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BLOCH TYPE SPACES ON THE UNIT BALL OF A HILBERT SPACE

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Abstract. We initiate the study of Bloch type spaces on the unit ball of a Hilbert space. As applications, the Hardy-Littlewood theorem in infinite-dimensional Hilbert spaces and characterizations of some holomorphic function spaces related to the Bloch type space are presented.

Keywords: Bloch type space; Lipschitz space; Hardy-Littlewood theorem; Hilbert space

MSC 2010: 32A18, 46E15

1. Introduction

The classical Bloch space of holomorphic functions on the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$ was extended to the higher dimension cases. In 1975, Hahn introduced the notion of Bloch functions on bounded homogeneous domains in $\mathbb{C}^n$ using the terminology from differential geometry [7]. In [19], [20], Timoney studied further Bloch functions on bounded homogeneous domains in terms of the Bergman metric. In [11], Krantz and Ma considered function theoretic and functional analytic properties of Bloch functions on a strongly pseudoconvex domain. To have a more complete insight on the theory of the Bloch space in the finite dimensional space, see the book by Zhu [25].

Recently, Bloch functions on the unit ball of an infinite-dimensional complex Hilbert space have been studied by Blasco, Galindo and Miralles [1]. In this article, we will continue the study in [1] and consider Bloch type spaces on the unit ball of a Hilbert space. Especially, we give four semi-norms of the Bloch type space

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and show their equivalences and present some other equivalent characterizations for Bloch functions from the geometric perspective which are the infinite-dimensional generalization of [19], Theorem 3.4.

Let $\mathbb{B}$ be the open unit ball of the complex Hilbert space E and let $\partial\mathbb{B}$ be the unit sphere. The class of all holomorphic functions $f\colon \mathbb{B} \to \mathbb{C}$ is denoted by $H(\mathbb{B})$. Denote by $\mathrm{Aut}(\mathbb{B})$ the group of all biholomorphic mappings of $\mathbb{B}$ onto itself. For $0 \leqslant \alpha < \infty$, let H^∞_α be the space of holomorphic functions $f \in H(\mathbb{B})$ satisfying

$$\sup_{x\in\mathbb{B}} (1-\|x\|^2)^\alpha |f(x)| < \infty.$$

We abbreviate $H^\infty = H^\infty_0$ for $\alpha = 0$.

The classical α-Bloch space is the space of holomorphic functions $F\colon \mathbb{D} \to \mathbb{C}$ satisfying

$$\|F\|_{\mathcal{B}^\alpha(\mathbb{D})} := \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |F'(z)| < \infty.$$

Now we introduce four semi-norms of the Bloch type space for $f \in H(\mathbb{B})$.

Denote

$$\|f\|_{1,\alpha} := \sup_{x\in\mathbb{B}} (1-\|x\|^2)^\alpha \|Df(x)\|,$$
$$\|f\|_{2,\alpha} := \sup_{x\in\mathbb{B}} (1-\|x\|^2)^\alpha |\mathcal{R}f(x)|,$$
$$\|f\|_{3,\alpha} := \sup_{y\in\partial\mathbb{B}} \|f_y\|_{\mathcal{B}^\alpha(\mathbb{D})},$$

where $\mathcal{R}f(x) = Df(x)(x)$, $f_y(z) = f(zy)$ for $z \in \mathbb{D}$.

The Möbius transforms of $\mathbb{B}$ are holomorphic mappings φ_a, $a \in \mathbb{B}$, given by

$$\varphi_a(x) = (P_a + s_a Q_a)(m_a(x)),$$

where $s_a = \sqrt{1-\|a\|^2}$, $P_a(x) = (\langle x,a\rangle/\langle a,a\rangle)a$, $Q_a = \mathrm{Id} - P_a$ and $m_a(x) = (a-x)/(1-\langle x,a\rangle)$.

Define

$$\|f\|_{4,\alpha} := \sup_{x\in\mathbb{B}} (1-\|x\|^2)^{\alpha-1} \|\widetilde{\nabla} f(x)\|,$$

where $\widetilde{\nabla} f(x) = Df\circ\varphi_x(0)$ with $\varphi_x \in \mathrm{Aut}(\mathbb{B})$.

Note that, by [1], Lemma 3.5

$$\|\widetilde{\nabla} f(x)\| = \sup_{w\neq 0} \frac{(1-\|x\|^2)|Df(x)(w)|}{\sqrt{(1-\|x\|^2)\|w\|^2 + |\langle w,x\rangle|^2}}.$$

Hence, we have

$$\|f\|_{4,\alpha} = \sup_{x\in\mathbb{B}} \sup_{w\neq 0} \frac{(1-\|x\|^2)^\alpha |Df(x)(w)|}{\sqrt{(1-\|x\|^2)\|w\|^2 + |\langle w,x\rangle|^2}}.$$

For $\alpha > 0$, denote

$$\mathcal{B}^\alpha = \{f \in H(\mathbb{B})\colon\ \|f\|_{1,\alpha} < \infty\},$$

and

$$T_\alpha = \{f \in H(\mathbb{B})\colon\ \|f\|_{4,\alpha} < \infty\}.$$

Equipped with the norm $\|f\|_\alpha = |f(0)| + \|f\|_{1,\alpha}$ for $f \in \mathcal{B}^\alpha$, the Bloch type space $\mathcal{B}^\alpha$ becomes a Banach space as usual. Denote the class of Bloch functions defined on $\mathbb{B}$ by $\mathcal{B}$ instead of $\mathcal{B}^1$. The little Bloch space will be denoted by $\mathcal{B}_0$; it consists of functions $f \in \mathcal{B}$ such that

$$\lim_{\|x\|\to 1^-} \|\widetilde{\nabla} f(x)\| = 0.$$

Now our main results can be described as follows.

Theorem 1. *Let $\alpha > 0$ and let $\mathbb{B}$ be the open unit ball of the complex Hilbert space E. If f is a complex-valued holomorphic function on $\mathbb{B}$, then the three seminorms $\|f\|_{1,\alpha}$, $\|f\|_{2,\alpha}$, and $\|f\|_{3,\alpha}$ are equivalent.*

Theorem 2. *Let $\mathbb{B}$ be the open unit ball of the complex Hilbert space E with $\dim E \geqslant 2$ and let f be a complex-valued holomorphic function on $\mathbb{B}$.*
(i) *If $0 < \alpha < 1/2$, then $f \in T_\alpha$ if and only if f is constant.*
(ii) *If $\alpha = 1/2$, then $f \in T_{1/2}$ if and only if $|Df(x)(y)|$ is bounded for all $x \in \mathbb{B}$ and $y \in \partial\mathbb{B}$ with $\langle x, y\rangle = 0$.*
(iii) *If $\alpha > 1/2$, then the two seminorms $\|f\|_{1,\alpha}$ and $\|f\|_{4,\alpha}$ are equivalent.*

Note that the results for $\alpha = 1$ in Theorems 1 and 2 have been obtained in [1] and the condition $\dim E \geqslant 2$ in Theorem 2 (i) and (ii) cannot be deleted in general.

From Theorem 1 and its proof, the interested reader can give some equivalent characterizations for the little Bloch type space $\mathcal{B}_0$. It is worth mentioning that the approach for the finite-dimensional case depends usually on the integral representation for holomorphic functions. However, it may fail for the infinite-dimensional setting. Hence we need to overcome the restriction of dimension in achieving our main results.

In this paper, we add some more equivalent characterizations for Bloch functions. To this end, we now give the definition of a schlicht disk for holomorphic functions defined in Hilbert spaces as in the case of several complex variables.

Definition 3. Let f be a holomorphic function on a domain Ω in E. For $z_0 \in \mathbb{C}$, the disk

$$D(z_0, r) = \{z \in \mathbb{C}\colon\ |z - z_0| < r\}$$

is called a schlicht disk in the range of f if there exists a holomorphic mapping $g\colon\ \mathbb{D} \to \Omega$ such that $f \circ g(z) = z_0 + rz$.

Following the paper [12], a holomorphic function $f\colon\ \mathbb{B} \to \mathbb{C}$ is said to be normal if

$$M_f = \sup\Big\{ \frac{(1 - \|x\|^2)\|Df(x)\|}{1 + |f(x)|^2} \colon\ x \in \mathbb{B} \Big\} < \infty.$$

With these two concepts in hand, our second main result can be established as follows.

Theorem 4. *Let $f \in H(\mathbb{B})$. Then the following conditions are equivalent:*

(i) *f is a Bloch function;*

(ii) *the radii of schlicht disks in the range of f are bounded above;*

(iii) *the family*

$$\{f \circ g\colon\ g \in H(\mathbb{D}, \mathbb{B})\}$$

is a family of Bloch functions with uniformly bounded Bloch norm;

(iv) *the family*

$$\big\{f \circ g - f \circ g(0)\colon\ g \in H(\mathbb{D}, \mathbb{B})\big\}$$

is a normal family in the sense of Montel;

(v) *the family*

$$\mathcal{F}_f := \{h = f \circ \phi - f \circ \phi(0)\colon\ \phi \in Aut(\mathbb{B})\}$$

is a family of normal functions such that M_h is uniformly bounded.

Remark 5. The Bloch space on bounded symmetric domains in arbitrary complex Banach spaces was considered in [5]. Although we treat only Bloch functions defined on the unit ball of a Hilbert space, all results in Theorem 4 can be proved in the more general setting of bounded symmetric domains.

The remaining part of this paper is organized as follows. Theorems 1, 2 and 4 are proved in Section 2. In Section 3, the Hardy-Littlewood theorem in infinite-dimensional Hilbert spaces is established as an application of Theorem 1. In addition, we give some equivalent characterizations for holomorphic function spaces related to the Bloch type space which are the generalizations of the main results in [4], [23] in infinite-dimensional Hilbert spaces.

2. Proof of Theorems 1, 2 and 4

Throughout this paper, denote by C an absolute positive constant and by $C(\alpha)$ a positive constant depending on α only. They may have different values at different places.

In order to prove Theorems 1 and 2, we establish two lemmas.

Lemma 6. *Let $\alpha \geqslant 0$ and let $f\colon \mathbb{B} \to \mathbb{C}$ be a holomorphic function.*

(i) *If $|f(x)| \leqslant (1-\|x\|^2)^{-\alpha}$ for all $x \in \mathbb{B}$, then*

$$|Df(x)(y)| \leqslant C(\alpha)(1-\|x\|^2)^{-\alpha-1/2}$$

for all $x \in \mathbb{B}$ and $y \in \partial\mathbb{B}$ with $\langle x, y\rangle = 0$.

(ii) *If $|Df(x)(y)| \leqslant (1-\|x\|^2)^{-\alpha}$ for all $x \in \mathbb{B}$, $y \in \partial\mathbb{B}$ with $\langle x, y\rangle = 0$, then*

$$|\mathcal{R}f(x)| \leqslant C(\alpha)(1-\|x\|^2)^{-\alpha-1/2} \quad \forall\, x \in \mathbb{B}.$$

(iii) *If $|f(x)| \leqslant (1-\|x\|^2)^{-\alpha}$ for all $x \in \mathbb{B}$, then*

$$|\mathcal{R}f(x)| \leqslant C(\alpha)(1-\|x\|^2)^{-\alpha-1} \quad \forall\, x \in \mathbb{B}.$$

Proof. (i) Let $x = rx' \in \mathbb{B}$, $y \in \partial\mathbb{B}$ be such that $x' \in \partial\mathbb{B}$ with $\langle x, y\rangle = 0$. Let us consider the holomorphic function $F\colon B^2 \to \mathbb{C}$ given by $F(z_1, z_2) = f(z_1x' + z_2y)$, where B^2 is the open unit ball of $\mathbb{C}^2$. By assumption, we get $|F(z_1, z_2)| \leqslant (1-|z_1|^2-|z_2|^2)^{-\alpha}$. By [18], Lemma 6.4.6, it holds that

$$\left|\frac{\partial F}{\partial z_2}(r, 0)\right| \leqslant C(\alpha)(1-r^2)^{-\alpha-1/2},$$

that is

$$|Df(x)(y)| \leqslant C(\alpha)(1-\|x\|^2)^{-\alpha-1/2}.$$

(ii) Based on the result for $\mathbb{C}^2$ (cf. [21], Lemma 1(a)), we obtain the desired estimate applying the same method as in (i).

(iii) There is just a corollary from (i) and (ii). $\square$

Lemma 7. *Let $\alpha \geqslant 0$ and let $f\colon \mathbb{B} \to \mathbb{C}$ be a holomorphic function. If*

$$|D\mathcal{R}f(x)(y)| \leqslant (1-\|x\|^2)^{-\alpha-1/2}$$

for all $x \in \mathbb{B}$ and $y \in \partial\mathbb{B}$ with $\langle x, y\rangle = 0$, then

$$|Df(x)(y)| \leqslant C(\alpha)(1-\|x\|^2)^{-\alpha}$$

for all $x \in \mathbb{B}$ and $y \in \partial\mathbb{B}$ with $\langle x, y\rangle = 0$.

Proof. For $f \in H(\mathbb{B})$, we rewrite it as $\sum\limits_{n=0}^{\infty} P_n(x)$, where P_n is an n-homogeneous polynomial, that is, the restriction to the diagonal of a continuous n-linear form on the n-fold space $E \times \ldots \times E$. Then $\mathcal{R}P_n = nP_n$ and DP_n is $(n-1)$-homogeneous $(n \geqslant 1)$, so that

$$(D\mathcal{R}P_n)(tx') = n(DP_n)(tx') = nt^{n-1}(DP_n)(x'),$$

where $x' \in \partial B_E$, $0 \leqslant t < 1$.

Hence,

$$\int_0^r (D\mathcal{R}P_n)(tx')\,\mathrm{d}t = r^n(DP_n)(x') = r(DP_n)(rx'),$$

which leads to

$$rDf(rx')(y) = \int_0^r (D\mathcal{R}f)(tx')(y)\,\mathrm{d}t.$$

It follows by assumption that for $r \in [1/2, 1)$, $y \in \partial\mathbb{B}$ with $\langle x', y\rangle = 0$,

$$|Df(rx')(y)| \leqslant 2\int_0^r |(D\mathcal{R}f)(tx')(y)|\,\mathrm{d}t \leqslant 2\int_0^r (1-t^2)^{-\alpha-1/2}\,\mathrm{d}t,$$

then

$$(1-\|x\|^2)^\alpha |Df(x)(y)| \leqslant 2\int_0^r (1-t)^{-1/2}\,\mathrm{d}t = 4(1-\sqrt{1-r}) < 4$$

for $\|x\| \in [1/2, 1)$ and $y \in \partial\mathbb{B}$ with $\langle x, y\rangle = 0$.

Hence, by the maximum principle for holomorphic functions,

$$|Df(x)(y)| \leqslant 4\Big(\frac{4}{3}\Big)^\alpha$$

for $\|x\| \in [0, 1/2]$ and $y \in \partial\mathbb{B}$ with $\langle x, y\rangle = 0$, as desired. $\square$

We now are in a position to prove Theorems 1 and 2.

Proof of Theorem 1. Let $x \in \mathbb{B}$ be fixed. For any $y \in \partial\mathbb{B}$, by the projection theorem, we can write $y = z_1 x + z_2 x_1$ for some $z_1, z_2 \in \mathbb{C}$, $x_1 \in \partial\mathbb{B}$ with $\langle x, x_1\rangle = 0$.

Note that

$$|z_1|^2\|x\|^2 + |z_2|^2 = \|y\|^2 = 1.$$

Hence, for $1/2 \leqslant \|x\| < 1$, we have

$$|Df(x)(y)| \leqslant 2|\mathcal{R}f(x)| + |Df(x)(x_1)|. \tag{2.1}$$

Suppose that $\|f\|_{2,\alpha} = 1$, then by Lemma 6 (i) we have

$$|D\mathcal{R}f(x)(x_1)| \leqslant C(\alpha)(1-\|x\|^2)^{-\alpha-1/2},$$

so by Lemma 7

$$|Df(x)(x_1)| \leqslant C(\alpha)(1-\|x\|^2)^{-\alpha}. \tag{2.2}$$

Hence, by (2.1) and (2.2) for $1/2 \leqslant \|x\| < 1$,

$$\|Df(x)\| = \sup_{y\in\partial\mathbb{B}} |Df(x)(y)| \leqslant (2+C(\alpha))(1-\|x\|^2)^{-\alpha}.$$

By the maximum principle for holomorphic mappings in Banach spaces, we have

$$\|f\|_{1,\alpha} \leqslant 2 + C(\alpha).$$

Consequently,

$$\|f\|_{1,\alpha} \leqslant C(\alpha)\|f\|_{2,\alpha}. \tag{2.3}$$

It is clear that

$$\|f\|_{2,\alpha} \leqslant \|f\|_{1,\alpha}. \tag{2.4}$$

Hence the two seminorms $\|\cdot\|_{1,\alpha}$ and $\|\cdot\|_{2,\alpha}$ are equivalent by (2.3) and (2.4).

Notice that $zf_y'(z) = \mathcal{R}f(zy)$ for any holomorphic f defined on $\mathbb{B}$, so we have that

$$\|f\|_{2,\alpha} \leqslant \|f\|_{3,\alpha}. \tag{2.5}$$

For $1/2 \leqslant |z| < 1$, we have

$$(1-|z|^2)^\alpha |f_y'(z)| \leqslant 2(1-|z|^2)^\alpha |\mathcal{R}f(zy)|.$$

Hence

$$\sup_{|z|\geqslant 1/2} (1-|z|^2)^\alpha |f_y'(z)| \leqslant 2 \sup_{\|x\|\geqslant 1/2} (1-\|x\|^2)^\alpha |\mathcal{R}f(x)| \leqslant 2\|f\|_{2,\alpha}.$$

Combining this with the maximum principle for holomorphic functions, one can show that

$$\|f\|_{3,\alpha} \leqslant 2\Big(\frac{4}{3}\Big)^\alpha \|f\|_{2,\alpha}. \tag{2.6}$$

Hence, by (2.5) and (2.6), the two semi-norms $\|\cdot\|_{2,\alpha}$ and $\|\cdot\|_{3,\alpha}$ are equivalent. $\square$

Remark 8. From the proof of Theorem 1, we see that the semi-norms $\|f\|_{2,\alpha}$ and $\|f\|_{3,\alpha}$ are equivalent for any holomorphic function f defined on the unit ball of a Banach space.

P r o o f of Theorem 2. (i) Let $0 < \alpha < 1/2$ and $f \in T_\alpha$. It holds that

$$|Df(x)(y)| \leqslant \|f\|_{4,\alpha}(1 - \|x\|^2)^{1/2-\alpha}$$

for all $x \in \mathbb{B}$ and $y \in \partial\mathbb{B}$ with $\langle x, y\rangle = 0$.

Now let us show first that

$$g(x) := Df(x)(y) \equiv 0, \quad x \in \mathbb{B} \text{ and } y \in E \text{ with } \langle x, y\rangle = 0. \tag{2.7}$$

To this end, for fixed $x \in \partial\mathbb{B}$ and $y \in E$ such that $\langle x, y\rangle = 0$, we consider the slice function $h(z) = g(zx)$ on $\mathbb{D}$ satisfying the relation

$$\limsup_{z\to\partial\mathbb{D}} |h(z)| = 0.$$

Applying the maximum principle to the holomorphic mapping h, we see that $h(z) \equiv 0$ on $\mathbb{D}$, so (2.7) follows.

Let $x \in \mathbb{B}$ be fixed. For every $w \in \partial\mathbb{B}$, by the projection theorem, we can write $w = zx + y$ with $\langle x, y\rangle = 0$ for some $z \in \mathbb{C}$, $y \in \mathbb{B}$. Hence, it follows that

$$|z|\|x\| \leqslant \|w\| = 1. \tag{2.8}$$

Now we have, by (2.7),

$$|Df(x)(w)| \leqslant |Df(x)(zx)| + |Df(x)(y)| = |z||\mathcal{R}f(x)|.$$

Combining this with (2.8), we obtain

$$\|Df(x)\| = \sup_{w\in\partial\mathbb{B}} |Df(x)(w)| \leqslant \frac{1}{\|x\|}|\mathcal{R}f(x)| \leqslant \|Df(x)\|,$$

which forces that $Df(x)$ and $\bar{x}$ are complex linear.

Note that $Df\colon\, \mathbb{B} \to E^*$ is holomorphic and thus $Df(x) \equiv 0$ on $\mathbb{B}$. Hence, f is constant, as desired.

(ii) Suppose that $|Df(x)(y)| \leqslant C$ for all $x \in \mathbb{B}$ and $y \in \partial\mathbb{B}$ with $\langle x, y\rangle = 0$. Let us show that $f \in T_{1/2}$.

By Lemma 6 (ii), it follows that

$$|\mathcal{R}f(x)| \leqslant C(1 - \|x\|^2)^{-1/2} \quad \forall\, x \in \mathbb{B}. \tag{2.9}$$

For fixed $x \in \mathbb{B}$, every $w \in E$ can be decomposed as $w = zx + y$ with $\langle x, y\rangle = 0$ for some $z \in \mathbb{C}$. Hence

$$(1 - \|x\|^2)\|w\|^2 + |\langle w, x\rangle|^2 = |z|^2\|x\|^2 + (1 - \|x\|^2)\|y\|^2.$$

It follows that

$$\begin{aligned}\frac{|Df(x)(w)|\sqrt{1-\|x\|^2}}{\sqrt{(1-\|x\|^2)\|w\|^2 + |\langle w, x\rangle|^2}} &\leqslant \frac{|z|\|\mathcal{R}f(x)\|\sqrt{1-\|x\|^2}}{\sqrt{|z|^2\|x\|^2 + (1-\|x\|^2)\|y\|^2}} \\ &\quad + \frac{|Df(x)(y)|\sqrt{1-\|x\|^2}}{\sqrt{|z|^2\|x\|^2 + (1-\|x\|^2)\|y\|^2}} \\ &\leqslant \frac{1}{\|x\|}|\mathcal{R}f(x)|\sqrt{1-\|x\|^2} + \Big|Df(x)\Big(\frac{y}{\|y\|}\Big)\Big|.\end{aligned}$$

Combining this with (2.9), we obtain that, for $1/2 \leqslant \|x\| < 1$,

$$\frac{|Df(x)(w)|\sqrt{1-\|x\|^2}}{\sqrt{(1-\|x\|^2)\|w\|^2 + |\langle w, x\rangle|^2}} \leqslant C. \tag{2.10}$$

Furthermore, $\|Df(x)\|$ is bounded for $\|x\| < 1/2$ by the maximum principle for holomorphic mappings. Then

$$\frac{|Df(x)(w)|\sqrt{1-\|x\|^2}}{\sqrt{(1-\|x\|^2)\|w\|^2 + |\langle w, x\rangle|^2}} \leqslant \|Df(x)\| \leqslant C. \tag{2.11}$$

Hence, by (2.10) and (2.11), $f \in T_{1/2}$.

Conversely, supposing $f \in T_{1/2}$, then for $x \in \mathbb{B}$ and $y \in \partial\mathbb{B}$ with $\langle x, y\rangle = 0$ we have

$$|Df(x)(y)| = \frac{|Df(x)(y)|\sqrt{1-\|x\|^2}}{\sqrt{(1-\|x\|^2)\|y\|^2 + |\langle y, x\rangle|^2}}.$$

Hence, $|Df(x)(y)|$ is bounded.

(iii) Note that $T_\alpha \subseteq \mathcal{B}^\alpha$. It remains to show $\mathcal{B}^\alpha \subseteq T_\alpha$ for $\alpha > 1/2$. Let us show first that if a holomorphic function $F\colon B^2 \to \mathbb{C}$ satisfies

$$\|F\|_{\mathcal{B}^\alpha(B^2)} = \sup_{z=(z_1,z_2)\in B^2} (1 - |z|^2)^\alpha \|DF(z)\| < \infty,$$

then

$$\Big|\frac{\partial F}{\partial z_2}(z_1, 0)\Big|(1 - |z_1|^2)^{\alpha - 1/2} \leqslant C(\alpha)\|F\|_{\mathcal{B}^\alpha(B^2)} \quad \forall\, z_1 \in \mathbb{D}. \tag{2.12}$$

To this end, set $s = \frac{1}{\sqrt{3}}(1-|z_1|^2)^{1/2}$ for $z_1 \in \mathbb{D}$. It holds that

$$\frac{\partial^2 F}{\partial z_2 \partial z_1}(z_1, 0) = \frac{1}{2\pi \mathrm{i}} \int_{|w|=s} \frac{\partial F}{\partial z_1}(z_1, w) \frac{\mathrm{d}w}{w^2},$$

then

$$\Bigl|\frac{\partial^2 F}{\partial z_2 \partial z_1}(z_1, 0)\Bigr| \leqslant \frac{\|F\|_{\mathcal{B}^\alpha(B^2)}}{s(1-|z_1|^2-s^2)^\alpha} = \frac{3^{\alpha+1/2}\|F\|_{\mathcal{B}^\alpha(B^2)}}{2^\alpha(1-|z_1|^2)^{\alpha+1/2}} \leqslant \frac{3^{\alpha+1/2}\|F\|_{\mathcal{B}^\alpha(B^2)}}{2^\alpha(1-|z_1|)^{\alpha+1/2}}.$$

Combining this with the formula

$$\frac{\partial F}{\partial z_2}(z_1, 0) - \frac{\partial F}{\partial z_2}(0,0) = z_1 \int_0^1 \frac{\partial^2 F}{\partial z_2 \partial z_1}(t z_1, 0)\, \mathrm{d}t,$$

we obtain

$$\Bigl|\frac{\partial F}{\partial z_2}(z_1, 0)\Bigr| \leqslant \Bigl|\frac{\partial F}{\partial z_2}(0,0)\Bigr| + \frac{3^{\alpha+1/2}\|F\|_{\mathcal{B}^\alpha(B^2)}}{2^\alpha\bigl(\alpha - \frac{1}{2}\bigr)}\bigl((1-|z_1|)^{-\alpha+1/2} - 1\bigr).$$

Hence, for $\alpha > 1/2$,

$$(1-|z_1|)^{\alpha-1/2}\Bigl|\frac{\partial F}{\partial z_2}(z_1, 0)\Bigr| \leqslant \Bigl(1 + \frac{3^{\alpha+1/2}}{2^\alpha\bigl(\alpha-\frac{1}{2}\bigr)}\Bigr)\|F\|_{\mathcal{B}^\alpha(B^2)},$$

and (2.12) follows. Based on this result and applying the method used in Lemma 6 (i), we can easily obtain that, for $f \in \mathcal{B}^\alpha$,

$$(1-\|x\|^2)^{\alpha-1/2}|Df(x)(y)| \leqslant C(\alpha)\|f\|_{1,\alpha}$$

for $x \in \mathbb{B}$ and $y \in \partial\mathbb{B}$ with $\langle x, y\rangle = 0$.

Notice that for any $w = zx + y \in E$ with $z \in \mathbb{C}$, $x \in \mathbb{B}$ and $y \in E$ such that $\langle x, y\rangle = 0$,

$$\frac{|Df(x)(w)|(1-\|x\|^2)^\alpha}{\sqrt{(1-\|x\|^2)\|w\|^2 + |\langle w, x\rangle|^2}} \leqslant \|Df(x)\|(1-\|x\|^2)^\alpha + \Bigl|D(x)\Bigl(\frac{y}{\|y\|}\Bigr)\Bigr|(1-\|x\|^2)^{\alpha-1/2},$$

which shows $\mathcal{B}^\alpha \subseteq T_\alpha$ for $\alpha > 1/2$, as desired. □

Proof of Theorem 4. (i)⇒(ii) Suppose that f is a Bloch function on $\mathbb{B}$. Let $D(z_0, r)$ be a schlicht disk in the range of f. Then there exists a holomorphic function $g\colon \mathbb{D} \to \mathbb{B}$ such that

$$f \circ g(z) = z_0 + rz.$$

Applying the Schwarz lemma for holomorphic functions (cf. [6], page 287), we have

$$\frac{(1-\|g(0)\|^2)\|Dg(0)\|^2+|\langle g(0),Dg(0)\rangle|^2}{(1-\|g(0)\|^2)^2}\leqslant 1.$$

Let $g(0)=x$ and $Dg(0)=w$. Then

$$r=|Df(x)(w)|\leqslant\frac{|Df(x)(w)|(1-\|x\|^2)}{\sqrt{(1-\|x\|^2)\|w\|^2+|\langle w,x\rangle|^2}}\leqslant\|\widetilde{\nabla}f(x)\|,$$

which shows that the radii of the schlicht disks in the range of f are bounded above by $Q_f:=\sup\limits_{x\in\mathbb{B}}\|\widetilde{\nabla}f(x)\|$.

(ii)$\Rightarrow$(i) Suppose the radii of the schlicht disks in the range of f are bounded above by R. For any fixed $y\in\partial\mathbb{B}$, define $g\colon\ \mathbb{D}\to\mathbb{B}$ by $g(z)=zy$. Fix $x\in\mathbb{B}$. By Bloch's theorem, the holomorphic function $f\circ\varphi_x\circ g$ has a schlicht disk in its range of radius

$$B|(f\circ\varphi_x\circ g)'(0)|=B|Df\circ\varphi_x(0)(y)|,$$

where B denotes Bloch's constant.

Therefore, by assumption, it follows that

$$|Df\circ\varphi_x(0)(y)|\leqslant\frac{R}{B}$$

for all $x\in\mathbb{B}$ and $y\in\partial\mathbb{B}$, thus f is a Bloch function, as desired.

(ii)$\Leftrightarrow$(iii) In the proofs above, we have

$$R\leqslant Q_f\leqslant\frac{R}{B}.$$

Note that the schlicht disks in the range of f are exactly those disks which are schlicht disks in the range of $f\circ g$ for some $g\in H(\mathbb{D},\mathbb{B})$. The desired result follows.

(iii)$\Leftrightarrow$(iv) Following the same arguments as in (6)$\Leftrightarrow$(7) in [19], Theorem 3.4, one can prove our result and we omit its details here.

(v)$\Rightarrow$(i) Note that any $h\in\mathcal{F}_f$ satisfies $h(0)=0$. By hypothesis, we have that

$$\{\|Df\circ\phi(0)\|\colon\ \phi\in\operatorname{Aut}(\mathbb{B})\}=\{\|Dh(0)\|\colon\ h\in\mathcal{F}_f\}$$

is bounded, as claimed.

(i)$\Rightarrow$(v) The inequality

$$\|Df\circ\phi(x)\|=\|Df\circ\phi\circ\varphi_x(0)(D\varphi_x(0))^{-1}\|\leqslant\frac{Q_f}{1-\|x\|^2},$$

implies that the family $\mathcal{F}_f$ is a family of normal functions such that M_h is uniformly bounded above by Q_f. Now the proof is complete. $\square$

3. Applications

Hardy and Littlewood [8] gave a characterization of the holomorphic Lipschitz space $\Lambda_\alpha(\mathbb{D})$ of order $\alpha \in (0,1]$ on the unit open disk $\mathbb{D}$ of $\mathbb{C}$, which states that a holomorphic function f on $\mathbb{D}$ satisfies

$$\sup_{\substack{z,w\in\mathbb{D}\\ z\neq w}} \frac{|f(z)-f(w)|}{|z-w|^\alpha} < \infty$$

if and only if

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{1-\alpha}|f'(z)| < \infty. \tag{3.1}$$

In [10], Krantz extended this result to harmonic functions. See also the version of the Hardy-Littlewood theorem for quaternionic slice regular functions [17]. As an application of Theorem 1, we first establish the Hardy-Littlewood theorem in the infinite-dimensional Hilbert space.

Denote

$$\operatorname{Lip}\alpha := \Big\{ f \in H(\mathbb{B})\colon \sup_{\substack{x,y\in\mathbb{B}\\ x\neq y}} \frac{|f(x)-f(y)|}{\|x-y\|^\alpha} < \infty \Big\}.$$

Theorem 9. *Let* $\alpha \in (0,1]$. *Then* $\operatorname{Lip}\alpha = \mathcal{B}^{1-\alpha}$.

P r o o f. Taking the same arguments as in [18], Lemma 6.4.8, one can show easily the inclusion $\mathcal{B}^{1-\alpha} \subseteq \operatorname{Lip}\alpha$.

Conversely, let $f \in \operatorname{Lip}\alpha$ and $x \in \partial\mathbb{B}$. Let us consider the holomorphic function $F\colon \mathbb{D} \to \mathbb{C}$ given by $F(z) = f(zx)$, which is in $\Lambda_\alpha(\mathbb{D})$. By the classical Hardy-Littlewood theorem, we have

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{1-\alpha}|F'(z)| < \infty,$$

which implies that

$$\sup_{x\in\mathbb{B}} (1-\|x\|^2)^{1-\alpha}|\mathcal{R}f(x)| < \infty.$$

From Theorem 1, it follows that $f \in \mathcal{B}^{1-\alpha}$. □

Holland and Walsh [9] further considered the Hardy-Littlewood theorem in the limit case $\alpha = 0$ for holomorphic Bloch spaces and proved that a holomorphic function f on $\mathbb{D}$ satisfies

$$\sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty$$

if and only if

$$\sup_{\substack{z,w\in\mathbb{D}\\ z\neq w}} \sqrt{(1-|z|^2)(1-|w|^2)}\Big|\frac{f(z)-f(w)}{z-w}\Big| < \infty. \tag{3.2}$$

The extensions to higher dimensions of the Holland-Walsh result as in (3.2) were obtained in [14], [16] for holomorphic functions in the unit ball of $\mathbb{C}^n$. Later, Pavlović found that the Holland-Walsh result holds even for an arbitrary C^1-function defined on the unit ball of $\mathbb{R}^n$, see [15].

In [23] Zhao gave a characterization of holomorphic Bloch type spaces on the unit ball of $\mathbb{C}^n$.

Theorem 10. *Let $0 < \alpha \leqslant 2$. Let λ be any real number satisfying the following conditions:*

(i) $0 \leqslant \lambda \leqslant \alpha$ *if* $0 < \alpha < 1$;
(ii) $0 < \lambda < 1$ *if* $\alpha = 1$;
(iii) $\alpha - 1 \leqslant \lambda \leqslant 1$ *if* $1 < \alpha \leqslant 2$.

Then a holomorphic function f on the open unit ball B^n of $\mathbb{C}^n$ is such that

$$\sup_{z\in B^n} (1-|z|^2)^{\alpha}|\nabla f(z)| < \infty$$

if and only if

$$\sup_{\substack{z,w\in B^n\\ z\neq w}} (1-|z|^2)^{\lambda}(1-|w|^2)^{\alpha-\lambda}\frac{|f(z)-f(w)|}{|z-w|} < \infty.$$

For more relative equivalent characterizations of Bloch type functions in the finite-dimensional Euclidean space, we refer to [2], [3], [13], [22] and references therein.

In [23], Zhao also offered some examples to show that the conditions on α and λ in Theorem 10 cannot be improved. In fact, Theorem 10 does hold for $\alpha = \lambda = 0$ and holds for any C^1-function defined on the unit ball of any infinite dimensional Hilbert space. Recently, Dai and Wang in [4] revealed the reason in the theory why some equivalent characterizations of the Bloch type space require extra conditions for α. In the present paper, we will show the main results in [4] still hold for holomorphic functions defined in any infinite-dimensional Hilbert space.

Denote

$$S_{\alpha,\lambda} := \Big\{f\in H(\mathbb{B})\colon \sup_{\substack{x,y\in\mathbb{B}\\ x\neq y}} (1-\|x\|^2)^{\lambda}(1-\|y\|^2)^{\alpha-\lambda}\frac{|f(x)-f(y)|}{\|x-y\|} < \infty\Big\}.$$

Theorem 11.

(i) *Let α, λ be any real numbers satisfying the following conditions:*

 (a) $0 < \lambda < \alpha - 1$ *if* $1 < \alpha \leqslant 2$;

 (b) $0 < \lambda \leqslant \alpha/2$ *if* $\alpha > 2$.

 Then $S_{\alpha,\lambda} = \mathcal{B}^{\lambda+1}$.

(ii) *Let α, λ be any real numbers satisfying the following conditions:*

 (a) $1 < \lambda < \alpha$ *if* $1 < \alpha \leqslant 2$;

 (b) $\alpha/2 < \lambda \leqslant \alpha$ *if* $\alpha > 2$.

 Then $S_{\alpha,\lambda} = \mathcal{B}^{\alpha-\lambda+1}$.

(iii) *Let* $\alpha \geqslant 1$. *Then* $S_{\alpha,\lambda} = H^\infty$ *for* $\lambda = 0$ *or* $\lambda = \alpha$.

Due to Theorems 10 and 11, the space $S_{\alpha,\lambda}$ is described completely for all cases $0 \leqslant \lambda \leqslant \alpha$.

In order to prove Theorem 11, we first generalize a result in [24] by Theorem 1 as follows.

Lemma 12. *Let* $\alpha > 1$. *Then* $\mathcal{B}^\alpha = H^\infty_{\alpha-1}$.

Proof. Let $x \in \partial\mathbb{B}$ and $f \in H^\infty_{\alpha-1}$ with $\|f\|_{H^\infty_{\alpha-1}} = 1$. Let us consider the holomorphic function $F\colon \mathbb{D} \to \mathbb{C}$ given by $F(z) = f(zx)$. Then $(1-|z|^2)^{\alpha-1}|F(z)| \leqslant 1$. By [24], Proposition 7, there exists a constant $C > 0$ such that

$$(1-|z|^2)^\alpha |F'(z)| \leqslant C \quad \forall\, z \in \mathbb{D},$$

that is

$$(1-|z|^2)^\alpha |Df(zx)(x)| \leqslant C \quad \forall\, z \in \mathbb{D},$$

which implies

$$(1-\|y\|^2)^\alpha |\mathcal{R}f(y)| \leqslant C \quad \forall\, y \in \mathbb{B}.$$

By virtue of Theorem 1, if follows that $f \in \mathcal{B}^\alpha$.

Conversely, let $f \in \mathcal{B}^\alpha$, then we have $F \in \mathcal{B}^\alpha$, which is also in $H^\infty_{\alpha-1}$ by [24], Proposition 7 again. Consequently, we have $f \in H^\infty_{\alpha-1}$. $\square$

Proof of Theorem 11. (i) Let $f \in \mathcal{B}^{\lambda+1}$. For $x, y \in \mathbb{B}$, we choose a path $\gamma(t) = tx + (1-t)y$, $t \in [0,1]$ connecting x and y. Then it follows that

$$\begin{aligned}|f(x) - f(y)| &= \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))\,\mathrm{d}t\right| = \left|\int_0^1 Df(\gamma(t))(x-y)\,\mathrm{d}t\right| \\ &\leqslant \int_0^1 |Df(\gamma(t))(x-y)|\,\mathrm{d}t \leqslant \int_0^1 \|Df(\gamma(t))\|\|x-y\|\,\mathrm{d}t\end{aligned}$$

$$\leqslant C\|x-y\| \int_0^1 \frac{1}{(1-\|\gamma(t)\|^2)^{\lambda+1}} \, \mathrm{d}t$$

$$\leqslant C\|x-y\| \int_0^1 \frac{1}{(1-t\|x\|-(1-t)\|y\|)^{\lambda+1}} \, \mathrm{d}t.$$

From the proof of [4], Theorem 3.1, we have

$$\int_0^1 \frac{1}{(1-t\|x\|-(1-t)\|y\|)^{\lambda+1}} \, \mathrm{d}t \leqslant \frac{C}{(1-\|x\|^2)^{\lambda}(1-\|y\|^2)^{\alpha-\lambda}}.$$

Hence,

$$|f(x)-f(y)| \leqslant \frac{C\|x-y\|}{(1-\|x\|^2)^{\lambda}(1-\|y\|^2)^{\alpha-\lambda}},$$

which shows that $f \in S_{\alpha,\lambda}$.

Conversely, the method in [4], Theorem 3.1 can be applied word by word to prove the inclusion $S_{\alpha,\lambda} \subseteq \mathcal{B}^{\lambda+1}$ by Lemma 12.

(ii) It follows by using (i). It is easy to check (iii) if we can prove that

$$|f(x)-f(a)| \leqslant 2\frac{\|x-a\|}{1-\|x\|} \quad \forall\, x, a \in \mathbb{B}, \tag{3.3}$$

for holomorphic functions $f \in H^{\infty}$ with $\|f\|_{H^{\infty}} = 1$.

Let us show inequality (3.3). From the Schwarz lemma for holomorphic functions, we have

$$|f(x)-f(0)| \leqslant 2\|x\| \quad \forall\, x \in \mathbb{B}.$$

Applying this inequality to the holomorphic function $f \circ \varphi_a$, we conclude that

$$|f(x)-f(a)| \leqslant 2\|\varphi_a(x)\| \leqslant 2\frac{\|a-x\|}{|1-\langle x,a\rangle|} \leqslant 2\frac{\|x-a\|}{1-\|x\|} \quad \forall\, x, a \in \mathbb{B},$$

as desired. □

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