
\end{aligned}$$

*In this case, the Lorentzian metric* $\langle \cdot, \cdot \rangle$ *on* $\mathfrak{p}$ *can be chosen in the same way as that in case* (3) *and the set of automorphisms can be chosen as* $\{\mathrm{id}, \sigma\}$, *where* $\sigma$ *is defined by*

$$\sigma(A) = -A, \quad \sigma(B) = B, \quad \sigma(C) = -C,$$
$$\sigma(\varepsilon_1) = -\varepsilon_1, \quad \sigma(\varepsilon_2) = \varepsilon_2, \quad \sigma(\varepsilon_3) = -\varepsilon_3.$$

The classification of 3-dimensional pseudo-Riemannian weakly symmetric manifolds follows from Theorem 5.1, Theorem 3.1, Theorem 3.2, and the classification of Riemannian weakly symmetric manifolds in dimension 3 given in [4]. The rest of this section is devoted to proving Theorem 5.1.

Proof of Theorem 5.1. First note that $\varrho(H) \subset \mathrm{SO}(2,1)$, where $\mathrm{SO}(2,1)$ is defined to be the identity component of the group of linear isometries of the inner product on $\mathfrak{p}$. If $\varrho(H) \neq \mathrm{SO}(2,1)$, then we have $\dim \varrho(H) \leqslant \frac{1}{2}(3-1)(3-2) = 1$ by a result of Montgomery and Samelson (see [9]). Therefore we have the following cases:

*Case 1*: $\dim \varrho(H) = 0$. The situation here is very similar to the 2-dimensional case. In fact, we can prove in exactly the same way as in the above section that $\mathfrak{g}$ is a 3-dimensional abelian Lie algebra.

*Case 2*: $\dim \varrho(H) = 1$. In this case, we have $\dim H = 1$ since $\varrho$ is a faithful representation. So in the weakly symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h})$, $\dim \mathfrak{h} = 1$. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$

be the corresponding reductive decomposition. Then there exists a Lorentzian inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{p}$ which is invariant under the actions of $\varrho(H)$. Thus, there exists a basis $\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}$ of $\mathfrak{p}$ satisfying

$$\langle\varepsilon_1,\varepsilon_1\rangle = \langle\varepsilon_2,\varepsilon_2\rangle = -\langle\varepsilon_3,\varepsilon_3\rangle = 1 \text{ and others zero}$$

such that for any $X\in\mathfrak{h}$ the matrix of $\operatorname{ad}X$ is $\begin{pmatrix} 0 & c & a\\ -c & 0 & b\\ a & b & 0\end{pmatrix}$ with respect to $\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}$.

First assume that $c=0$. If $a=b=0$, then for any $X\neq 0$, $X\in\mathfrak{h}$, $e^{t\operatorname{ad}X}$ acts on $\mathfrak{p}$ as the identity transformation for any $t\in\mathbb{R}$. Thus, there must be some $j_0$ such that $\sigma_{j_0}(Y)=-Y$ for any $Y\in\mathfrak{p}$. It follows that for any $X,Y\in\mathfrak{p}$

$$-[X,Y] = \sigma_{j_0}[X,Y] = [\sigma_{j_0}(X),\sigma_{j_0}(Y)] = [-X,-Y] = [X,Y].$$

Hence, $\mathfrak{p}$ is an abelian ideal of $\mathfrak{g}$. Assume that neither $a$ nor $b$ is zero. Let

$$\varepsilon_1' = \frac{1}{\sqrt{a^2+b^2}}(b\varepsilon_1 - a\varepsilon_2),\quad \varepsilon_2' = \frac{1}{\sqrt{a^2+b^2}}(a\varepsilon_1 + b\varepsilon_2),\quad \varepsilon_3' = \varepsilon_3.$$

Then we have

$$\langle\varepsilon_1',\varepsilon_1'\rangle = \langle\varepsilon_2',\varepsilon_2'\rangle = -\langle\varepsilon_3',\varepsilon_3'\rangle = 1 \text{ and others zero}$$

and the matrix of $(\operatorname{ad}X)/\sqrt{a^2+b^2}$ is $\begin{pmatrix} 0&0&0\\ 0&0&1\\ 0&1&0\end{pmatrix}$ with respect to $\{\varepsilon_1',\varepsilon_2',\varepsilon_3'\}$.

Next assume that $c\neq 0$. If $a=b=0$, then the matrix of $\operatorname{ad}(X/c)$ is $\begin{pmatrix} 0&1&0\\ -1&0&0\\ 0&0&0\end{pmatrix}$. Now we further assume that neither $a$ nor $b$ is zero. Let

$$\varepsilon_1' = \frac{1}{\sqrt{a^2+b^2}}(b\varepsilon_1 - a\varepsilon_2),\quad \varepsilon_2' = \frac{1}{\sqrt{a^2+b^2}}(a\varepsilon_1 + b\varepsilon_2),\quad \varepsilon_3' = \varepsilon_3.$$

Then we have

$$\langle\varepsilon_1',\varepsilon_1'\rangle = \langle\varepsilon_2',\varepsilon_2'\rangle = -\langle\varepsilon_3',\varepsilon_3'\rangle = 1,\quad \langle\varepsilon_i',\varepsilon_j'\rangle = 0\quad\text{for } i\neq j$$

and the matrix of $\operatorname{ad}X$ with respect to $\{\varepsilon_1',\varepsilon_2',\varepsilon_3'\}$ is

$$\begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & \sqrt{a^2+b^2}\\ 0 & \sqrt{a^2+b^2} & 0\end{pmatrix}.$$

If $\sqrt{a^2+b^2}=1$, then the matrix is equal to $\begin{pmatrix} 0&1&0\\ -1&0&1\\ 0&1&0\end{pmatrix}$. If $0<\sqrt{a^2+b^2}<1$, let

$$\varepsilon_1'' = \frac{\varepsilon_1' + \sqrt{a^2+b^2}\varepsilon_3'}{\sqrt{1-(a^2+b^2)}},\quad \varepsilon_2'' = \varepsilon_2',\quad \varepsilon_3'' = \frac{\sqrt{a^2+b^2}\varepsilon_1' + \varepsilon_3'}{\sqrt{1-(a^2+b^2)}}.$$

Then we have

$$\langle \varepsilon_1'', \varepsilon_1'' \rangle = \langle \varepsilon_2'', \varepsilon_2'' \rangle = -\langle \varepsilon_3'', \varepsilon_3'' \rangle = 1, \quad \langle \varepsilon_i'', \varepsilon_j'' \rangle = 0 \quad \text{for } i \neq j,$$

and the matrix of $(\operatorname{ad} X)/\sqrt{1-(a^2+b^2)}$ is $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ with respect to $\{\varepsilon_1'', \varepsilon_2'', \varepsilon_3''\}$. If $1 < \sqrt{a^2+b^2}$, let

$$\varepsilon_1'' = \frac{\sqrt{a^2+b^2}\varepsilon_1' + \varepsilon_3'}{\sqrt{a^2+b^2-1}}, \quad \varepsilon_2'' = \varepsilon_2', \quad \varepsilon_3'' = \frac{\varepsilon_1' + \sqrt{a^2+b^2}\varepsilon_3'}{\sqrt{a^2+b^2-1}}.$$

Then we have

$$\langle \varepsilon_1'', \varepsilon_1'' \rangle = \langle \varepsilon_2'', \varepsilon_2'' \rangle = -\langle \varepsilon_3'', \varepsilon_3'' \rangle = 1, \quad \langle \varepsilon_i'', \varepsilon_j'' \rangle = 0 \quad \text{for } i \neq j,$$

and the matrix of $(\operatorname{ad} X)/\sqrt{a^2+b^2-1}$ is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ with respect to $\{\varepsilon_1'', \varepsilon_2'', \varepsilon_3''\}$.

In summary, we have proved:

**Proposition 5.1.** *Let $(M, \langle \cdot, \cdot \rangle)$ be a reductive Lorentzian weakly symmetric space and $(\mathfrak{g}, \mathfrak{h})$ be the corresponding Lorentzian weakly symmetric Lie algebra with a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where $\dim \mathfrak{h} = 1$ and $\dim \mathfrak{p} = 3$. Then there exists a basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ of $\mathfrak{p}$ and $X \neq 0$, $X \in \mathfrak{h}$ such that*

$$\langle \varepsilon_1, \varepsilon_1 \rangle = \langle \varepsilon_2, \varepsilon_2 \rangle = -\langle \varepsilon_3, \varepsilon_3 \rangle = 1, \quad \langle \varepsilon_1, \varepsilon_1 \rangle = 0 \quad \textit{for } i \neq j,$$

*and the matrix of* $\operatorname{ad} X$ *is*

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

*with respect to* $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$.

If the matrix of $\operatorname{ad} X$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then $e^{\operatorname{ad} tX}$ acts as the identity transformation on $\mathfrak{p}$ for any $t \in \mathbb{R}$. Therefore there exists a $j_0$ such that $\sigma_{j_0}(Y) = -Y$ for any $Y \in \mathfrak{p}$. Thus for any $Y_1, Y_2 \in \mathfrak{g}$,

$$-[Y_1, Y_2] = \sigma_{j_0}[Y_1, Y_2] = [\sigma_{j_0}(Y_1), \sigma_{j_0}(Y_2)] = [-Y_1, -Y_2] = [Y_1, Y_2].$$

That is, $\mathfrak{g}$ is an abelian Lie algebra of dimension 4.

If the matrix of $\operatorname{ad} X$ is
$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
then for the basis $\{X, \varepsilon_1, \varepsilon_2, \varepsilon_3\}$ of $\mathfrak{g}$,

$$[X, \varepsilon_1] = -\varepsilon_2, \quad [X, \varepsilon_2] = \varepsilon_1, \quad [X, \varepsilon_3] = 0.$$

Let $V$ be the subspace of $\mathfrak{p}$ spanned by $\{\varepsilon_1, \varepsilon_2\}$. It is easy to see that $\operatorname{ad} X(\mathfrak{g}) \subset V$. Now we determine the Lie brackets $[\varepsilon_i, \varepsilon_j]$ for $i, j = 1, 2, 3$. Taking into account the skew-symmetry of the Lie bracket, it is enough to determine $[\varepsilon_1, \varepsilon_2]$, $[\varepsilon_1, \varepsilon_3]$ and $[\varepsilon_2, \varepsilon_3]$. By the Jacobi identity, we have

$$[X, [\varepsilon_1, \varepsilon_3]] = [[X, \varepsilon_1], \varepsilon_3] + [\varepsilon_1, [X, \varepsilon_3]].$$

Thus $[X, [\varepsilon_1, \varepsilon_3]] = -[\varepsilon_2, \varepsilon_3]$. It follows that

$$[\varepsilon_2, \varepsilon_3] = a\varepsilon_1 + b\varepsilon_2. \tag{5.1}$$

Similarly, we have $[X, [\varepsilon_2, \varepsilon_3]] = [\varepsilon_1, \varepsilon_3]$. Taking into account equation (5.1), we have

$$[\varepsilon_1, \varepsilon_3] = b\varepsilon_1 - a\varepsilon_2. \tag{5.2}$$

Now from the identity

$$[X, [\varepsilon_1, \varepsilon_2]] = [[X, \varepsilon_1], \varepsilon_2] + [\varepsilon_1, [X, \varepsilon_2]] = [-\varepsilon_2, \varepsilon_2] + [\varepsilon_1, \varepsilon_1] = 0,$$

we know that $[\varepsilon_1, \varepsilon_2] \in \operatorname{span}(X, \varepsilon_3)$. Therefore we can write

$$[\varepsilon_1, \varepsilon_2] = dX + c\varepsilon_3. \tag{5.3}$$

Clearly, there exists a $i$ such that

$$e^{\operatorname{ad} tX} \sigma_i(\varepsilon_1 + x\varepsilon_2 + x^2\varepsilon_3) = -(\varepsilon_1 + x\varepsilon_2 + x^2\varepsilon_3) \tag{5.4}$$

for infinitely many $x \in \mathbb{R}$. Assume that

$$\sigma_i(\varepsilon_1) = a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3, \quad \sigma_i(\varepsilon_2) = b_1\varepsilon_1 + b_2\varepsilon_2 + b_3\varepsilon_3, \quad \sigma_i(\varepsilon_3) = c_1\varepsilon_1 + c_2\varepsilon_2 + c_3\varepsilon_3.$$

Plugging this into equation (5.4) and comparing the coefficients of $\varepsilon_3$, we have

$$a_3 + xb_3 + x^2c_3 = -x^2.$$

It follows that

$$a_3 = b_3 = 0, \quad c_3 = -1.$$

Since $\sigma_i$ keeps the Lorentzian metric invariant, we have

$$c_1 = c_2 = 0, \quad a_1^2 + a_2^2 = b_1^2 + b_2^2 = 1, \quad a_1 b_1 + a_2 b_2 = 0.$$

Applying $\sigma_i$ to both sides of $[\varepsilon_1, \varepsilon_3] = b\varepsilon_1 - a\varepsilon_2$, we get

$$-(a_1 b + a_2 a)\varepsilon_1 + (a_1 a - a_2 b)\varepsilon_2 = (a_1 b - b_1 a)\varepsilon_1 + (a_2 b - b_2 a)\varepsilon_2.$$

It follows that

$$2a_1 b = (-a_2 + b_1)a, \quad 2a_2 b = (a_1 + b_2)a.$$

Then we have

$$2b = 2a_1^2 b + 2a_2^2 b = (-a_2 + b_1)a_1 a + (a_1 + b_2)a_2 a = (a_1 b_1 + a_2 b_2)a = 0.$$

Therefore the Lie brackets of the Lie algebra $\mathfrak{g}$ are determined by

$$[X, \varepsilon_1] = -\varepsilon_2, \quad [X, \varepsilon_2] = \varepsilon_1, \quad [X, \varepsilon_3] = 0,$$
$$[\varepsilon_1, \varepsilon_2] = dX + c\varepsilon_3, \quad [\varepsilon_1, \varepsilon_3] = -a\varepsilon_2, \quad [\varepsilon_2, \varepsilon_3] = a\varepsilon_1,$$

where $a, b, c, d \in \mathbb{R}$. If $d \neq 0$, we set

$$\varepsilon_3' = X + \frac{c}{d}\varepsilon_3 \quad \text{and} \quad \mathfrak{p}' = \mathrm{span}(\varepsilon_1, \varepsilon_2, \varepsilon_3'),$$

then we have $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}'$, $[\mathfrak{h}, \mathfrak{p}'] \subset \mathfrak{p}'$, and the above brackets are still valid. Therefore it is enough to study the case when $d = 0$. Now we have the following subcases.

*Subcase 2.1*: $c = 0$. In this case, let $\varepsilon_3' = aX + \varepsilon_3$ and $\mathfrak{p}' = \mathrm{span}(\varepsilon_1, \varepsilon_2, \varepsilon_3')$. Then $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}'$, where $\mathfrak{p}'$ is an abelian ideal of $\mathfrak{g}$, and the action of $X$ on $\mathfrak{p}'$ is given by

$$[X, \varepsilon_1] = -\varepsilon_2, \quad [X, \varepsilon_2] = \varepsilon_1, \quad [X, \varepsilon_3'] = 0.$$

Let $\langle \cdot, \cdot \rangle$ be the Lorentzian metric on $\mathfrak{p}'$ defined by

$$\langle \varepsilon_1, \varepsilon_1 \rangle = \langle \varepsilon_2, \varepsilon_2 \rangle = -\langle \varepsilon_3', \varepsilon_3' \rangle = 1, \quad \langle \varepsilon_i, \varepsilon_j \rangle = 0 \quad \text{for } i \neq j$$

and let $\sigma$ be the endomorphism on $\mathfrak{g}$ defined by

$$\sigma(X) = -X, \quad \sigma(\varepsilon_1) = -\varepsilon_1, \quad \sigma(\varepsilon_2) = \varepsilon_2, \quad \sigma(\varepsilon_3') = -\varepsilon_3'.$$

Then it is easy to check that $\langle\cdot,\cdot\rangle$ is $\operatorname{ad} X$-invariant, that $\sigma$ keeps the Lie brackets invariant, i.e., it is a Lie algebra automorphism, and that $\sigma^2 = \mathrm{id}$.

Now we prove that $(\mathfrak{g}, \mathfrak{h})$ is a Lorentzian weakly symmetric Lie algebra with respect to $S = \{\mathrm{id}, \sigma\}$. For this, we only need to check (WSL3) in Definition 3.1. Note that $e^{\operatorname{ad} tX}(\varepsilon_3') = \varepsilon_3'$. Thus $e^{\operatorname{ad} tX}$ keeps $V_1 = \mathrm{span}(\varepsilon_1, \varepsilon_2)$ invariant and on $V_1$ it has the matrix

$$\begin{pmatrix} \cos t & \sin t \\ -\cos t & \sin t \end{pmatrix}$$

with respect to the basis $\{\varepsilon_1, \varepsilon_2\}$. Hence $e^{\operatorname{ad} tX}$ is a rotation of $t$ angle since $\{\varepsilon_1, \varepsilon_2\}$ is an orthonormal basis with respect to $\langle\cdot,\cdot\rangle$. Now for any element $\varepsilon = x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3'$ in $\mathfrak{p}$ we have $\sigma(\varepsilon) = -x_1\varepsilon_1 + x_2\varepsilon_2 - x_3\varepsilon_3'$. There exists $t_0 \in \mathbb{R}$ such that

$$e^{\operatorname{ad} t_0 X}(-x_1\varepsilon_1 + x_2\varepsilon_2) = -x_1\varepsilon_1 - x_2\varepsilon_2$$

since $-x_1\varepsilon_1 + x_2\varepsilon_2 \in V_1$ has the same length as $-x_1\varepsilon_1 - x_2\varepsilon_2$. Then

$$e^{\operatorname{ad} t_0 X}\sigma(\varepsilon) = e^{\operatorname{ad} t_0 X}(-x_1\varepsilon_1 + x_2\varepsilon_2 - x_3\varepsilon_3') = -x_1\varepsilon_1 - x_2\varepsilon_2 - x_3\varepsilon_3' = -\varepsilon.$$

Therefore $(\mathfrak{g}, \mathfrak{h})$ is a Lorentzian weakly symmetric Lie algebra.

*Subcase 2.2*: $c \neq 0$ and $a = 0$. Without loss of generality, we can assume that $c = 1$ by replacing $\varepsilon_3$ by $c\varepsilon_3$ if necessary. Then $\mathfrak{p}$ is a three-dimensional Heisenberg Lie algebra with the Lie brackets

$$[\varepsilon_1, \varepsilon_2] = \varepsilon_3, \quad [\varepsilon_1, \varepsilon_3] = [\varepsilon_2, \varepsilon_3] = 0,$$

and the action of $X$ on $\mathfrak{p}$ is given by

$$[X, \varepsilon_1] = -\varepsilon_2, \quad [X, \varepsilon_2] = \varepsilon_1, \quad [X, \varepsilon_3] = 0.$$

Let $\langle\cdot,\cdot\rangle$ be the Lorentzian metric on $\mathfrak{p}$ defined by

$$\langle\varepsilon_1, \varepsilon_1\rangle = \langle\varepsilon_2, \varepsilon_2\rangle = -\langle\varepsilon_3, \varepsilon_3\rangle = 1, \quad \langle\varepsilon_i, \varepsilon_j\rangle = 0 \quad \text{for } i \neq j,$$

and let $\sigma$ be the endomorphism on $\mathfrak{g}$ defined by

$$\sigma(X) = -X, \quad \sigma(\varepsilon_1) = -\varepsilon_1, \quad \sigma(\varepsilon_2) = \varepsilon_2, \quad \sigma(\varepsilon_3) = -\varepsilon_3.$$

Then a similar argument as in Subcase 2.1 shows that $(\mathfrak{g}, \mathfrak{h})$ is a Lorentzian weakly symmetric Lie algebra with respect to $S = \{\mathrm{id}, \sigma\}$.

*Subcase 2.3*: $ac > 0$. Let $\varepsilon_1' = \varepsilon_1/\sqrt{ac}$, $\varepsilon_2' = \varepsilon_2/\sqrt{ac}$ and $\varepsilon_3' = \varepsilon_3/a$. Then $\mathfrak{p}$ is an ideal of $\mathfrak{g}$ with the brackets

$$[\varepsilon_1', \varepsilon_2'] = \varepsilon_3', \quad [\varepsilon_3', \varepsilon_1'] = \varepsilon_2', \quad [\varepsilon_2', \varepsilon_3'] = \varepsilon_1',$$

and the action of $X$ on $\mathfrak{p}$ is given by

$$[X, \varepsilon_1'] = -\varepsilon_2', \quad [X, \varepsilon_2'] = \varepsilon_1', \quad [X, \varepsilon_3'] = 0.$$

Clearly, $\mathfrak{p}$ is isomorphic with the real compact simple Lie algebra $\mathfrak{su}(2)$. Let $\langle \cdot, \cdot \rangle$ be the Lorentzian metric on $\mathfrak{p}$ defined by

$$\langle \varepsilon_1', \varepsilon_1' \rangle = \langle \varepsilon_2', \varepsilon_2' \rangle = -\langle \varepsilon_3', \varepsilon_3' \rangle = 1 \quad \text{and} \quad \langle \varepsilon_i', \varepsilon_j' \rangle = 0 \quad \text{for } i \neq j,$$

and let $\sigma$ be the endomorphism on $\mathfrak{g}$ defined by

$$\sigma(X) = -X, \quad \sigma(\varepsilon_1') = -\varepsilon_1', \quad \sigma(\varepsilon_2') = \varepsilon_2', \quad \sigma(\varepsilon_3') = -\varepsilon_3'.$$

Then a similar argument as in Subcase 2.1 shows that $(\mathfrak{g}, \mathfrak{h})$ is a Lorentzian weakly symmetric Lie algebra with respect to $S = \{\mathrm{id}, \sigma\}$.

*Subcase 2.4*: $ac < 0$. Let $\varepsilon_1' = \varepsilon_1/\sqrt{-ac}$, $\varepsilon_2' = \varepsilon_2/\sqrt{-ac}$ and $\varepsilon_3' = -\varepsilon_3/a$. Then $\mathfrak{p}$ is an ideal of $\mathfrak{g}$ with the brackets

$$[\varepsilon_1', \varepsilon_2'] = \varepsilon_3', \quad [\varepsilon_1', \varepsilon_3'] = \varepsilon_2', \quad [\varepsilon_2', \varepsilon_3'] = -\varepsilon_1',$$

and the action of $X$ on $\mathfrak{p}$ is given by

$$[X, \varepsilon_1'] = -\varepsilon_2', \quad [X, \varepsilon_2'] = \varepsilon_1', \quad [X, \varepsilon_3'] = 0.$$

Clearly, $\mathfrak{p}$ is isomorphic with the real simple Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. Let $\langle \cdot, \cdot \rangle$ be the Lorentzian metric on $\mathfrak{p}$ defined by

$$\langle \varepsilon_1', \varepsilon_1' \rangle = \langle \varepsilon_2', \varepsilon_2' \rangle = -\langle \varepsilon_3', \varepsilon_3' \rangle = 1 \quad \text{and} \quad \langle \varepsilon_i', \varepsilon_j' \rangle = 0 \quad \text{for } i \neq j,$$

and let $\sigma$ be the endomorphism on $\mathfrak{g}$ defined by

$$\sigma(X) = -X, \quad \sigma(\varepsilon_1') = -\varepsilon_1', \quad \sigma(\varepsilon_2') = \varepsilon_2', \quad \sigma(\varepsilon_3') = -\varepsilon_3'.$$

Similarly as in Subcase 2.1, $(\mathfrak{g}, \mathfrak{h})$ is a Lorentzian weakly symmetric Lie algebra with respect to $S = \{\mathrm{id}, \sigma\}$.

If the matrix of $\mathrm{ad}X$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

then with respect to the basis $\{X, \varepsilon_1, \varepsilon_2, \varepsilon_3\}$ of $\mathfrak{g}$,

$$[X, \varepsilon_1] = 0, \quad [X, \varepsilon_2] = \varepsilon_3, \quad [X, \varepsilon_3] = \varepsilon_2.$$

Let $V$ be the subspace of $\mathfrak{p}$ spanned by $\{\varepsilon_2, \varepsilon_3\}$. It is easy to see that $\mathrm{ad}\, X(\mathfrak{g}) \subset V$. Now we determine the Lie brackets $[\varepsilon_i, \varepsilon_j]$ for $i, j = 1, 2, 3$. Taking into account the skew-symmetry of the Lie bracket, it is enough to determine $[\varepsilon_1, \varepsilon_2]$, $[\varepsilon_1, \varepsilon_3]$ and $[\varepsilon_2, \varepsilon_3]$. By the Jacobi identity, we have

$$[X, [\varepsilon_1, \varepsilon_3]] = [[X, \varepsilon_1], \varepsilon_3] + [\varepsilon_1, [X, \varepsilon_3]].$$

Thus $[X, [\varepsilon_1, \varepsilon_3]] = [\varepsilon_1, \varepsilon_2]$. It follows that

$$[\varepsilon_1, \varepsilon_2] = a\varepsilon_2 + b\varepsilon_3. \tag{5.5}$$

Similarly, we have $[X, [\varepsilon_1, \varepsilon_2]] = [\varepsilon_1, \varepsilon_3]$. Therefore we have

$$[\varepsilon_1, \varepsilon_3] = b\varepsilon_2 + a\varepsilon_3. \tag{5.6}$$

Now it follows from the identity

$$[X, [\varepsilon_2, \varepsilon_3]] = [[X, \varepsilon_2], \varepsilon_3] + [\varepsilon_2, [X, \varepsilon_3]] = [\varepsilon_3, \varepsilon_3] + [\varepsilon_2, \varepsilon_2] = 0,$$

that $[\varepsilon_2, \varepsilon_3] \in \mathrm{span}(X, \varepsilon_1)$. Therefore we can write

$$[\varepsilon_2, \varepsilon_3] = dX + c\varepsilon_1. \tag{5.7}$$

Suppose $\sigma_j(X) = a_j^X X$. Since $[X, \varepsilon_2] = \varepsilon_3$, we have

$$\sigma_j(\varepsilon_3) = \sigma_j([X, \varepsilon_2]) = [\sigma_j(X), \sigma_j(\varepsilon_2)] = a_j^X [X, \sigma_j(\varepsilon_2)] \subset \mathrm{ad}\, X(\mathfrak{g}) \subset V.$$

Similarly, $\sigma_j(\varepsilon_2) \subset V$. Clearly, there exists a $i$ such that

$$e^{\mathrm{ad}\, tX} \sigma_i(\varepsilon_1 + x\varepsilon_2 + x\varepsilon_3) = -(\varepsilon_1 + x\varepsilon_2 + x\varepsilon_3) \tag{5.8}$$

for infinitely many $x \in \mathbb{R}$. Assume that

$$\sigma_i(\varepsilon_1) = a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3, \quad \sigma_i(\varepsilon_2) = b_2\varepsilon_2 + b_3\varepsilon_3, \quad \sigma_i(\varepsilon_3) = c_2\varepsilon_2 + c_3\varepsilon_3.$$

Plugging the above equations into equation (5.8) and comparing the coefficients of $\varepsilon_1$, we have

$$a_1 = -1.$$

Since $\sigma_i$ keeps the Lorentzian metric invariant, we have

$$a_2 = a_3 = 0, \quad b_2^2 - b_3^2 = 1, \quad c_2^2 - c_3^2 = -1, \quad b_2c_2 - b_3c_3 = 0. \tag{5.9}$$

Applying $\sigma_i$ to both sides of $[\varepsilon_1, \varepsilon_2] = a\varepsilon_2 + b\varepsilon_3$, we get

$$-(b_2a + b_3b)\varepsilon_2 - (b_2b + b_3a)\varepsilon_3 = (b_2a + c_2b)\varepsilon_2 + (b_3a + c_3b)\varepsilon_3.$$

It follows that

$$2b_2a = -(c_2 + b_3)b, \quad 2b_3a = -(b_2 + c_3)b. \tag{5.10}$$

Then we have

$$2a = 2b_2^2a - 2b_3^2a = -b_2(c_2 + b_3)b + b_3(b_2 + c_3)b = (b_3c_3 - b_2c_2)b = 0.$$

Now we assert that $b = 0$. In fact, if $b \neq 0$, then by equation (5.10) we have

$$b_2 = -c_3, \quad b_3 = -c_2.$$

Then it follows from equation (5.8) and the fact that $[X, \varepsilon_2 - \varepsilon_3] = -(\varepsilon_2 - \varepsilon_3)$ that

$$\begin{aligned} -(\varepsilon_1 + x\varepsilon_2 + x\varepsilon_3) = e^{\operatorname{ad} tX}\sigma_i(\varepsilon_1 + x\varepsilon_2 + x\varepsilon_3) &= e^{\operatorname{ad} tX}(-\varepsilon_1 + x(b_2 - b_3)(\varepsilon_2 - \varepsilon_3)) \\ &= -\varepsilon_1 + e^{-t}x(b_2 - b_3)(\varepsilon_2 - \varepsilon_3), \end{aligned}$$

which is a contradiction. Thus $b = 0$. A similar argument shows that

$$b_2 = c_3, \quad b_3 = c_2.$$

Applying $\sigma_i$ to both sides of $[X, \varepsilon_2] = \varepsilon_3$, we have

$$a_i^X b_2 = b_2, \quad a_i^X b_3 = b_3,$$

which implies that $a_i^X = 1$. Now applying $\sigma_i$ to both sides of equation (5.7), we have

$$dX + c\varepsilon_1 = da_i^X X - c\varepsilon_1.$$

It follows that $c = 0$. Consequently, Lie brackets of the Lie algebra $\mathfrak{g}$ are determined by

$$[X, \varepsilon_1] = 0, \quad [X, \varepsilon_2] = \varepsilon_3, \quad [X, \varepsilon_3] = \varepsilon_2, \quad [\varepsilon_1, \varepsilon_2] = [\varepsilon_1, \varepsilon_3] = 0, \quad [\varepsilon_2, \varepsilon_3] = dX.$$

Let $\langle \cdot, \cdot \rangle$ be the Lorentzian metric on $\mathfrak{p}$ defined by

$$\langle \varepsilon_1, \varepsilon_1 \rangle = \langle \varepsilon_2, \varepsilon_2 \rangle = -\langle \varepsilon_3, \varepsilon_3 \rangle = 1 \quad \text{and} \quad \langle \varepsilon_i, \varepsilon_j \rangle = 0 \quad \text{for } i \neq j,$$

and let $\sigma$ be the endomorphism on $\mathfrak{g}$ defined by

$$\sigma(X) = X, \quad \sigma(\varepsilon_1) = -\varepsilon_1, \quad \sigma(\varepsilon_2) = -\varepsilon_2, \quad \sigma(\varepsilon_3) = -\varepsilon_3.$$

Then it is easy to check that $(\mathfrak{g}, \mathfrak{h})$ is a Lorentzian weakly symmetric Lie algebra with respect to $S = \{\mathrm{id}, \sigma\}$.

If the matrix of $\mathrm{ad}X$ is

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

then with respect to the basis $\{X, \varepsilon_1, \varepsilon_2, \varepsilon_3\}$ of $\mathfrak{g}$,

$$[X, \varepsilon_1] = -\varepsilon_2, \quad [X, \varepsilon_2] = \varepsilon_1 + \varepsilon_3, \quad [X, \varepsilon_3] = \varepsilon_2.$$

Assume that

$$[\varepsilon_1, \varepsilon_2] = aX + b\varepsilon_1 + c\varepsilon_2 + d\varepsilon_3. \tag{5.11}$$

Since $[X, [\varepsilon_1, \varepsilon_2]] = [[X, \varepsilon_1], \varepsilon_2] + [\varepsilon_1, [X, \varepsilon_2]] = [\varepsilon_1, \varepsilon_3]$, we have

$$[\varepsilon_1, \varepsilon_3] = c\varepsilon_1 + (d - b)\varepsilon_2 + c\varepsilon_3. \tag{5.12}$$

Since $[X, [\varepsilon_1, \varepsilon_3]] = [[X, \varepsilon_1], \varepsilon_3] + [\varepsilon_1, [X, \varepsilon_3]] = -[\varepsilon_2, \varepsilon_3] + [\varepsilon_1, \varepsilon_2]$, we have

$$[\varepsilon_2, \varepsilon_3] = aX + (2b - d)\varepsilon_1 + c\varepsilon_2 + b\varepsilon_3. \tag{5.13}$$

Clearly, there exists a $i$ such that

$$e^{\mathrm{ad}\, tX} \sigma_i(\varepsilon_1 + x\varepsilon_2 + (1 - x^2)\varepsilon_3) = -(\varepsilon_1 + x\varepsilon_2 + (1 - x^2)\varepsilon_3) \tag{5.14}$$

for infinitely many $x \in \mathbb{R}$. Here we can require that $x$ is bigger than any fixed positive real number. Assume that

$$\sigma_i(\varepsilon_1) = a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3, \quad \sigma_i(\varepsilon_2) = b_1\varepsilon_1 + b_2\varepsilon_2 + b_3\varepsilon_3, \quad \sigma_i(\varepsilon_3) = c_1\varepsilon_1 + c_2\varepsilon_2 + c_3\varepsilon_3.$$

Clearly $\{\varepsilon_1' = \varepsilon_1 + \varepsilon_3, \varepsilon_2, \varepsilon_3\}$ is another basis of $\mathfrak{p}$ and the action of $\operatorname{ad} X$ is

$$[X, \varepsilon_1'] = 0, \quad [X, \varepsilon_2] = \varepsilon_1', \quad [X, \varepsilon_3] = \varepsilon_2.$$

Let $f_i(x) = a_i + b_i x + c_i(1 - x^2)$ for $i = 1, 2, 3$. It follows that

$$\begin{aligned} e^{\operatorname{ad} tX}\sigma_i(\varepsilon_1 + x\varepsilon_2 + (1 - x^2)\varepsilon_3) &= e^{\operatorname{ad} tX}(f_1(x)\varepsilon_1 + f_2(x)\varepsilon_2 + f_3(x)\varepsilon_3) \\ &= e^{\operatorname{ad} tX}(f_1(x)\varepsilon_1' + f_2(x)\varepsilon_2 + (f_3(x) - f_1(x))\varepsilon_3) \\ &= -(\varepsilon_1 + x\varepsilon_2 + (1 - x^2)\varepsilon_3) = -\varepsilon_1' - x\varepsilon_2 + x^2\varepsilon_3. \end{aligned}$$

Comparing the coefficients of $\varepsilon_3$, we have $f_3(x) - f_1(x) = x^2$, i.e.,

$$a_3 - a_1 - 1 + (b_3 - b_1)x + (c_3 - c_1 + 1)(1 - x^2) = 0.$$

It follows that

$$a_3 = a_1 + 1, \quad b_3 = b_1, \quad c_3 = c_1 - 1.$$

Since $\sigma_i$ keeps $\langle \varepsilon_1, \varepsilon_1 \rangle$, $\langle \varepsilon_2, \varepsilon_2 \rangle$ and $\langle \varepsilon_3, \varepsilon_3 \rangle$, then we have

$$a_2^2 = 2a_1 + 2, \quad b_2^2 = 1, \quad c_2^2 = -2c_1. \tag{5.15}$$

Since $\sigma_i$ keeps $\langle \varepsilon_1, \varepsilon_2 \rangle$ and $\langle \varepsilon_2, \varepsilon_3 \rangle$, we have

$$a_2 b_2 = b_1, \quad b_2 c_2 = -b_1. \tag{5.16}$$

It follows that $a_2 = b_1 b_2$ and $c_2 = -b_1 b_2$. Let $\sigma_i(X) = a_i^X X$. Applying $\sigma_i$ to both sides of $[X, \varepsilon_2] = \varepsilon_1 + \varepsilon_3$, we have

$$a_i^X b_2(\varepsilon_1 + \varepsilon_3) = (a_1 + c_1)\varepsilon_1 + (a_2 + c_2)\varepsilon_2 + (a_3 + c_3)\varepsilon_3 = -(\varepsilon_1 + \varepsilon_3).$$

It implies that $a_i^X b_2 = -1$. That is, $a_i^X = -b_2$. Comparing the coefficients of $\varepsilon_1'$ in equation (5.14), we have

$$a_1 + b_1 x + c_1(1 - x^2) + \frac{t}{2}(a_2 + b_2 x + c_2(1 - x^2)) + \frac{t^2}{6}(a_3 - a_1 + (c_3 - c_1)(1 - x^2)) = -1,$$

that is,

$$t^2 x^2 + 3t(b_2 x - c_2 x^2) + 6(b_1 x - c_1 x^2) = 0. \tag{5.17}$$

Then the determinant of equation (5.17) is not less than 0, i.e.,

$$9(b_2 x - c_2 x^2)^2 - 24x^2(b_1 x - c_1 x^2) = -c_2^2 x^2 - 2b_1 x + 3 \geqslant 0.$$

By the choice of $x$, we must have $c_2 = 0$. It follows that

$$a_1 = c_3 = -1, \quad a_2 = a_3 = b_1 = b_3 = c_1 = c_2 = 0.$$

Comparing the coefficients of $\varepsilon_2$ in equation (5.14), we have

$$tx^2 + 2(b_1 + 1)x = 0.$$

Together with equation (5.17) we must have $b_2 = -1$. Then $a_i^X = 1$. That is,

$$\sigma_i(X) = X, \quad \sigma_i(\varepsilon_1) = -\varepsilon_1, \quad \sigma_i(\varepsilon_2) = -\varepsilon_2, \quad \sigma_i(\varepsilon_3) = -c_3\varepsilon_3.$$

Applying $\sigma_i$ to both sides of $[\varepsilon_1, \varepsilon_2] = aX + b\varepsilon_1 + c\varepsilon_2 + d\varepsilon_3$ we have

$$b = c = d = 0.$$

Up to now, we know that Lie brackets of the Lie algebra $\mathfrak{g}$ are determined by

$$[X, \varepsilon_1] = -\varepsilon_2, \quad [X, \varepsilon_2] = \varepsilon_1 + \varepsilon_3, \quad [X, \varepsilon_3] = \varepsilon_2,$$
$$[\varepsilon_1, \varepsilon_3] = 0, \quad [\varepsilon_1, \varepsilon_2] = [\varepsilon_2, \varepsilon_3] = aX.$$

Let $\langle \cdot, \cdot \rangle$ be the Lorentzian metric on $\mathfrak{p}$ defined by

$$\langle \varepsilon_1, \varepsilon_1 \rangle = \langle \varepsilon_2, \varepsilon_2 \rangle = -\langle \varepsilon_3, \varepsilon_3 \rangle = 1 \quad \text{and others zero}$$

and let $\sigma$ be the endomorphism on $\mathfrak{g}$ defined by

$$\sigma(X) = X, \quad \sigma(\varepsilon_1) = -\varepsilon_1, \quad \sigma(\varepsilon_2) = -\varepsilon_2, \quad \sigma(\varepsilon_3) = -\varepsilon_3.$$

It is easy to check that $(\mathfrak{g}, \mathfrak{h})$ is a Lorentzian weakly symmetric Lie algebra with respect to $S = \{\mathrm{id}, \sigma\}$.

*Case 3*: $\varrho(H) = \mathrm{SO}(2,1)$. Let $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ be a basis of $\mathfrak{p}$ satisfying

$$\langle \varepsilon_1, \varepsilon_1 \rangle = \langle \varepsilon_2, \varepsilon_2 \rangle = -\langle \varepsilon_3, \varepsilon_3 \rangle = 1 \quad \text{and others zero.}$$

Since $\varrho$ is faithful, we know that there exists a basis $\{A, B, C\}$ of $H$ such that

$$\begin{aligned} &[A, \varepsilon_1] = -\varepsilon_2, \quad &&[A, \varepsilon_2] = \varepsilon_1, \quad &&[A, \varepsilon_3] = 0, \\ &[B, \varepsilon_1] = \varepsilon_3, &&[B, \varepsilon_2] = 0, &&[B, \varepsilon_3] = \varepsilon_1, \\ &[C, \varepsilon_1] = 0, &&[C, \varepsilon_2] = \varepsilon_3, &&[C, \varepsilon_3] = \varepsilon_2. \end{aligned}$$

It follows that

$$[A, B] = -C, \quad [A, C] = B, \quad [B, C] = A.$$

Assume that $[\varepsilon_1, \varepsilon_3] = a_1 A + a_2 B + a_3 C + a_4 \varepsilon_1 + a_5 \varepsilon_2 + a_3 \varepsilon_3$. Since $[A, [\varepsilon_1, \varepsilon_3]] = -[\varepsilon_2, \varepsilon_3]$, we get

$$[\varepsilon_2, \varepsilon_3] = -a_3 B + a_2 C - a_5 \varepsilon_1 + a_4 \varepsilon_2.$$

Since $[A, [\varepsilon_2, \varepsilon_3]] = [\varepsilon_1, \varepsilon_3]$, we get

$$[\varepsilon_1, \varepsilon_3] = a_2 B + a_3 C + a_4 \varepsilon_1 + a_5 \varepsilon_2.$$

Since $[C, [\varepsilon_1, \varepsilon_3]] = [\varepsilon_1, \varepsilon_2]$, we get

$$[\varepsilon_1, \varepsilon_2] = -a_2 A + a_5 \varepsilon_3.$$

By $[C, [\varepsilon_1, \varepsilon_2]] = [\varepsilon_1, \varepsilon_3]$, we get

$$[\varepsilon_1, \varepsilon_3] = a_2 B + a_5 \varepsilon_2.$$

By $[B, [\varepsilon_1, \varepsilon_2]] = -[\varepsilon_2, \varepsilon_3]$, we get

$$[\varepsilon_2, \varepsilon_3] = a_2 C - a_5 \varepsilon_1.$$

Let $\sigma$ be the endomorphism on $\mathfrak{g}$ defined by

$$\sigma(A) = -A,\ \sigma(B) = B,\ \sigma(C) = -C,\ \sigma(\varepsilon_1) = -\varepsilon_1,\ \sigma(\varepsilon_2) = \varepsilon_2,\ \sigma(\varepsilon_3) = -\varepsilon_3.$$

For any element $\varepsilon = x_1 \varepsilon_1 + x_2 \varepsilon_2 + x_3 \varepsilon_3$ in $\mathfrak{p}$ we have $\sigma(\varepsilon) = -x_1 \varepsilon_1 + x_2 \varepsilon_2 - x_3 \varepsilon_3$. There exists $t_0 \in \mathbb{R}$ such that

$$e^{\operatorname{ad} t_0 A}(-x_1 \varepsilon_1 + x_2 \varepsilon_2) = -x_1 \varepsilon_1 - x_2 \varepsilon_2$$

since $-x_1 \varepsilon_1 + x_2 \varepsilon_2 \in V_1$ has the same length as $-x_1 \varepsilon_1 - x_2 \varepsilon_2$. Then

$$e^{\operatorname{ad} t_0 X} \sigma(\varepsilon) = e^{\operatorname{ad} t_0 X}(-x_1 \varepsilon_1 + x_2 \varepsilon_2 - x_3 \varepsilon_3) = -x_1 \varepsilon_1 - x_2 \varepsilon_2 - x_3 \varepsilon_3 = -\varepsilon.$$

Therefore $(\mathfrak{g}, \mathfrak{h})$ is a Lorentzian weakly symmetric Lie algebra.

The proof of Theorem 5.1 is now completed. □

## 6. Symmetry of low dimensional pseudo-Riemannian manifolds

This section studies some problems related to the symmetry of pseudo-Riemannian manifolds based on the classification of the 3-dimensional pseudo-Riemannian weakly symmetric manifolds in the previous section.

We first generalize the definition of $k$-fold symmetric Riemmanian manifolds to the pseudo-Riemannian case. Let $(M,Q)$ be an $n$-dimensional connected pseudo-Riemannian manifold, where $n \geqslant 3$. If given any point $p \in M$ and $k$ tangent vectors $\xi_1, \xi_2, \ldots, \xi_k$, there exists isometry $\sigma$ of $(M,Q)$ such that $\sigma(p) = p$ and $\mathrm{d}\sigma|_p(\xi_i) = -\xi_i$, $i = 1, 2, \ldots, k$, then $(M,Q)$ is called $k$-fold symmetric. Obviously, a 1-fold symmetric space is just a weakly symmetric space, and if $k \geqslant 2$, then a $k$-fold symmetric space is globally symmetric. It is the main result of [5] that any 2-fold symmetric Riemannian manifold must be globally symmetric.

Let $G/H$ be a homogenous space with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of $G$ and $H$, respectively. If $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h},\mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{h}$, then there exists an involution $\sigma$ of $\mathfrak{g}$ such that

$$\sigma|_{\mathfrak{h}} = \mathrm{id}, \quad \sigma|_{\mathfrak{p}} = -\mathrm{id}.$$

It follows that $G/H$ is a symmetric space. This observation combined with Theorems 4.1 and 5.1 implies the following theorems.

**Theorem 6.1.** *Any reductive Lorentzian weakly symmetric coset space of dimension* 2 *is a symmetric space.*

**Theorem 6.2.** *Let $G/H$ be a reductive Lorentzian weakly symmetric coset space in dimension* 3*. Then $G/H$ is a symmetric space if and only if $G/H$ is one of cases* (1), (2), (3), (7), (8), *and case* (9) *with $b = 0$ in Theorem* 5.1.

Now we can prove the following theorem.

**Theorem 6.3.** *Any reductive* 2*-fold symmetric pseudo-Riemannian weakly symmetric manifold is globally symmetric.*

P r o o f. It suffices to prove that the spaces in cases (4), (5), (6), and (9) with $b \neq 0$ in Theorem 5.1 are not 2-fold symmetric.

First assume that a space in cases (4) and (5) in Theorem 5.1 is 2-fold symmetric. Then for $\varepsilon_2$ and $2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$ there exists linear isometry $\sigma$ (induced by the isotropy representation of the space) of $\mathfrak{p}$ such that

$$\sigma(\varepsilon_2) = -\varepsilon_2, \quad \sigma(2\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = -2\varepsilon_1 - \varepsilon_2 - \varepsilon_3.$$

Then $\sigma(2\varepsilon_1+\varepsilon_3)=-2\varepsilon_1-\varepsilon_3$. Since $2\varepsilon_1+\varepsilon_3$, $\varepsilon_2$, $\varepsilon_1+2\varepsilon_3$ form an orthogonal basis of $\mathfrak{p}$, we have

$$\sigma(\varepsilon_1+2\varepsilon_3)=\pm(\varepsilon_1+2\varepsilon_3).$$

If $\sigma(\varepsilon_1+2\varepsilon_3)=-(\varepsilon_1+2\varepsilon_3)$, then

$$\sigma(\varepsilon)=-\varepsilon \quad \forall\,\varepsilon\in\mathfrak{p}. \tag{6.1}$$

If $\sigma(\varepsilon_1+2\varepsilon_3)=\varepsilon_1+2\varepsilon_3$, then

$$\sigma(\varepsilon_1)=-\frac{5}{3}\varepsilon_1-\frac{4}{3}\varepsilon_3,\quad \sigma(\varepsilon_2)=-\varepsilon_2,\quad \sigma(\varepsilon_3)=\frac{4}{3}\varepsilon_1+\frac{5}{3}\varepsilon_3. \tag{6.2}$$

If $\sigma$ satisfies identity (6.1), then the Lorentzian space must be globally symmetric, which is a contradiction to Theorem 6.2. If $\sigma$ satisfies identity (6.2), then it contradicts the action of $\sigma$ on the bracket $[\varepsilon_2,\varepsilon_3]$ and the fact that the action of $\sigma$ is induced by the adjoint action of the isotropy representation, hence must keep the Lie brackets of the elements.

Next assume that a space in case (6) in Theorem 5.1 is 2-fold symmetric. Then for $\varepsilon_1$ and $\varepsilon_2+\varepsilon_3$ there exists a linear isometry $\sigma$ on $\mathfrak{p}$ such that

$$\sigma(\varepsilon_1)=-\varepsilon_1,\quad \sigma(\varepsilon_2+\varepsilon_3)=-\varepsilon_2-\varepsilon_3.$$

It contradicts the action of $\sigma$ on the bracket $[\varepsilon_1,\varepsilon_2+\varepsilon_3]$.

Finally assume that a space in case (9) for $b\neq 0$ in Theorem 5.1 is 2-fold symmetric. Then for $\varepsilon_1$ and $\varepsilon_2+\varepsilon_3$ there exists a linear isometry $\sigma$ of $\mathfrak{p}$ such that

$$\sigma(\varepsilon_1)=-\varepsilon_1,\quad \sigma(\varepsilon_2+\varepsilon_3)=-\varepsilon_2-\varepsilon_3.$$

It contradicts the action of $\sigma$ on the bracket $[\varepsilon_1,\varepsilon_2+\varepsilon_3]$.

The theorem is now proved. □

*References*


[1] *J. Berndt, L. Vanhecke*: Geometry of weakly symmetric spaces. J. Math. Soc. Japan *48* (1996), 745–760. zbl MR doi

[2] *Z. Chen, J. A. Wolf*: Pseudo-Riemannian weakly symmetric manifolds. Ann. Global Anal. Geom. *41* (2012), 381–390. zbl MR doi

[3] *V. del Barco, G. P. Ovando*: Isometric actions on pseudo-Riemannian nilmanifolds. Ann. Global Anal. Geom. *45* (2014), 95–110. zbl MR doi

[4] *S. Deng*: An algebraic approach to weakly symmetric Finsler spaces. Can. J. Math. *62* (2010), 52–73. zbl MR doi

[5] *S. Deng*: On the symmetry of Riemannian manifolds. J. Reine Angew. Math. *680* (2013), 235–256. zbl MR doi

[6] *S. Helgason*: Differential Geometry, Lie Groups, and Symmetric Spaces. Pure and Applied Mathematics 80, Academic Press, New York, 1978. zbl MR
[7] *S. Kobayashi, K. Nomizu*: Foundations of Differential Geometry. Vol. I. Interscience Publishers, John Wiley & Sons, New York, 1963. zbl MR
[8] *A. Selberg*: Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. J. Indian Math. Soc., New Ser. *20* (1956), 47–87. zbl MR
[9] *H.-C. Wang*: Two-point homogeneous spaces. Ann. Math. *55* (1952), 177–191. zbl MR doi
[10] *J. A. Wolf*: Harmonic Analysis on Commutative Spaces. Mathematical Surveys and Monographs 142, American Mathematical Society, Providence, 2007. zbl MR doi
[11] *O. S. Yakimova*: Weakly symmetric Riemannian manifolds with a reductive isometry group. Sb. Math. *195* (2004), 599–614 (In English. Russian original.); translation from Mat. Sb. *195* (2004), 143-160. zbl MR doi
[12] *W. Ziller*: Weakly symmetric spaces. Topics in Geometry. In Memory of Joseph D'Atri (Gindikin, S. et al., eds.). Progr. Nonlinear Differ. Equ. Appl. 20, Birkhäuser, Boston, 1996, pp. 355–368. zbl MR doi

*Authors' address*: Bo Zhang, Zhiqi Chen, Shaoqiang Deng (correspodning author), School of Mathematical Sciences and LPMC, Nankai University, No. 94 Weijin Road, Nankai District, Tianjin 300071, P. R. China, e-mail: `zhangbo@mail.nankai.edu.cn`, `chenzhiqi@nankai.edu.cn`, `dengsq@nankai.edu.cn`.