Czechoslovak Mathematical Journal

Wei Ding; Jiao Chen; Yaoming Niu

Note on duality of weighted multi-parameter Triebel-Lizorkin spaces

Czechoslovak Mathematical Journal, Vol. 69 (2019), No. 3, 763–779

Persistent URL: http://dml.cz/dmlcz/147788

Czechoslovak Mathematical Journal, 69 (144) (2019), 763–779

NOTE ON DUALITY OF WEIGHTED MULTI-PARAMETER TRIEBEL-LIZORKIN SPACES

Wei Ding, Nantong, Jiao Chen, Chongqing, Yaoming Niu, Baotou

Received November 4, 2017. Published online February 18, 2019.

Abstract. We study the duality theory of the weighted multi-parameter Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}(\omega;\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. This space has been introduced and the result

$$(\dot{F}_p^{\alpha,q}(\omega;\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}))^* = \mathrm{CMO}_p^{-\alpha,q'}(\omega;\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$$

for $0<p\leqslant 1$ has been proved in Ding, Zhu (2017). In this paper, for $1<p<\infty$, $0<q<\infty$ we establish its dual space $\dot{H}_p^{\alpha,q}(\omega;\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$.

Keywords: Triebel-Lizorkin space; duality; weighted multi-parameter

MSC 2010: 42B25, 42B35

1. Introduction

The classical theory of harmonic analysis may be described as centering around the Hardy-Littlewood maximal operator and its relationship with certain singular integral operators which commute with the usual dilations on $\mathbb{R}^m$, given by $\delta(x)=(\delta x_1,\dots,\delta x_m)$, $\delta>0$. If this isotropic dilations are replaced by more general non-isotropic groups of dilations, then many non-isotropic variants of the classical theories can be produced, such as the multi-parameter pure product theory, corresponding to the dilations $\delta\colon x\to(\delta_1x_1,\delta_2x_2)$, $x=(x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, $\delta=(\delta_1,\delta_2)$, $\delta_1,\delta_2>0$, which has been developed over the past decades. We refer the reader to [2], [3], [4], [5], [9], [11], [12], [13], [14], [17], [18], [21], [22], [24], [25], [27], [28], [29], [30], [31], [32], [37].

The first author is supported by NNSF of China grants (11501308, 11771223, 11801049) and Jiangsu Government Scholarship for Overseas Studies. The third author is supported by NNSF of China grants (11661061).

DOI: 10.21136/CMJ.2019.0509-17

Triebel-Lizorkin spaces form a unifying class of function spaces encompassing many well-studied classical function spaces such as Lebesgue spaces, Hardy spaces, Lipschitz spaces, and the space BMO. For more information see Triebel [35], Frazier, Jawerth [15] for one-parameter Triebel-Lizorkin spaces, Bownik [1] for anisotropic Triebel-Lizorkin spaces, Li, et al. [26] for weighted anisotropic Triebel-Lizorkin spaces and Yuan, et al. [37] for a unified framework.

The pure weighted multi-parameter Triebel-Lizorkin spaces were first introduced in [31], and reintroduced in [8]. To be precise, let $\psi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$, $i = 1, 2$ with

$$\operatorname{supp} \widehat{\psi^{(i)}} \subseteq \left\{\xi \in \mathbb{R}^{n_i} \colon \tfrac{1}{2} \leqslant |\xi_i| \leqslant 2\right\} \tag{1.1}$$

and

$$\sum_{j \in \mathbb{Z}} |\widehat{\psi^{(i)}}(2^{-j}\xi_i)|^2 = 1 \quad \forall\, \xi_i \in \mathbb{R}^{n_i} \setminus \{0\}. \tag{1.2}$$

From (1.1), one has the moment condition

$$\int_{\mathbb{R}^{n_i}} \psi^{(i)}(x) x^{\delta} \,\mathrm{d}x = 0 \tag{1.3}$$

for all multi-indices $\delta \in \mathbb{N}^{n_i}$. Denote

$$\mathcal{S}_0(\mathbb{R}^{n_1+n_2}) = \left\{ f \in \mathcal{S}(\mathbb{R}^{n_1+n_2}) \colon \int_{\mathbb{R}^{n_1+n_2}} f(x) x^{\alpha} \,\mathrm{d}x = 0 \quad \forall\, |\alpha| \geqslant 0 \right\}.$$

The following discrete Calderón's identity is an extension of Lemma 2.1 in [15] to multi-parameter. One can prove it similarly as in the proof of Theorem 1.3 in [23].

Theorem 1.1. *Suppose that* $\psi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$, $i = 1, 2$ *are functions satisfying conditions* (1.1)–(1.2). *Then*

$$f(x_1, x_2) = \sum_{j,k \in \mathbb{Z}} \sum_{I \times J \in \Pi_j^{n_1} \times \Pi_k^{n_2}} |I||J| (\psi_{j,k} * f)(x_I, x_J) \times \psi_{j,k}(x_1 - x_I, x_2 - x_J), \tag{1.4}$$

where the series converges in $L^2(\mathbb{R}^{n_1+n_2})$, $\mathcal{S}_0(\mathbb{R}^{n_1+n_2})$ *and* $\mathcal{S}_0'(\mathbb{R}^{n_1+n_2})$.

Here and after, for $i = 1, 2$ and any $j \in \mathbb{Z}$, denote $\Pi_j^{n_i} = \{I \colon I$ are dyadic cubes in $\mathbb{R}^{n_i}$ with the side length $l(I) = 2^{-j}$, and the left lower corners of I are $x_I = 2^{-j} l$, $l \in \mathbb{Z}^{n_i}\}$, $\mathcal{D}_{n_i} = \bigcup_j \Pi_j^{n_i}$ and $\mathcal{D} = \mathcal{D}_{n_1} \times \mathcal{D}_{n_2}$. Set

$$\psi_{j,k}(x_1, x_2) = \psi_j^{(1)}(x_1) \psi_k^{(2)}(x_2),$$
$$\psi_j^{(1)}(x_1) = 2^{jn_1} \psi^{(1)}(2^j x_1), \quad \psi_k^{(2)}(x_2) = 2^{kn_2} \psi^{(2)}(2^k x_2).$$

We now recall some definitions of product weights in two-parameter setting. For $1 < p < \infty$, a nonnegative locally integrable function $w \in A_p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ if there exists a constant $C > 0$ such that

$$\sup_{R\in\mathcal{D}} \left(\frac{1}{|R|}\int_R \omega(x)\,\mathrm{d}x\right)\left(\frac{1}{|R|}\int_R \omega(x)^{-1/(p-1)}\,\mathrm{d}x\right)^{p-1} < \infty.$$

We say $\omega \in A_1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ if there exists a constant $C > 0$ such that

$$M_s\omega(x) \leqslant C\omega(x)$$

for almost every $x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, where M_s is the strong maximal function defined by

$$M_s f(x) = \sup_{x\in R\in\mathcal{D}} \frac{1}{|R|}\int_R |f(v)|\,\mathrm{d}v.$$

We also define $M_\mu^s f$, the strong maximal function with respect to measure μ, by

$$M_\mu^s f(x) = \sup_{x\in R\in\mathcal{D}} \frac{1}{\mu(R)}\int_R |f(y)|\,\mathrm{d}\mu(y).$$

At last we define $A_\infty = A_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ by

$$A_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = \bigcup_{1\leqslant p<\infty} A_p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}).$$

If $\omega \in A_\infty$, then $q_\omega = \inf\{q\colon\ \omega \in A_q\}$ is called the critical index of ω. Notice that if $\omega \in A_\infty$, then $q_\omega < \infty$.

With the discrete Calderón's identity, the following weighted multi-parameter Triebel-Lizorkin space was introduced in [8], [31].

Definition 1.1. Let $0 < p,\ q < \infty$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, $\omega \in A_\infty$, and $\psi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$, $i = 1, 2$ satisfying conditions (1.1)–(1.2). The weighted multi-parameter Triebel-Lizorkin space $\dot{F}_p^{\alpha,q} = \dot{F}_p^{\alpha,q}(\omega; \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is defined by

$$\dot{F}_p^{\alpha,q}(\omega; \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = \{f \in \mathcal{S}_0'(\mathbb{R}^{n_1+n_2})\colon\ \|f\|_{\dot{F}_p^{\alpha,q}(\omega;\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} < \infty\},$$

where

$$\|f\|_{\dot{F}_p^{\alpha,q}(\omega;\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} = \left\|\left(\sum_{j,k\in\mathbb{Z}} 2^{-(j\alpha_1+k\alpha_2)q} \sum_{I\times J\in\Pi_j^{n_1}\times\Pi_k^{n_2}} |\psi_{j,k} * f(x_I, x_J)|^q \chi_I(x_1)\chi_J(x_2)\right)^{1/q}\right\|_{L^p(\omega)}.$$

The corresponding discrete weighted multi-parameter Triebel-Lizorkin space $\dot{f}_p^{\alpha,q} = \dot{f}_p^{\alpha,q}(\omega;\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ is defined to be the set of all complex-valued sequences $s=\{s_R\}_R$ satisfying

$$(1.5)\qquad \|s\|_{\dot{f}_p^{\alpha,q}} = \Big\|\Big(\sum_{R\in\mathcal{D}}(|I|^{\alpha_1/n_1}|J|^{\alpha_2/n_2}|s_R|\tilde{\chi}_R(x))^q\Big)^{1/q}\Big\|_{L^p(\omega)} < \infty,$$

where $\tilde{\chi}_R(x) = |R|^{-1/2}\chi_R(x)$.

This weighted multi-parameter Triebel-Lizorkin type space is well defined, since it has been proved in [8] that $\dot{F}_p^{\alpha,q}$ is independent of the choice of the functions $\psi^{(1)}$ and $\psi^{(2)}$. This space also can be characterized by its continuous form [8], that is

$$\Big\|\Big(\sum_{j,k\in\mathbb{Z}}2^{-(j\alpha_1+k\alpha_2)q}\sum_{I\times J\in\Pi_j^{n_1}\times\Pi_k^{n_2}}|\psi_{j,k}*f(x_I,x_J)|^q\chi_I(x_1)\chi_J(x_2)\Big)^{1/q}\Big\|_{L^p(\omega)}$$
$$\approx \Big\|\Big(\sum_{j,k\in\mathbb{Z}}2^{-(j\alpha_1+k\alpha_2)q}|\psi_{j,k}*f|^q\Big)^{1/q}\Big\|_{L^p(\omega)}.$$

Though there have been extensive works on dual spaces of multi-parameter Hardy spaces (see [3], [14], [19], [20], [23], etc.), the duality of Triebel-Lizorkin spaces has almost been studied in the one-parameter settings started in [15], [35], see also [1] for anisotropic Triebel-Lizorkin spaces, [26] for weighted anisotropic Triebel-Lizorkin spaces. Recently, there has been some progress in the dual of multi-parameter Triebel-Lizorkin spaces. One can see [7] for the dual of multi-parameter Triebel-Lizorkin spaces associated with the composition of two singular integral operators, and see [8] for the dual of weighted multi-parameter Triebel-Lizorkin spaces. We want to point out that in [8], the dual of $\dot{F}_p^{\alpha,q}(\omega;\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ was only discussed when $0<p\leqslant 1$. Hence, our goal is to complete this work.

Definition 1.2. Let $0<p<\infty$, $0<q\leqslant\infty$, $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2$, $\omega\in A_\infty$, and $\psi^{(i)}\in\mathcal{S}(\mathbb{R}^{n_i})$, $i=1,2$ satisfying conditions (1.1)–(1.2). The weighted multi-parameter Triebel-Lizorkin-type space $\dot{H}_p^{\alpha,q} = \dot{H}_p^{\alpha,q}(\omega;\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ is defined to be the set of all $f\in\mathcal{S}_0'(\mathbb{R}^{n_1+n_2})$ such that

$$\|f\|_{\dot{H}_p^{\alpha,q}(\omega;\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$$
$$= \Big\|\Big(\sum_{j,k\in\mathbb{Z}}2^{-(j\alpha_1+k\alpha_2)q}\sum_{I\times J\in\Pi_j^{n_1}\times\Pi_k^{n_2}}\Big|\psi_{j,k}*f(x_I,x_J)\frac{|R|}{\omega(R)}\Big|^q\chi_I(x)\chi_J(y)\Big)^{1/q}\Big\|_{L^p(\omega)}<\infty.$$

The corresponding discrete weighted multi-parameter Triebel-Lizorkin-type space $\dot{h}_p^{\alpha,q} = \dot{h}_p^{\alpha,q}(\omega;\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ is defined to be the set of all complex-valued sequences

$s = \{s_R\}_R$ such that

$$\|s\|_{\dot{h}_p^{\alpha,q}} = \left\| \left(\sum_{R\in\mathcal{D}} \left(|I|^{\alpha_1/n_1} |J|^{\alpha_2/n_2} |s_R| \frac{|R|}{\omega(R)} \tilde{\chi}_R(x) \right)^q \right)^{1/q} \right\|_{L^p(\omega)} < \infty, \tag{1.6}$$

where $\tilde{\chi}_R(x) = |R|^{-1/2}\chi_R(x)$.

Let q' denote the conjugate of q, such that $1/q + 1/q' = 1$ when $1 \leqslant q \leqslant \infty$; if $0 < q < 1$, it is also convenient to let $q' = \infty$. One of the main theorems of this paper is the following.

Theorem 1.2. *Let $1 < p < \infty$, $0 < q < \infty$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, $\omega \in A_\infty$. Then*

$$(\dot{F}_p^{\alpha,q})^* = \dot{H}_{p'}^{-\alpha,q'}.$$

Namely, if $g \in \dot{H}_{p'}^{-\alpha,q'}$, then the map l_g given by $l_g(f) = \langle f, g\rangle$ and defined initially for $f \in \mathcal{S}_0(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, extends to a continuous linear functional on $\dot{F}_p^{\alpha,q}$ with $\|l_g\| \lesssim \|g\|_{\dot{H}_{p'}^{-\alpha,q'}}$. Conversely, every $l \in (\dot{F}_p^{\alpha,q})^*$ satisfies $l = l_g$ for some $g \in \dot{H}_{p'}^{-\alpha,q'}$ with $\|l_g\| \approx \|g\|_{\dot{H}_{p'}^{-\alpha,q'}}$.

In order to prove the above duality theorem, following Frazier and Jawerth in the one-parameter case [15], we should first do these in the corresponding discrete weighted multi-parameter sequence spaces.

Theorem 1.3. *Suppose $1 < p < \infty$, $0 < q < \infty$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, $\omega \in A_\infty$. Then*

$$(\dot{f}_p^{\alpha,q})^* = \dot{h}_{p'}^{-\alpha,q'}.$$

More precisely, l is a bounded linear functional on $\dot{f}_p^{\alpha,q}$ if and only if l is of the form

$$l(s) = \langle s, t\rangle = \sum_{R\in\mathcal{D}} s_R \bar{t}_R \quad \forall\, s = \{s_R\}_{R\in\mathcal{D}} \in \dot{f}_p^{\alpha,q}$$

for some sequence $t = \{t_R\}_{R\in\mathcal{D}}$ with

$$\|l\|_{(\dot{f}_p^{\alpha,q})^*} \approx \|t\|_{\dot{h}_{p'}^{-\alpha,q'}}.$$

The organization of the paper is as follows. In Section 2, we introduce the multi-parameter ψ-transform S_ψ and its inverse ψ-transform T_ψ. We prove that S_ψ, T_ψ are all bounded. In Section 3, we establish the duality of the sequence space $\dot{f}_p^{\alpha,q}$ and obtain $(\dot{f}_p^{\alpha,q})^* = \dot{h}_{p'}^{-\alpha,q'}$. In Section 4, we obtain the dual of the space $\dot{F}_p^{\alpha,q}$. In Section 5, we concern the imbedding theorems which are used in Section 3 to establish the duality of the sequence space $\dot{f}_p^{\alpha,q}$ in the case $1 < p < \infty$, $0 < q < 1$.

Finally, we make some conventions. Throughout the paper, C denotes a positive constant that is independent of the main parameters involved, but whose value may vary from line to line. Constants with subscript, such as C_1, do not change in different occurrences. We denote $f \leqslant Cg$ by $f \lesssim g$. If $f \lesssim g \lesssim f$, we write $f \approx g$.

2. Multi-parameter ψ-transform

For any $R \in \Pi_j^{n_1} \times \Pi_k^{n_2}$ set $\psi_R(x) = |R|^{1/2}\psi_{j,k}(x_1 - x_I, x_2 - x_J)$. Then by (2.2), it is easy to have

$$f(x) = \sum_{R\in\mathcal{D}} \langle f, \psi_R\rangle\, \psi_R(x). \tag{2.1}$$

Definition 2.1. Suppose that $\psi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$, $i = 1, 2$ are functions satisfying condition (1.3) and set $\psi(x_1, x_2) = \psi^{(1)}(x_1)\psi^{(2)}(x_2)$. Define the multi-parameter ψ-transform S_ψ as the map taking $f \in \mathcal{S}_0'(\mathbb{R}^{n_1+n_2})$ to the sequence $S_\psi f = \{(S_\psi f)_R\}_R$, where $(S_\psi f)_R = \langle f, \psi_R\rangle$. Define the inverse multi-parameter ψ-transform T_ψ as the map taking a sequence $s = \{s_R\}_R$ to $T_\psi s = \sum\limits_R s_R\psi_R(x)$.

By (2.1), for $f \in \mathcal{S}_0(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, $g \in \mathcal{S}_0'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ one has

$$\langle f, g\rangle = \left\langle \sum_{R\in\mathcal{D}} (S_\psi f)_R\psi_R(x), g \right\rangle = \langle S_\psi f, S_\psi g\rangle\,. \tag{2.2}$$

For a sequence $s = s_R$ one also has the following identity:

$$\langle S_\psi f, s\rangle = \sum_{R\in\mathcal{D}} \langle f, \psi_R\rangle\, s_R = \left\langle f, \sum_{R\in\mathcal{D}} s_R\psi_R \right\rangle = \langle f, T_\psi s\rangle\,. \tag{2.3}$$

The following generalization of the fundamental result of Theorem 2.2 in [15] holds.

Theorem 2.1. *Suppose* $0 < p < \infty$, $0 < q \leqslant \infty$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, $\omega \in A_\infty$, *and* $\psi^i \in \mathcal{S}(\mathbb{R}^{n_i})$, $i = 1, 2$, *are functions satisfying condition* (1.3). *The operators* $S_\psi\colon \dot{F}_p^{\alpha,q} \to \dot{f}_p^{\alpha,q} (\dot{H}_p^{\alpha,q} \to \dot{h}_p^{\alpha,q})$ *and* $T_\psi\colon \dot{f}_p^{\alpha,q} \to \dot{F}_p^{\alpha,q} (\dot{h}_p^{\alpha,q} \to \dot{H}_p^{\alpha,q})$ *are bounded and* $T_\psi \circ S_\psi$ *is the identity on* $\dot{F}_p^{\alpha,q} (\dot{H}_p^{\alpha,q})$.

Before proving Theorem 2.1, we need the following two lemmas. The following almost orthogonality estimates can be seen in Appendix K of [16].

Lemma 2.1. *Let* $\psi^{(i)}, \phi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$, $i = 1, 2$ *be functions satisfying condition* (1.3). *Then given any positive integers* L, M *there exists a constant* $C = C(L, M)$ *such that*

$$|\psi_{j,k} * \phi_{j',k'}(x_1, x_2)| \leqslant C 2^{-|j-j'|L} 2^{-|k-k'|L} \frac{2^{(j\wedge j')n_1}}{(1 + 2^{j\wedge j'}|x_1|)^M} \frac{2^{(k\wedge k')n_2}}{(1 + 2^{k\wedge k'}|x_2|)^M},$$

where $t \wedge s = \min\{t, s\}$.

Lemma 2.2. *Let* $I, I'; J, J'$ *be dyadic cubes in* $\mathbb{R}^{n_1}$ *and* $\mathbb{R}^{n_2}$*, respectively, such that* $l(I) = 2^{-j}$, $l(J) = 2^{-k}$, *and* $l(I') = 2^{-j'}$, $l(J') = 2^{-k'}$. *Then for any* $u, u^* \in I$, $v, v^* \in J$ *we have*

$$\sum_{I',J'} \frac{2^{(j\wedge j')n_1} 2^{(k\wedge k')n_2} |I'||J'|}{(1 + 2^{j\wedge j'}|u - x_{I'}|)^M (1 + 2^{k\wedge k'}|v - x_{J'}|)^M} |\phi_{j',k'} * f(x_{I'}, y_{J'})|$$
$$\leqslant C_1 \left\{ M_s \left(\sum_{I'} \sum_{J'} |\phi_{j',k'} * f(x_{I'}, y_{J'})| \chi_{I'} \chi_{J'} \right)^r (u^*, v^*) \right\}^{1/r},$$

where $C_1 = 2^{(1-1/r)[n_1(j\wedge j' - j') + n_2(k\wedge k' - k')]}$ *and* $0 < r < 1$, *which can be arbitrarily small if* M *is big enough.*

The proof of Lemma 2.2 can be found in [23], [34].

Lemma 2.3 (Theorem 1.12 in [6])**.** *Suppose that* $1 < p, q < \infty$, $\omega \in A_p(\mathbb{R}^n)$. *Then*

$$\|\{M(f_i)\}_i\|_{L^p_\omega(l^q)} \leqslant C_{n,p,q,\omega} \|\{f_i\}_i\|_{L^p_\omega(l^q)},$$

where M *denotes the Hardy-Littlewood maximal operator and*

$$L^p_\omega(l^q) = \left\{ f = \{f_v\} \colon \ \|f\|_{L^p(l^q)} = \left\| \left(\sum_v |f_v|^q \right)^{1/q} \right\|_{L^p(\omega)} < \infty \right\}.$$

Remark 2.1. Since product weighted $\omega \in A_p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ implies $\omega(\cdot, y) \in A_p(\mathbb{R}^{n_1})$, $\omega(x, \cdot) \in A_p(\mathbb{R}^{n_2})$, and the strong maximal operator $M_s \leqslant M \circ M$, by iteration, Lemma 2.3 also holds for M_s.

P r o o f of Theorem 2.1. The identity is obvious and the boundedness of S_ψ is also immediate since

$$\|S_\psi(f)\|_{\dot{f}_p^{\alpha,q}} = \|f\|_{\dot{F}_p^{\alpha,q}}, \|S_\psi(f)\|_{\dot{h}_p^{\alpha,q}} = \|f\|_{\dot{H}_p^{\alpha,q}}$$

from the definitions.

We now outline the proof of T_ψ's boundedness. Details can be seen in [7], [23]. For a sequence $s = \{s_R\}_{R\in\mathcal{D}}$ let $f(x) = T_\psi s = \sum\limits_R s_R\psi_R(x)$. Then by Lemma 2.1,

$$\begin{aligned}&|\psi_{j,k} * \psi_{j',k'}(x_{I'} - x_I, x_{J'} - x_J)|\\ &\quad\leqslant C2^{-|j-j'|L}2^{-|k-k'|L}\frac{2^{(j\wedge j')n_1}2^{(k\wedge k')n_2}}{(1+2^{j\wedge j'}|x_{I'}-x_I|)^M(1+2^{k\wedge k'}|x_{J'}-x_J|)^M}.\end{aligned}$$

Hence, using Lemma 2.2 for any $u'\in I', v'\in J'$ one has

$$\begin{aligned}&|f * \psi_{j',k'}(x_{I'}, x_{J'})|\\ &\quad\lesssim \sum_{j,k} 2^{-|j-j'|L}2^{-|k-k'|L}C_1\Big\{M_s\Big(\sum_{R\in\Pi_{j,k}}|R|^{-1/2}|s_R|\chi_I\chi_J\Big)^r(u',v')\Big\}^{1/r}\end{aligned}$$

for $r>0$ which can be sufficiently small if one chooses M big enough. Summing over j', k' and $R\in\Pi_j^{n_1}\times\Pi_k^{n_2}$, one has

$$\begin{aligned}&\Big(\sum_{j',k'\in\mathbb{Z}} 2^{-(j'\alpha_1+k'\alpha_2)q}\sum_{R'\in\Pi_{j'}^{n_1}\times\Pi_{k'}^{n_2}}|\psi_{j',k'} * f(x_{I'},x_{J'})|^q\chi_{I'}(x')\chi_{J'}(y')\Big)^{1/q}\\ &\quad\leqslant C\Big(\sum_{j',k'\in\mathbb{Z}} 2^{-(j'\alpha_1+k'\alpha_2)q}\Big[\sum_{j,k}2^{-|j-j'|L}2^{-|k-k'|L}\\ &\qquad\times C_1\Big\{M_s\Big(\sum_{R\in\Pi_j^{n_1}\times\Pi_k^{n_2}}|R|^{-1/2}|s_R|\chi_I\chi_J\Big)^r(u',v')\Big\}^{1/r}\Big]^q\Big)^{1/q}.\end{aligned}$$

Then by the inequality $\big(\sum\limits_l a_l\big)^q\leqslant\sum\limits_l a_l^q$, if $0<q\leqslant 1$, or the Cauchy inequality with exponents $q, q', 1/q+1/q'=1$, if $q>1$, we obtain

$$\begin{aligned}&\Big(\sum_{j',k'\in\mathbb{Z}} 2^{-(j'\alpha_1+k'\alpha_2)q}\sum_{R'\in\Pi_{j'}^{n_1}\times\Pi_{k'}^{n_2}}|\psi_{j',k'} * f(x_{I'},x_{J'})|^q\chi_{I'}(x')\chi_{J'}(y')\Big)^{1/q}\\ &\quad\lesssim\Big(\sum_{j,k\in\mathbb{Z}}2^{-(j\alpha_1+k\alpha_2)q}\Big\{M_s\Big(\sum_{R\in\Pi_j^{n_1}\times\Pi_k^{n_2}}|R|^{-1/2}|s_R|\chi_I\chi_J\Big)^r(u',v')\Big\}^{q/r}\Big)^{1/q}.\end{aligned}$$

Applying Lemma 2.3 provided $r<\min\{p/q_\omega, q, 1\}$, we complete the proof of T_ψ's boundedness from $\dot{f}_p^{\alpha,q}$ to $\dot{F}_p^{\alpha,q}$. Since the proof of T_ψ's boundedness from $\dot{h}_p^{\alpha,q}$ to $\dot{H}_p^{\alpha,q}$ is similar, we omit it. □

3. Duality of $\dot f_p^{\alpha,q}$

P r o o f of Theorem 1.3. For any $s \in \dot f_p^{\alpha,q}$, $t \in \dot f_{p'}^{\alpha,q'}$ we have

$$
\begin{aligned}
&\Big|\sum_{R\in\mathcal D} s_R\bar t_R\Big| \\
&\quad\leqslant \int \sum_{R\in\mathcal D} |I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R|\tilde\chi_R(x)|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R|\tilde\chi_R(x)\frac{|R|}{\omega(R)}\omega(x)\,\mathrm dx \\
&\quad\leqslant \int \Big(\sum_{R\in\mathcal D}(|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R|\tilde\chi_R(x))^q\Big)^{1/q} \\
&\qquad\qquad \times \Big(\sum_{R\in\mathcal D}(|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R|\frac{|R|}{\omega(R)}\tilde\chi_R(x))^{q'}\Big)^{1/q'}\omega(x)\,\mathrm dx \\
&\quad\leqslant \|s\|_{\dot f_p^{\alpha,q}}\|t\|_{\dot h_{p'}^{-\alpha,q'}}
\end{aligned}
$$

by duality if $1\leqslant q<\infty$, or by imbedding $\ell^q \hookrightarrow \ell^1$ if $0<q<1$. This yields that t is a continuous linear functional on $\dot f_p^{\alpha,q}$, and

$$\|t\|_{(\dot f_p^{\alpha,q})^*} \leqslant \|t\|_{\dot h_{p'}^{-\alpha,q'}}.$$

For the converse direction, we spilt its proof into 2 cases: $(p,q)\in(1,\infty)\times[1,\infty)$, $(1,\infty)\times(0,1)$.

Case 1: $(p,q)\in(1,\infty)\times[1,\infty)$. This case is elementary. Take any $l\in(\dot f_p^{\alpha,q})^*$. Then there exists a sequence $t=\{t_R\}_R$ such that $l(s)=\sum_R s_R\bar t_R$ for any $s=\{s_R\}_R\in\dot f_p^{\alpha,q}$. Now we need a well-known result that

$$(L^p(l^q))^* = L^{p'}(l^{q'}) \tag{3.1}$$

if $1<p<\infty$, $0<q<\infty$, where $L^p(l^q)=L^p_\omega(l^q)$ with the pairing $\langle f,g\rangle = \int\sum_v f_v(x)\overline{g}_v(x)\omega(x)\,\mathrm dx$ for $f\in L^p(l^q)$, $g\in L^{p'}(l^{q'})$ (see, e.g. [35]). Let $I\colon \dot f_p^{\alpha,q}\to L^p(l^q)$ be defined as

$$I(s)=\{f_{j,k}\}_{j,k\in\mathbb Z}\quad\text{where}\quad f_{j,k}=\sum_{R\in\Pi_{j,k}}|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}s_R\tilde\chi_R(x).$$

Clearly, the map I is a linear isometry onto a subspace of $L^p(l^q)$. By the Hahn-Banach Theorem, there exists $\tilde l\in(L^p(l^q))^*$ such that $\tilde l\circ I=l$ and $\|\tilde l\|=\|l\|_{(\dot f_p^{\alpha,q})^*}$.

By (3.1), $\tilde{l}(f) = \langle f, g \rangle$ for some $g \in L^{p'}(l^{q'})$ with $\|g\|_{L^{p'}(l^{q'})} \leqslant \|l\|_{(\dot{f}_p^{\alpha,q})^*}$. Hence

$$\begin{aligned} l(s) = \tilde{l}(I(s)) &= \int \sum_{j,k} f_{j,k}(x) \overline{g}_{j,k}(x) \omega(x)\,\mathrm{d}x \\ &= \int \sum_{j,k} \Big(\sum_{R \in \Pi_{j,k}} |I|^{\alpha_1/(m-1)} |J|^{\alpha_2} s_R \tilde{\chi}_R(x) \Big) \overline{g}_{j,k}(x) \omega(x)\,\mathrm{d}x \\ &= \sum_{j,k} \sum_{R \in \Pi_{j,k}} s_R \Big(|I|^{\alpha_1/(m-1)} |J|^{\alpha_2} |R|^{-1/2} \int_R \overline{g}_{j,k}(x) \omega(x)\,\mathrm{d}x \Big) \\ &= \sum_{j,k} \sum_{R \in \Pi_{j,k}} s_R t_R = \langle t, s \rangle \end{aligned}$$

for all $s \in \dot{f}_p^{\alpha,q}$, where $t = \{t_R\}_R$ with $t_R = |I|^{\alpha_1/(m-1)} |J|^{\alpha_2} |R|^{-1/2} \int_R \overline{g}_{j,k}(x) \omega(x)\,\mathrm{d}x$ for $R \in \Pi_{j,k}$. Then

$$\begin{aligned} \|t\|_{\dot{h}_{p'}^{-\alpha,q'}} &= \bigg\| \Big(\sum_{j,k} \sum_{R \in \Pi_{j,k}} \Big(\frac{1}{\omega(R)} \int_R g_{j,k}(x) \omega(x)\,\mathrm{d}x \Big)^{q'} \chi_R(x) \Big)^{1/q'} \bigg\|_{L^{p'}(\omega)} \\ &\leqslant \|\{M^s_{\omega(x)\,\mathrm{d}x}(g_{j,k})\}\|_{L^{p'}(l^{q'})} \lesssim \|g\|_{L^{p'}(l^{q'})} \leqslant \|l\|_{(\dot{f}_p^{\alpha,q})^*}. \end{aligned}$$

So we complete the proof of Case 1.

Case 2: $(p, q) \in (1, \infty) \times (0, 1)$. In this case, $L^p(l^q)$ is not a normed space, hence, we can not use the Hahn-Banach theorem.

Take $l \in (\dot{f}_p^{\alpha,q})^*$. Then there exists a sequence $t = \{t_R\}$ such that for any $s = \{s_R\}_R \in \dot{f}_p^{\alpha,q}$,

$$\begin{aligned} |l(s)| &= \Big| \sum_R s_R \bar{t}_R \Big| \leqslant C \|s\|_{\dot{f}_p^{\alpha,q}} \\ &= C \bigg\| \Big(\sum_{R \in \mathcal{D}} (|I|^{\alpha_1/(m-1)} |J|^{\alpha_2} |s_R| \tilde{\chi}_R(x))^q \Big)^{1/q} \bigg\|_{L^p(\omega)}. \end{aligned} \tag{3.2}$$

If we prove the estimates

$$\|t\|_{\dot{h}_{p'}^{-\alpha,\infty}} = \Big\| \sup_{R \in \mathcal{D}} |I|^{-\alpha_1/(m-1)} |J|^{-\alpha_2} |t_R| \frac{|R|}{\omega(R)} \tilde{\chi}_R(x) \Big\|_{L^{p'}(\omega)} < \infty,$$

we can complete the proof.

Define $\Pi = \{R \in \mathcal{D},\ t_R \neq 0\}$ and let

$$u_R = s_R \bar{t}_R, \quad c_R = \frac{|I|^{\alpha_1/(m-1)} |J|^{\alpha_2}}{|R|^{1/2} |t_R|}$$

for $R \in \Pi$. We may assume that $s_R \bar{t}_R \geqslant 0$ for all $R \in \mathcal{D}$ by choosing proper s_R, moreover we can assume $s_R = 0$ if $R \notin \Pi$. Then (3.2) can be rewritten as

$$\|u\|_{\ell^1} \leqslant c \left\| \left(\sum_{R \in \Pi} |u_R|^q (c_R)^q \chi_R \right)^{1/q} \right\|_{L^p}$$

for all $u = \{u_R\}_{R \in \Pi}$. Then (ii) of Theorem 5.1 with $0 < q < r = 1 < p < \infty$ yields

$$\int \sup_{R \in \Pi} ((c_R) \omega(R))^{p/(1-p)} \chi_R \omega(x) \, \mathrm{d}x < \infty,$$

that is,

$$\int \sup_{R \in \Pi} \left(|I|^{-\alpha_1/(m-1)} |J|^{-\alpha_2} |t_R| \frac{|R|}{\omega(R)} \tilde{\chi}_R \right)^{p/(p-1)} \omega(x) \, \mathrm{d}x < \infty,$$

which completes the proof. □

4. Duality of $\dot{F}_p^{\alpha,q}$

In this section we derive the dual Theorem 1.2 using Theorem 1.3 and Theorem 2.1. It is known from Proposition 6 in [31] that $\mathcal{S}_0(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is dense in $\dot{F}_p^{\alpha,q}$ for $0 < p$, $q < \infty$.

P r o o f of Theorem 1.2. Let $g \in \dot{H}_{p'}^{-\alpha,q'}$, $f \in \mathcal{S}_0(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $1 < p < \infty$, $0 < q < \infty$. Firstly, by identity (2.2) one has $\langle f, g \rangle = \langle S_\psi f, S_\psi g \rangle$. Hence

$$|\langle f, g \rangle| \leqslant \|S_\psi f\|_{\dot{f}_p^{\alpha,q}} \|S_\psi g\|_{\dot{h}_{p'}^{-\alpha,q'}} \lesssim \|f\|_{\dot{F}_p^{\alpha,q}} \|g\|_{\dot{H}_{p'}^{-\alpha,q'}}$$

by Theorem 2.1. This proves that $\|l_g\| \lesssim \|g\|_{\dot{H}_{p'}^{-\alpha,q'}}$.

Conversely, suppose $l \in (\dot{F}_p^{\alpha,q})^*$. Then $l_1 \equiv l \circ T_\psi \in (\dot{f}_p^{\alpha,q})^*$, so by Theorem 1.3, there exists $t = \{t_R\}_R \in \dot{h}_{p'}^{-\alpha,q'}$ such that

$$l_1(s) = \langle s, t \rangle = \sum_R s_R \bar{t}_R$$

for all $s = \{s_R\}_R \in \dot{f}_p^{\alpha,q}$. Moreover, $\|t\|_{\dot{h}_{p'}^{-\alpha,q'}} \approx \|l_1\| \lesssim \|l\|$ for the boundedness of T_ψ. Note that $l_1 \circ S_\psi = l \circ T_\psi \circ S_\psi = l$ since $T_\psi \circ S_\psi$ is the identity by Theorem 2.1. Then letting $g = T_\psi(t)$ and $f \in \mathcal{S}_0(\mathbb{R}^{n_1} \times \mathbb{R}^{n_1})$, one has

$$l(f) = l_1(S_\psi(f)) = \langle S_\psi(f), t \rangle = \langle f, T_\psi(t) \rangle = \langle f, g \rangle$$

by (2.3), which implies that $l = l_g$, and by Theorem 2.1 again, one has

$$\|g\|_{\dot{H}_{p'}^{-\alpha,q'}} = \|T_\psi(t)\|_{\dot{H}_{p'}^{-\alpha,q'}} \lesssim \|t\|_{\dot{h}_{p'}^{-\alpha,q'}} \lesssim \|l\|.$$

This completes the proof. □

5. Imbedding theorem

In this section we give a characterization of imbedding ℓ^r spaces into $\dot{f}_p^{\alpha,q}$ and imbedding $\dot{f}_p^{\alpha,q}$ into ℓ^r spaces. This result was first established by Verbisky [36] in the dyadic cubes with respect to an arbitrary positive locally finite measure on the Euclidean space, and was generalized by Bownik [1] to discrete anisotropic Triebel-Lizorkin sequence space.

Theorem 5.1. *Assume that* Π *is any subfamily of* $\mathcal{D}$, $\{c_R\}_{R\in\Pi}$ *is any positive sequence, and* $\omega \in A_p$.

(i) *Suppose* $0 < p < r \leqslant q \leqslant \infty$. *Then the inequality*

$$\Big\| \Big(\sum_{R\in\Pi} |s_R|^q (c_R)^q \chi_R \Big)^{1/q} \Big\|_{L^p(\omega)} \leqslant C \|s\|_{\ell^r} \tag{5.1}$$

holds for all scalar sequences $s = \{s_R\}_{R\in\Pi}$ *if and only if*

$$\int \sup_{R\in\Pi} ((c_R)^r \omega(R))^{p/(r-p)} \chi_R(x) \omega(x) \,\mathrm{d}x < \infty. \tag{5.2}$$

(ii) *Suppose* $0 < q \leqslant r < p < \infty$. *Then the inequality*

$$\Big\| \Big(\sum_{R\in\Pi} |s_R|^q (c_R)^q \chi_R \Big)^{1/q} \Big\|_{L^p(\omega)} \geqslant C \|s\|_{\ell^r} \tag{5.3}$$

holds for all scalar sequences $s = \{s_R\}_{R\in\Pi}$ *if and only if* (5.2) *holds.*

To establish Theorem 5.1 we will follow the original approach of Verbitsky [36]. Thus, we invite the following known results.

Lemma 5.1 (Theorem 1 (i), (ii) of [36])**.** *Let* $0 < p < r \leqslant q \leqslant \infty$. *Then*

$$\Big\| \Big(\sum_i |s_i|^q \varphi_i^q \Big)^{1/q} \Big\|_{L^p(\omega)} \leqslant C \|s\|_{\ell^r}$$

holds if

$$\int \sup_i (\phi_i^{r-p}(x) \|\phi_i\|^p_{L^p(\omega)})^{p/(r-p)} \omega(x) \,\mathrm{d}x < \infty.$$

Suppose $0 < q \leqslant r < p < \infty$. *Then*

$$\Big\| \Big(\sum_i |s_i|^q \varphi_i^q \Big)^{1/q} \Big\|_{L^p(\omega)} \geqslant C \|s\|_{\ell^r}$$

holds if

$$\int \sup_i (\phi_i^{p-r}(x) / \|\phi_i\|^p_{L^p(\omega)})^{p/(p-r)} \omega(x) \,\mathrm{d}x < \infty.$$

Lemma 5.2 (Theorem 1.1 of [33]). *Let $0 < p < r < \infty$, I be any index set, and let $\{\varphi_i\}_{i\in I}$ be a family in $L^p(\omega)$. Then the inequality*

$$\Big\|\sup_{i\in I}|s_i|\varphi_i\Big\|_{L^p(\omega)} \leqslant C\|s\|_{\ell^r}$$

holds for all scalar sequences $s = \{s_i\}_{i\in I} \in \ell^r$ if and only if there exists a non-negative measurable function $F \geqslant 0$ with $\int F(x)\omega(x)\,\mathrm{d}x \leqslant 1$ such that

$$\sup_{i\in I}\|F^{-1/p}\varphi_i\|_{L^{r,\infty}(\mu)} < \infty,$$

where $L^{r,\infty}(\mu)$ is a weak-L^r with respect to the measure $\mathrm{d}\mu(x) = F(x)\omega(x)\,\mathrm{d}x$ defined as

$$\|f\|_{L^{r,\infty}(\mu)} = \sup_{t>0} t\mu(\{x \in \mathbb{R}^{n_1}\times\mathbb{R}^{n_2} \colon |f(x)| > t\})^{1/r} < \infty$$

for $f \in L^{r,\infty}(\mu)$.

Lemma 5.3 (Remark 2 of [36]). *If $0 < q = r < p < \infty$, then*

$$\Big\|\Big(\sum_{i\in I}|s_i|^q\phi_i^q\Big)^{1/q}\Big\|_{L^p(\omega)} \geqslant C\|s\|_{\ell^r}$$

holds if and only if there exists $F \geqslant 0$ such that

$$\int F(x)\omega(x)\,\mathrm{d}x \leqslant 1, \quad \text{and} \quad \inf_i\|F^{-1/p}\phi_i\|_{L^r(\mu)} > 0,$$

where $\mathrm{d}\mu(x) = F(x)\omega(x)\,\mathrm{d}x$.

Proof of Theorem 5.1. Let $\varphi_R(x) = c_R\chi_R(x)$.

Part (i): Firstly, (5.2) $\Rightarrow$ (5.1) is a direct consequence of the first part of Lemma 5.1 since $\int(c_R\chi_R(x))^p\omega(x)\,\mathrm{d}x = (c_R)^p\omega(R)$. Now suppose that (5.1) holds for $p < r$. By imbedding $\ell^q \hookrightarrow \ell^\infty$ and Lemma 5.2, there exists a non-negative measurable function $F \geqslant 0$ with $\int F(x)\omega(x)\,\mathrm{d}x \leqslant 1$ such that

$$\sup_{R\in\Pi}\|F^{-1/p}c_R\chi_R\|_{L^{r,\infty}(\mu)} = \sup_{R\in\Pi}c_R\|F^{-1/p}\chi_R\|_{L^{r,\infty}(\mu)} < \infty, \tag{5.4}$$

where $\mathrm{d}\mu = F(x)\omega(x)\,\mathrm{d}x$. Let $f = F^{-1/p}\chi_R$, then $\|f\|_{L^p(\mu)} = \omega(R)^{1/p}$. Suppose $p < s < r$, and $1/s = t/p + (1-t)/r$ with $0 < t < 1$. Applying the well-known interpolation inequality (see [16], Proposition 1.1.14)

$$\|f\|_{L^s(\mu)} \leqslant C\|f\|^t_{L^{p,\infty}(\mu)}\|f\|^{1-t}_{L^{r,\infty}(\mu)},$$

one has for any $R \in \Pi$,

$$\left(\int_R F^{-s/p+1}\omega(x)\,\mathrm{d}x\right)^{1/s} \leqslant C\omega(R)^{t/p}\|F^{-1/p}\chi_R\|^{1-t}_{L^{r,\infty}(\mu)}.$$

Letting $\delta = s/p - 1$ and combining the above inequality with (5.4), we obtain

$$(c_R\omega(R)^{1/r})^{pr/(r-p)}\left(\frac{1}{\omega(R)}\int_R F^{-\delta}\omega(x)\,\mathrm{d}x\right)^{1/\delta} \leqslant C < \infty.$$

On the other hand, by the Hölder inequality with exponents $(\delta+\varepsilon)/\varepsilon$, $(\delta+\varepsilon)/\delta$ one has

$$\left(\frac{1}{\omega(R)}\int_R F^{-\delta}\omega(x)\,\mathrm{d}x\right)^{1/\delta}\left(\frac{1}{\omega(R)}\int_R F^{\varepsilon}\omega(x)\,\mathrm{d}x\right)^{1/\varepsilon} \geqslant 1$$

for all $\delta, \varepsilon > 0$. Hence

$$(c_R\omega(R)^{1/r})^{pr/(r-p)} \leqslant C\left(\frac{1}{\omega(R)}\int_R F^{\varepsilon}\omega(x)\,\mathrm{d}x\right)^{1/\varepsilon} \leqslant C(M^s_{\omega(x)\,\mathrm{d}x}(F^{\varepsilon})(x))^{1/\varepsilon}$$

for $x \in R$. It is known that $M^s_{\omega(x)\,\mathrm{d}x}$ is bounded on $L^{1/\varepsilon}(\omega)$ for $0 < \varepsilon < 1$ when $\omega \in A_\infty$ (see [10]), hence we have

$$\begin{aligned}\int \sup_{R\in\Pi}((c_R)^r\omega(R))^{p/(r-p)}\chi_R\omega(x)\,\mathrm{d}x &\lesssim \int (M^s_{\omega(x)\,\mathrm{d}x}(F^{\varepsilon})(x))^{1/\varepsilon}\omega(x)\,\mathrm{d}x\\ &\lesssim \int F(x)\omega(x)\,\mathrm{d}x < \infty.\end{aligned}$$

This completes the proof (i) of Theorem 5.1.

Part (ii): It is easy to see that (5.2) $\Rightarrow$ (5.3) is a direct consequence of the second part of Lemma 5.1. Now suppose that (5.3) holds. We first prove (5.2) for $q = r$ following the original argument of Verbitsky [36]. By Lemma 5.3, there exists $F \in L^1(\omega)$, $F \geqslant 0$, such that

$$\inf_{R\in\Pi}\int F^{1-r/p}(c_R\chi_R)^r\omega(x)\,\mathrm{d}x = \inf_{R\in\Pi}(c_R)^r\int_R F^{1-r/p}\omega(x)\,\mathrm{d}x > 0.$$

It follows from the above inequality that

$$\begin{aligned}&\int \sup_{R\in\Pi}((c_R)^r\omega(R))^{p/(r-p)}\chi_R\omega(x)\,\mathrm{d}x\\ &\qquad\leqslant \int \sup_{R\in\Pi}\left(\frac{1}{\omega(R)}\int_R F^{1-r/p}\omega(y)dy\right)^{p/(p-r)}\chi_R\omega(x)\,\mathrm{d}x\\ &\qquad\leqslant \int (M^s_{\omega(x)\,\mathrm{d}x}(F^{1-r/p})(x))^{p/(p-r)}\omega(x)\,\mathrm{d}x \leqslant C\int F(x)\omega(x)\,\mathrm{d}x < \infty.\end{aligned}$$

When $q < r$, we use the argument of Bownik [1] by taking advantage of the already established duality of $\dot{f}_p^{\alpha,1}$, $p > 1$ in Section 3. Note that by duality

$$\|s\|_{\ell^r} = \sup_{t=\{t_R\}} \frac{(\sum |s_R|^q |t_R|^q)^{1/q}}{\|t\|_{\ell^{rq/(r-q)}}}.$$

Hence, equation (5.3) is equivalent to the inequality

$$\left(\sum_{R\in\Pi} |s_R|^q |t_R|^q\right)^{1/q} \leqslant C \left\| \left(\sum_{R\in\Pi} |s_R|^q (c_R)^q \chi_R\right)^{1/q} \right\|_{L^p(\omega)} \|t\|_{\ell^{rq/(r-q)}}. \tag{5.5}$$

On the other hand, since $1 < p/q < \infty$, by the already established duality $(\dot{f}_{p/q}^{\alpha,1})^* = \dot{h}_{p/(p-q)}^{-\alpha,\infty}$, one has for $\alpha = (n_1/2, n_2/2)$,

$$\begin{aligned} \sup_{u=\{u_R\}} \frac{|\sum u_R \overline{v}_R|}{\|\sum u_R \chi_R\|_{L^{p/q}(\omega)}} &= \sup_{u=\{u_R\}} \frac{|\langle u, v\rangle|}{\|u\|_{\dot{f}_{p/q}^{\alpha,1}}} = \|v\|_{\dot{h}_{p/(p-q)}^{-\alpha,\infty}} \\ &= \left\| \sup_{R\in\mathcal{D}} |v_R| \omega(R)^{-1} \chi_R \right\|_{L^{p/(p-q)}(\omega)}. \end{aligned} \tag{5.6}$$

Let

$$v_R = \begin{cases} |t_R|^q (c_R)^{-q}, & R \in \Pi; \\ 0, & R \in \mathcal{D} \setminus \Pi \end{cases}$$

and

$$u_R = \begin{cases} |s_R|^q (c_R)^q, & R \in \Pi; \\ 0, & R \in \mathcal{D} \setminus \Pi. \end{cases}$$

Then (5.6) may be rewritten by taking the qth roots in the form

$$\sup_{s=\{s_R\}} \frac{\left|\sum_{R\in\Pi} |s_R|^q |t_R|^q\right|^{1/q}}{\left\| \left(\sum_{R\in\Pi} |s_R|^q (c_R)^q \chi_R\right)^{1/q} \right\|_{L^p(\omega)}} = \left\| \sup_{R\in\Pi} |t_R| (c_R)^{-1} \omega(R)^{-1/q} \chi_R \right\|_{L^{pq/(p-q)}(\omega)}. \tag{5.7}$$

Let $p_1 = pq/(p-q)$, $r_1 = rq/(r-q)$ and $\tilde{c}_R = (c_R)^{-1}\omega(R)^{-1/q}$. Combining (5.5) with (5.7) yields

$$\left\| \sup_{R\in\Pi} |t_R| (\tilde{c}_R) \chi_R \right\|_{L^{p_1}(\omega)} \leqslant C \|t\|_{\ell^{r_1}}$$

for all $t = \{t_R\}_R$. Using the facts that $p_1 r_1/(r_1 - p_1) = pr/(p-r)$, $p_1 < r_1$, and applying (i) of Theorem 5.1, we get from the preceding inequality

$$\int \sup_{R\in\Pi} ((\tilde{c}_R)^{r_1} \omega(R))^{p_1/(r_1-p_1)} \chi_R \omega(x)\, \mathrm{d}x = \int \sup_{R\in\Pi} ((c_R \chi_R)^r \omega(R))^{p/(r-p)}\, \mathrm{d}x < \infty.$$

Hence, (5.2) holds for $q < r$. We complete the proof. □

Acknowledgement. The authors would like to thank the referee very much for his/her helpful comments and suggestions.

References

[1] *M. Bownik*: Duality and interpolation of anisotropic Triebel-Lizorkin spaces. Math. Z. *259* (2008), 131–169. zbl MR doi

[2] *L. Carleson*: A counterexample for measures bounded on H^p for the bidisc. Mittag-Leffler Report. No. 7 (1974).

[3] *S.-Y. A. Chang, R. Fefferman*: A continuous version of duality of H^1 with BMO on the bidisc. Ann. Math. (2) *112* (1980), 179–201. zbl MR doi

[4] *S.-Y. A. Chang, R. Fefferman*: The Calderón-Zygmund decomposition on product domains. Am. J. Math. *104* (1982), 455–468. zbl MR doi

[5] *S.-Y. A. Chang, R. Fefferman*: Some recent developments in Fourier analysis and H^p theory on product domains. Bull. Am. Math. Soc., New Ser. *12* (1985), 1–43. zbl MR doi

[6] *D. Cruz-Uribe, J. M. Martell, C. Pérez*: Sharp weighted estimates for classical operators. Adv. Math. *229* (2012), 408–441. zbl MR doi

[7] *W. Ding, G. Lu*: Duality of multi-parameter Triebel-Lizorkin spaces associated with the composition of two singular integral operators. Trans. Am. Math. Soc. *368* (2016), 7119–7152. zbl MR doi

[8] *W. Ding, Y. Zhu*: Duality of weighted multiparameter Triebel-Lizorkin spaces. Acta Math. Sci., Ser. B, Engl. Ed. *37* (2017), 1083–1104. zbl MR doi

[9] *X. Fan, J. He, B. Li, D. Yang*: Real-variable characterizations of anisotropic product Musielak-Orlicz Hardy spaces. Sci. China, Math. *60* (2017), 2093–2154. zbl MR doi

[10] *R. Fefferman*: Strong differentiation with respect to measures. Am. J. Math. *103* (1981), 33–40. zbl MR doi

[11] *R. Fefferman*: Calderón-Zygmund theory for product domains: H^p spaces. Proc. Natl. Acad. Sci. USA *83* (1986), 840–843. zbl MR doi

[12] *R. Fefferman*: Harmonic analysis on product spaces. Ann. Math. (2) *126* (1987), 109–130. zbl MR doi

[13] *R. Fefferman, E. M. Stein*: Singular integrals on product spaces. Adv. Math. *45* (1982), 117–143. zbl MR doi

[14] *S. H. Ferguson, M. T. Lacey*: A characterization of product BMO by commutators. Acta Math. *189* (2002), 143–160. zbl MR doi

[15] *M. Frazier, B. Jawerth*: A discrete transform and decompositions of distribution spaces. J. Funct. Anal. *93* (1990), 34–170. zbl MR doi

[16] *L. Grafakos*: Classical and Modern Fourier Analysis. Pearson/Prentice Hall, Upper Saddle River, 2004. zbl MR

[17] *R. F. Gundy, E. M. Stein*: H^p theory for the polydisk. Proc. Natl. Acad. Sci. USA *76* (1979), 1026–1029. zbl MR doi

[18] *Y. Han, M.-Y. Lee, C.-C. Lin, Y.-C. Lin*: Calderón-Zygmund operators on product Hardy spaces. J. Funct. Anal. *258* (2010), 2834–2861. zbl MR doi

[19] *Y. Han, J. Li, G. Lu*: Duality of multiparameter Hardy spaces H^p on spaces of homogeneous type. Ann. Sc. Norm. Super. Pisa, Cl. Sci. *9* (2010), 645–685. zbl MR doi

[20] *Y. Han, J. Li, G. Lu*: Multiparameter Hardy space theory on Carnot-Carathéodory spaces and product spaces of homogeneous type. Trans. Am. Math. Soc. *365* (2013), 319–360. zbl MR doi

[21] *Y. Han, G. Lu, Z. Ruan*: Boundedness criterion of Journé's class of singular integrals on multiparameter Hardy spaces. J. Funct. Anal. *264* (2013), 1238–1268. zbl MR doi

[22] *Y. Han, G. Lu, Z. Ruan*: Boundedness of singular Integrals in Journé's class on weighted multiparameter Hardy spaces. J. Geom. Anal. *24* (2014), 2186–2228. zbl MR doi
[23] *Y. Han, C. Lin, G. Lu, Z. Ruan, E. T. Sawyer*: Hardy spaces associated with different homogeneities and boundedness of composition operators. Rev. Mat. Iberoam. *29* (2013), 1127–1157. zbl MR doi
[24] *J.-L. Journé*: Calderón-Zygmund operators on product spaces. Rev. Mat. Iberoam. *1* (1985), 55–91. zbl MR doi
[25] *J.-L. Journé*: Two problems of Calderón-Zygmund theory on product spaces. Ann. Inst. Fourier *38* (1988), 111–132. zbl MR doi
[26] *B. Li, M. Bownik, D. Yang, W. Yuan*: Duality of weighted anisotropic Besov and Triebel-Lizorkin spaces. Positivity *16* (2012), 213–244. zbl MR doi
[27] *B. D. Li, X. Fan, Z. W. Fu, D. Yang*: Molecular characterization of anisotropic Musielak-Orlicz Hardy spaces and their applications. Acta Math. Sin., Engl. Ser. *32* (2016), 1391–1414. zbl MR doi
[28] *J. Liu, D. Yang, W. Yuan*: Anisotropic Hardy-Lorentz spaces and their applications. Sci. China, Math. *59* (2016), 1669–1720. zbl MR doi
[29] *J. Liu, D. Yang, W. Yuan*: Anisotropic variable Hardy-Lorentz spaces and their real interpolation. J. Math. Anal. Appl. *456* (2017), 356–393. zbl MR doi
[30] *J. Liu, D. Yang, W. Yuan*: Littlewood-Paley characterizations of anisotropic Hardy-Lorentz spaces. Acta Math. Sci., Ser. B, Engl. Ed. *38* (2018), 1–33. zbl MR doi
[31] *G. Z. Lu, Y. P. Zhu*: Singular integrals and weighted Triebel-Lizorkin and Besov Spaces of arbitrary number of parameters. Acta Math. Sin., Engl. Ser. *29* (2013), 39–52. zbl MR doi
[32] *J. Pipher*: Journér's covering lemma and its extension to higher dimensions. Duke Math. J. *53* (1986), 683–690. zbl MR doi
[33] *G. Pisier*: Factorization of operators through $L^{p\infty}$ or L^{p1} and non-commutative generalizations. Math. Ann. *276* (1986), 105–136. zbl MR doi
[34] *Z. Ruan*: Weighted Hardy spaces in three-parameter case. J. Math. Anal. Appl. *367* (2010), 625–639. zbl MR doi
[35] *H. Triebel*: Theory of Function Spaces. Monographs in Mathematics 78, Birkhäuser, Basel, 1983. zbl MR doi
[36] *I. E. Verbitsky*: Imbedding and multiplier theorems for discrete Littlewood-Paley spaces. Pac. J. Math. *176* (1996), 529–556. zbl MR doi
[37] *W. Yuan, W. Sickel, D. Yang*: Morrey and Campanato Meet Besov, Lizorkin and Triebel. Lecture Notes in Mathematics 2005, Springer, Berlin, 2010. zbl MR doi

Authors' addresses: Wei Ding, School of Sciences, Nantong University, Nantong 226007, P. R. China, e-mail: dingwei@ntu.edu.cn; Jiao Chen (corresponding author), School of Mathematical Sciences, Chongqing Normal University, Chongqing 400000, P. R. China, e-mail: chenjiaobnu@163.com; Yaoming Niu, Faculty of Mathematics, Baotou Teachers' College of Inner Mongolia University of Science and Technology, Baotou 014030, P. R. China, e-mail: nymmath@126.com.