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ON THE CARDINALITY OF URYSOHN SPACES AND WEAKLY H-CLOSED SPACES

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Abstract. We introduce the cardinal invariant $\theta\text{-}aL'(X)$, related to $\theta\text{-}aL(X)$, and show that if X is Urysohn, then $|X| \leqslant 2^{\theta\text{-}aL'(X)\chi(X)}$. As $\theta\text{-}aL'(X) \leqslant aL(X)$, this represents an improvement of the Bella-Cammaroto inequality.

We also introduce the classes of firmly Urysohn spaces, related to Urysohn spaces, strongly semiregular spaces, related to semiregular spaces, and weakly H-closed spaces, related to H-closed spaces.

Keywords: Urysohn space; θ-closure; pseudocharacter; almost Lindelöf degree; cardinality; cardinal invariant

MSC 2010: 54A25, 54D10, 54D20

1. Introduction

We follow the notation from [8] and [9]. Recall that a *space* X is *Urysohn* if for every two distinct points $x, y \in X$ there are open sets U and V such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$.

Many researchers have worked on the cardinality of Urysohn and H-closed spaces, in particular considering cardinal invariants defined by using the θ-closure operator defined below (see, for instance, [1], [2], [4], [5], [6], [7], [10], [11]).

For a space X, we denote by $\chi(X)$ ($\psi(X)$, $\pi\chi(X), c(X), t(X)$) the *character* (respectively, *pseudocharacter*, *π-character*, *celluarity*, *tightness*), see [8].

The *θ-closure* of a set A in a space X is the set $\mathrm{cl}_\theta(A) = \{x \in X\colon$ for every neighborhood $U \ni x$, $\overline{U} \cap A \neq \emptyset\}$; A is said to be *θ-closed* if $A = \mathrm{cl}_\theta(A)$, see [13].

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The complement of a θ-closed set is called a θ-*open* set. Considering the fact that the θ-closure operator is not in general idempotent, Bella and Cammaroto defined in [4] the θ-*closed hull* of a subset A of a space X, denoted by $[A]_\theta$, that is the smallest θ-closed subset of X containing A.

The γ-*closure* (see [2]) of the set A in a space X is the set $\mathrm{cl}_\gamma(A) = \{x\colon$ for every open neighborhood of X, $\mathrm{cl}_\theta(\overline{U}) \cap A \neq \emptyset\}$. A is said to be γ-closed if $A = \mathrm{cl}_\gamma(A)$. The γ-closure operator is not in general idempotent.

If X is a Urysohn space, the θ-*pseudocharacter of a point* x in X (see [2]) is $\psi_\theta(x,X) = \min\{|\mathcal{U}|\colon \mathcal{U}$ is a family of open neighborhoods of x and $\{x\}$ is the intersection of the θ-closures of the closures of $\mathcal{U}\}$; the θ-*pseudocharacter of* X is $\psi_\theta(X) = \sup\{\psi_\theta(x,X)\colon\ x \in X\}$.

The *almost Lindelöf degree* of a subset Y of a space X is $aL(Y,X) = \min\{k\colon$ for every cover $\mathcal{V}$ of Y consisting of open subsets of X, there exists $\mathcal{V}' \subseteq \mathcal{V}$ such that $|\mathcal{V}'| \leqslant k$ and $\bigcup\{\overline{V}\colon\ V \in \mathcal{V}'\} = Y\}$. The function $aL(X,X)$ is called the *almost Lindelöf degree* of X and denoted by $aL(X)$ (see [14] and [9]). The *almost Lindelöf degree of* X *with respect to closed subsets* is $aL_C(X) = \min\{k\colon$ for every closed set $Y \subset X$ and every cover $\mathcal{V}$ of Y consisting of open subsets of X, there exists $\mathcal{V}' \subseteq \mathcal{V}$ such that $|\mathcal{V}'| \leqslant k$ and $\bigcup\{\overline{V}\colon\ V \in \mathcal{V}'\} = Y\}$. The *almost Lindelöf degree of* X *with respect to* θ-*closed subsets* is $aL_\theta(X) = \min\{k\colon$ for every θ-closed set $Y \subset X$ and every cover $\mathcal{V}$ of Y consisting of open subsets of X, there exists $\mathcal{V}' \subseteq \mathcal{V}$ such that $|\mathcal{V}'| \leqslant k$ and $\bigcup\{\overline{V}\colon\ V \in \mathcal{V}'\} = Y\}$. We have that the almost Lindelöf degree is hereditary with respect to θ-closed sets, so we have that $aL(X) = aL_\theta(X)$.

A space X is called H-*closed* if it has finite almost Lindelöf degree.

Let $(X,\tau) = X$ be a topological space. We say that a subset A of X is regular open (regular closed) if $A = \mathrm{int}(\overline{A})$ ($A = \overline{\mathrm{int}(A)}$). The family $\mathcal{B} = \{U\colon$ U is regular open in $X\}$ is a base for X. Space X equipped with the topology generated by the base $\mathcal{B}$ is called the *semiregularization* of X and is denoted by X_s. If $X = X_s$, then X is called *semiregular*.

For a subset A of a space X we will denote by $[A]^{\leqslant\lambda}$ the family of all subsets of A of cardinality $\leqslant \lambda$.

In Section 2 we introduce the notions of weakly H-closed spaces, strongly semiregular spaces and firmly Urysohn spaces and we prove that if X is a H-closed, strongly semiregular, firmly Urysohn space, then X is compact.

In Section 3 we construct two new cardinal invariants denoted by θ-$aL'(X)$, related to the θ-almost Lindelöf degree, and by $t_{\tilde{c}}(X)$, related to tightness, using an operator denoted by $\tilde{c}(\cdot)$ (for this new operator we have that if X is a space and $A \subset X$, then $\mathrm{cl}_\theta(A) \subseteq \tilde{c}(A) \subseteq \mathrm{cl}_\gamma(A)$). We prove some results concerning weakly H-closed spaces and, in particular, we prove that if X is a Urysohn space, then

$|X| \leqslant 2^{\theta\text{-}aL'(X)t_{\tilde{c}}(X)\psi_\theta(X)}$, which represents an improvement of the Bella-Cammaroto inequality (see [4]).

2. Weakly H-closed spaces and strong semiregular spaces

In [2] the cardinal invariant θ-$aL(X)$ was introduced, known as the *θ-almost Lindelöf degree of the space X*.

Definition 2.1. The *θ-almost Lindelöf degree* of a subset Y of a space X is θ-$aL(Y,X) = \min\{k\colon$ for every cover $\mathcal{V}$ of Y consisting of open subsets of X, there exists $\mathcal{V}' \subseteq \mathcal{V}$ such that $|\mathcal{V}'| \leqslant k$ and $\bigcup\{\mathrm{cl}_\theta(\overline{V})\colon V \in \mathcal{V}'\} = Y\}$.

The function θ-$aL(X,X)$ is the *θ-almost Lindelöf degree of the space* X and it is denoted by θ-$aL(X)$.

We have that θ-$aL(X) \leqslant aL(X)$ for every space X, and using a slight modification of Example 2.3 in [3] we prove that the previous inequality can be strict.

We say that a space is *weakly H-closed* if it has finite θ-almost Lindelöf degree. We obviously have that an H-closed space is weakly H-closed but the converse is not true as the following example shows:

Example 2.1. We consider Bing's tripod space $\mathbb{B}$. Let $\mathbb{B}$ be the set $\{(x,y) \in \mathbb{Q}^2\colon y \geqslant 0\}$ with the topology generated by the neighborhood base $\mathcal{V}_{(x,y)} = \{N_\varepsilon(x,y)\colon \varepsilon > 0\}$, where $N_\varepsilon(x,y) = \{(x,y)\} \cup \bigl(x - \frac{1}{\sqrt{3}}y - \varepsilon, x - \frac{1}{\sqrt{3}}y + \varepsilon\bigr) \times \{0\} \cup \bigl(x + \frac{1}{\sqrt{3}}y - \varepsilon, x + \frac{1}{\sqrt{3}}y + \varepsilon\bigr) \times \{0\}$. It is easily seen that X is not Urysohn and not H-closed but considering the fact that the θ-closure of the closure of every open set is the whole space we certainly have that X is weakly H-closed.

We also introduce a new cardinal function related to the weakly Lindelöf degree called the *θ-weakly Lindelöf degree.*

Definition 2.2. Let X be a topological space and Y a subset of X, the *θ-weakly Lindelöf degree of* Y with respect to X, denoted by *θ-$wL(Y,X)$*, is

$$\theta\text{-}wL(Y,X) = \min\Bigl\{k\colon \text{ for every open cover } \mathcal{U} \text{ of } Y \text{ by open subsets of } X \text{ there exists } \mathcal{V} \in [\mathcal{U}]^{\leqslant k} \text{ such that } Y = \mathrm{cl}_\theta\Bigl(\overline{\bigcup \mathcal{V}}\Bigr)\Bigr\}.$$

Definition 2.3. Let X be a topological space, the *θ-weakly Lindelöf degree of X*, denoted by *θ-$wL(X)$*, is *θ-$wL(X,X)$*.

Now we prove that for a θ-open subset U of a space X, θ-$wL(\overline{U},X) \leqslant \theta$-$wL(X)$.

Proposition 2.1. *If U is a θ-open subset of a space X, then θ-$wL(\overline{U},X) \leqslant$ θ-$wL(X)$.*

P r o o f. Let $k = \theta\text{-}wL(X)$, $U \subseteq X$ θ-open and $\mathcal{U}$ a cover of $\overline{U}$ by sets open in X. Then $\mathcal{U} \cup \{X \setminus \overline{U}\}$ is an open cover of X and since $k = \theta\text{-}wL(X)$ there exists $\mathcal{V} \in [\mathcal{U}]^{\leqslant k}$ such that $X = \mathrm{cl}_\theta(\overline{X \setminus \overline{U}}) \cup \mathrm{cl}_\theta(\overline{\bigcup \mathcal{V}})$. We want to prove that $U \cap \mathrm{cl}_\theta(\overline{X \setminus \overline{U}}) = \emptyset$. Suppose that there exists $x \in U \cap \mathrm{cl}_\theta(\overline{X \setminus \overline{U}})$. Point x is in U which is θ-open, so there exists an open set W such that $x \in W \subseteq \overline{W} \subseteq U$. Then $\overline{W} \cap \overline{X \setminus \overline{U}} \neq \emptyset$, so $U \cap X \setminus \mathrm{int}(\overline{U}) \neq \emptyset$, a contradiction. So U is contained in $\mathrm{cl}_\theta(\overline{\bigcup \mathcal{V}})$ and thus $\overline{U} \subseteq \mathrm{cl}_\theta(\overline{\bigcup \mathcal{V}})$, therefore $\theta\text{-}wL(\overline{U}, X) \leqslant k$. □

Corollary 2.1. *If X is weakly H-closed and U is θ-open in X, then for every open cover $\mathcal{U}$ of $\overline{U}$ there exist a finite subfamily $\mathcal{V}$ such that $\overline{U} \subseteq \bigcup\limits_{V \in \mathcal{V}} \mathrm{cl}_\theta(\overline{V})$.*

Definition 2.4. We say that a space X is *strongly semiregular* if the family $\mathcal{B} = \{X \setminus \overline{U} \colon U \text{ is a } \theta\text{-open subset of } X\}$ is a base for X.

It could be seen that $\mathcal{B}$ satisfies the requirements to be a base for some topology. X equipped with the topology generated by the base $\mathcal{B}$ is called the *strong semiregularization of* X and is denoted by X_{ss}.

It is natural to investigate the relation between the strong semiregularization X_{ss} and the semiregularization X_s of the space X.

Considering the fact that a θ-open set is open, the closure of an open set is regular closed and that the complement of a regular closed set is regular open, we have

Proposition 2.2. *A strongly semiregular space is semiregular.*

The converse of the above statement is not true.

E x a m p l e 2.2. Let $\mathbb{B}_s$ be the semiregularization of the space $\mathbb{B}$ of Example 2.1. We have that if U is a θ-open subset of $\mathbb{B}$, then $\mathbb{B}_s \setminus U$ is nowhere dense and so it has empty interior, hence the space is not strongly semiregular.

We can also observe that for a Hausdorff space X, it may happen that X_{ss} is not Hausdorff even though X_s is Hausdorff.

E x a m p l e 2.3. $\mathbb{B}_{ss}$ has the indiscrete topology whereas $\mathbb{B}_s$ is Hausdorff.

We now give the definition of a new axiom of separation.

Definition 2.5. A topological space X is *firmly Urysohn* if for every $x, y \in X$ with $x \neq y$, there exist two open subsets U, V of X such that $x \in U$, $y \in V$ and $\overline{U} \cap \mathrm{cl}_\theta(\overline{V}) = \emptyset$.

We know that a firmly Urysohn space is Urysohn but we ask:

Question 2.1. Does there exist a Urysohn not firmly Urysohn space?

We investigate the θ-closed subsets of a firmly Urysohn, weakly H-closed space and we prove:

Proposition 2.3. *If X is firmly Urysohn and weakly H-closed and U is a θ-open subset of X, then $H = \overline{U}$ is θ-closed.*

Proof. Fix $x \notin H$. For each $y \in H$, there are open sets U_y and V_y such that $x \in U_y$, $y \in V_y$, and $\mathrm{cl}_\theta(U_y) \cap \mathrm{cl}_\theta(\overline{V_y}) = \emptyset$. By Corollary 2.1, for the open cover $\{V_y \colon y \in H\}$ of H, there is a finite subset $F \subseteq H$ such that $H \subseteq \bigcup\limits_{y\in F} \mathrm{cl}_\theta(\overline{V_y})$. Let $U = \bigcap\limits_{y\in F} U_y$. Then $U \cap \bigcap\limits_{y\in F} \mathrm{cl}_\theta(\overline{V_y}) = \emptyset$. Thus, H is θ-closed. $\square$

We also investigate the relation between θ-open sets and regular spaces and we prove:

Proposition 2.4. *X has a base of θ-open sets if and only if X is regular.*

Proof. Suppose that X has a base consisting of θ-open sets. Let U be an open subset of X and $x \in U$, then there exists a θ-open subset V of X such that $x \in V \subseteq U$ and $x \notin X - V$. Subset V is θ-closed, so there exists an open subset W of X such that $x \in W$ and $\overline{W} \cap (X - V) = \emptyset$. This means $x \in W \subseteq \overline{W} \subseteq V \subseteq U$ if and only if X is a regular space. $\square$

Using the previous property, we find a connection with compact spaces. We recall an important result:

Lemma 2.1 ([12])**.** *A space X is H-closed and regular if and only if X is compact.*

If X is regular, then the θ-closure equals the closure and for this reason we can say that in a regular space the almost Lindelöf degree and the θ-almost Lindelöf degree coincide.

From Corollary 4.8(k) in [12], we prove:

Theorem 2.1. *If X is a firmly Urysohn, weakly H-closed and strongly semiregular space, then it is compact.*

Proof. Suppose that X is firmly Urysohn, weakly H-closed and strongly semiregular. We want to show that such a space X is regular because in a regular space we have that the closure and θ-closure are equal to each other and for this reason we have that the weakly H-closedness is equivalent to the H-closedness and we can apply Lemma 2.1. If X is strongly semiregular, then it has a base consisting of complements of closures of θ-open subsets of X which, by Proposition 2.3, are θ-open. So X has a base consisting of θ-open sets and for this reason it is regular. $\square$

We can observe that having a base consisting of complements of closures of θ-open sets is equivalent to having a base consisting of interiors of θ-closed sets.

We know that the character of a space is greater than or equal to the character of its semiregularization. We have the same result when considering the strongly semiregularization of the space X.

Proposition 2.5. *If X is a space and X_{ss} is the strong semiregularization of X, then $\chi(X_{ss}) \leqslant \chi(X)$.*

Proof. Let $x \in X$ and let $\mathcal{B}_x$ be an open neighborhood system of x such that $|\mathcal{B}_x| \leqslant \chi(X)$. It suffices to show that $\{\mathrm{int}([U]_\theta) \colon U \in \mathcal{B}_x\}$ is an open neighborhood system for x in X_{ss}. Let A be a θ-closed set such that $x \in \mathrm{int}(A)$. There is $U \in \mathcal{B}_x$ such that $U \subseteq \mathrm{int}(A) \subseteq A$. Thus, $x \in U \subseteq [U]_\theta \subseteq A$, and it follows that $x \in \mathrm{int}([U]_\theta) \subseteq \mathrm{int}(A)$. $\square$

3. Construction of the cardinal function θ-$aL'(\cdot)$ and of the operator $\tilde{c}(\cdot)$

In this section we modify a filter construction given in [7]. In that paper an operator $\hat{c}$ was constructed. Here we construct a related operator $\tilde{c}$.

Let X be a topological space, $x \in X$ and $\mathcal{F}_x$ the collection of all finite intersections C of regular closed sets such that $x \in C$. It is easy to prove that $\mathcal{F}_x$ is a filter base which can be extended to a filter $\mathcal{C}_x$ that is maximal in the collection of all finite intersections of regular closed sets, partially ordered by inclusion.

The maximal filter $\mathcal{C}_x$ has the following properties:

Proposition 3.1. *Let X be a topological space and $x \in X$. Every regular closed subset of X which meets every element of $\mathcal{C}_x$ is an element of $\mathcal{C}_x$.*

Proof. Let U be an open subset of X and $x \in X$. $\overline{U}$ is regular closed. Suppose $\overline{U}$ meets every element of $\mathcal{C}_x$. Then $\{\overline{U}\} \cup \mathcal{C}_x$ is a filter base that contains $\mathcal{C}_x$ which can be extended to a maximal filter $\mathcal{M}$. As $\mathcal{C}_x$ is maximal, we have $\mathcal{C} = \mathcal{M}$ and $\overline{U} \in \mathcal{C}_x$. $\square$

Proposition 3.2. *Let X be a Urysohn space, then for every $x, y \in X$ with $x \neq y$ we have that $\mathcal{C}_x \neq \mathcal{C}_y$.*

Proof. Let $x, y \in X$ with $x \neq y$. Since X is a Urysohn space, there exist two open subsets U, V of X such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$. We have that $\overline{U} \in \mathcal{C}_x$ and $\overline{V} \in \mathcal{C}_y$. If $\mathcal{C}_x = \mathcal{C}_y$, then $\overline{V} \in \mathcal{C}_x$ and $\overline{U} \cap \overline{V} = \emptyset \in \mathcal{C}_x$, a contradiction. $\square$

We now define new operators using the maximal filter $\mathcal{C}_x$.

Definition 3.1. For a space X and an open subset U of X, define:

$$\widetilde{U} = \{x \in X \colon \overline{U} \in \mathcal{C}_x\}.$$

In the following propositions we give several properties of $\widetilde{U}$.

Proposition 3.3. *Let X be a space and U an open subset of X. Then $\overline{U} \subseteq \widetilde{U} \subseteq \mathrm{cl}_\theta(\overline{U})$.*

Proof. If $x \in \overline{U}$, then $\overline{U} \in \mathcal{C}_x$ and $x \in \widetilde{U}$. Let V be an open subset of X such that $x \in \overline{V}$. By Proposition 3.1, $\overline{V} \cap \overline{U} \neq \emptyset$, hence $x \in \mathrm{cl}_\theta(\overline{U})$. $\square$

Question 3.1. If X is a space, does there exist an open subset U of X such that $\overline{U} \subsetneq \widetilde{U} \subsetneq \mathrm{cl}_\theta(\overline{U})$?

Proposition 3.4. *If X is a topological space and V, W are open subsets of X, then $\widetilde{V} \cup \widetilde{W} = \widetilde{V \cup W}$. In particular this operator distributes over finite unions.*

Proof. $\widetilde{V} \cup \widetilde{W} = \{x \in X \colon \overline{V} \in \mathcal{C}_x\} \cup \{x \in X \colon \overline{W} \in \mathcal{C}_x\} = \{x \in X \colon \overline{V} \cup \overline{W} = \overline{V \cup W} \in \mathcal{C}_x\} = \widetilde{V \cup W}$. $\square$

The analogue of the following proposition in the case of Hausdorff spaces is contained in the proof of Proposition 4.1 in [7].

Proposition 3.5. *X is a Urysohn space if and only if for every $x, y \in X$ with $x \neq y$ there exist open subsets U, V of X such that $\widetilde{U} \cap \widetilde{V} = \emptyset$.*

Proof. Suppose that for every $x, y \in X$ with $x \neq y$ there exist open subsets U, V of X such that $\widetilde{U} \cap \widetilde{V} = \emptyset$. We have that $\overline{U} \subseteq \widetilde{U}$ and $\overline{V} \subseteq \widetilde{V}$, so $\overline{U} \cap \overline{V} = \emptyset$. This means that X is a Urysohn space.

Conversely, suppose X is Urysohn, then for every $x, y \in X$ with $x \neq y$ there exist open subsets U, V of X such that $\overline{U} \cap \overline{V} = \emptyset$. We want to show that $\widetilde{U} \cap \widetilde{V} = \emptyset$. In order to have a contradiction, suppose there exists $z \in \widetilde{U} \cap \widetilde{V}$. From the definition of $\widetilde{U}$ and $\widetilde{V}$, we have $\overline{U}, \overline{V} \in \mathcal{C}_z$. $\mathcal{C}_z$ is a filter, therefore $\overline{U} \cap \overline{V} = \emptyset \in \mathcal{C}_z$, a contradiction. $\square$

For a space X and a subset A of X we define a new operator called the *$\tilde{c}$-closure* in this way:

Definition 3.2. Let X be a space and A a subset of X,

$$\tilde{c}(A) = \{x \in X \colon \widetilde{U} \cap A \neq \emptyset \text{ for every open subset } U \text{ of } X \text{ containing } x\}.$$

We say that $A \subseteq X$ is *$\tilde{c}$-closed* if $A = \tilde{c}(A)$.

We have the following propositions.

Proposition 3.6. *If X is a space and A is a subset of X, then we have* $\operatorname{cl}_\theta(A) \subseteq \tilde{c}(A) \subseteq \operatorname{cl}_\gamma(A)$.

Proof. If $x \in \operatorname{cl}_\theta(A)$, then for every open subset V of X such that $x \in V$, $\overline{V} \cap A \neq \emptyset$. Therefore $\widetilde{V} \cap A \neq \emptyset$ and $x \in \tilde{c}(A)$.

If $x \in \tilde{c}(A)$, then for every open subset U of X such that $x \in U$ we have that $\widetilde{U} \cap A \neq \emptyset$ and by Proposition 3.3 we have $\widetilde{U} \subseteq \operatorname{cl}_\theta(\overline{U})$. Thus for every open subset U of X such that $x \in U$ we have that $\operatorname{cl}_\theta(\overline{U}) \cap A \neq \emptyset$, so $x \in \operatorname{cl}_\gamma(A)$. □

Question 3.2. If X is a space, does there exist $A \subseteq X$ such that $\operatorname{cl}_\theta(A) \subsetneq \tilde{c}(A) \subsetneq \operatorname{cl}_\gamma(A)$?

Proposition 3.7. *If X is a space and U is an open subset of X, then* $\operatorname{cl}_\theta(\overline{U}) \subseteq \tilde{c}(\widetilde{U})$.

Proof. If U is an open subset of X, by definitions we have $\operatorname{cl}_\theta(\overline{U}) \subseteq \tilde{c}(\overline{U}) \subseteq \tilde{c}(\widetilde{U})$. □

Question 3.3. If X is a space, does there exist an open subset U of X such that $\operatorname{cl}_\theta(\overline{U}) \subsetneq \tilde{c}(\widetilde{U})$?

We investigate the relation between Urysohn spaces and the operator $\tilde{c}(\cdot)$ and we prove the following:

Proposition 3.8. *If X is a Urysohn space, then for all $x, y \in X$ with $x \neq y$ there exists an open subset U of X such that $x \in U$ and $y \notin \tilde{c}(\widetilde{U})$.*

Proof. Let $x, y \in X$ with $x \neq y$. X is a Urysohn space, so that by Proposition 3.5 there exist open subsets U, V of X such that $x \in U$, $y \in V$ and $\widetilde{V} \cap \widetilde{U} = \emptyset$. We have that $\tilde{c}(\widetilde{U}) = \{z \in X \colon \widetilde{W} \cap \widetilde{U} \neq \emptyset$ for every open subset U of $X\}$ and $y \in V$ but $\widetilde{V} \cap \widetilde{U} = \emptyset$, so $y \notin \tilde{c}(\widetilde{U})$. □

For a space X we define a new cardinal invariant $t_{\tilde{c}}(X)$ related to the tightness $t(X)$.

Definition 3.3. For a space X, the *$\tilde{c}$-tightness of a point* $x \in X$, denoted by $t_{\tilde{c}}(x, X)$, is

$$t_{\tilde{c}}(x, X) = \min\{k \colon \text{ for every } A \subseteq X, \text{ if } x \in \tilde{c}(A), \text{ there exists } B \in [A]^{\leqslant k} \text{such that } x \in \tilde{c}(B)\}.$$

The *$\tilde{c}$-tightness of the space* X, denoted by $t_{\tilde{c}}(X)$, is

$$t_{\tilde{c}}(X) = \sup_{x \in X} t_{\tilde{c}}(x, X).$$

A natural question is:

Question 3.4. Are $t_{\tilde{c}}(X)$ and $t(X)$ incomparable?

With the following proposition we prove that $t_{\tilde{c}}(X)$ is bounded above by the character.

Proposition 3.9. *If X is a space, then $t_{\tilde{c}}(X) \leqslant \chi(X)$.*

Proof. Let $x \in X$, $A \subseteq X$ such that $x \in \tilde{c}(A)$ and let $\mathcal{V}_x$ be a neighborhood system of x in X with $|\mathcal{V}_x| \leqslant \chi(x, X)$. Because $x \in \tilde{c}(A)$, for every $V \in \mathcal{V}_x$ we have that $\widetilde{V} \cap A \neq \emptyset$. Let $y_{\widetilde{V}} \in \widetilde{V} \cap A$ for every $V \in \mathcal{V}_x$. We put $B = \{y_{\widetilde{V}} \colon V \in \mathcal{V}_x\}$, so $B \subseteq A$, $x \in \tilde{c}(B)$ and $|B| \leqslant \chi(x, X)$. This proves that $t_{\tilde{c}}(x, X) \leqslant \chi(x, X)$. $\square$

Using the $\tilde{c}$-tightness and the θ-pseudocharacter we find a bound for the cardinality of the $\tilde{c}$-closure of a subset A of a space X.

Proposition 3.10. *Let X be a Urysohn space such that $t_{\tilde{c}}(X)\psi_\theta(X) \leqslant k$. Then for every $A \subseteq X$ we have that $|\tilde{c}(A)| \leqslant |A|^k$.*

Proof. Let $x \in \tilde{c}(A)$. Since $\psi_\theta(X) \leqslant k$, by Proposition 3.8 there exists a family $\{U_\alpha(x)\}_{\alpha<k}$ of neighborhoods of x such that $\{x\} = \bigcap\limits_{\alpha<k} \mathrm{cl}_\theta(\overline{U_\alpha(x)}) = \bigcap\limits_{\alpha<k} \tilde{c}(\widetilde{U_\alpha(x)})$. We want to prove that $x \in \tilde{c}(\widetilde{U_\alpha(x)} \cap A)$ for all $\alpha < k$. Let U be an open neighborhood of x and $\alpha < k$. Since $x \in \tilde{c}(A)$, we have that $\emptyset \neq \widetilde{U \cap U_\alpha(x)} \cap A \subseteq \widetilde{U} \cap \widetilde{U_\alpha(x)} \cap A$. This shows that $x \in \tilde{c}(\widetilde{U_\alpha(x)} \cap A)$. Since $t_{\tilde{c}}(X) \leqslant k$, there exists $A_\alpha \subset \widetilde{U_\alpha(x)} \cap A$ such that $|A_\alpha| \leqslant k$ and $x \in \tilde{c}(A_\alpha) \subseteq \tilde{c}(\widetilde{U_\alpha(x)})$. Then $\{x\} = \bigcap\limits_{\alpha<k} \tilde{c}(A_\alpha)$ and $\{A_\alpha\}_{\alpha<k} \in [[A]^{\leqslant k}]^{\leqslant k}$, so $|\tilde{c}(A)| \leqslant |[[A]^{\leqslant k}]^{\leqslant k}| = |A|^k$. $\square$

We now give another version of the Lindelöf and of the θ-almost Lindelöf degree using these new operators.

Definition 3.4. Let X be a topological space and Y a subset of X. We define $\widetilde{L}(Y, X)$ by

$$\widetilde{L}(Y, X) = \min\Big\{k \colon \text{ for every cover } \mathcal{U} \text{ of } Y \text{ by sets open in } X \\ \text{there exists } \mathcal{V} \in [\mathcal{U}]^{\leqslant k} \text{such that } X = \bigcup \widetilde{\mathcal{V}}\Big\}.$$

We put $\widetilde{L}(X, X) = \widetilde{L}(X)$.

We show now that if A is $\tilde{c}$-closed, then $\widetilde{L}(A, X) \leqslant \widetilde{L}(X)$.

Proposition 3.11. *If A is a $\tilde{c}$-closed subset of X, then $\widetilde{L}(A, X) \leqslant \widetilde{L}(X)$.*

P r o o f. Suppose that $\widetilde{L}(X) \leqslant k$ and let C be a $\tilde{c}$-closed subset of X. For every $x \in X \setminus C$ there exists an open subset U_x of X such that $\widetilde{U_x} \subseteq X \setminus C$. Let $\mathcal{U}$ be an open cover of C, then $\mathcal{V} = \mathcal{U} \cup \{U_x\colon\ x \in X \setminus C\}$ is an open cover of X. As $\widetilde{L}(X) \leqslant k$, there exists $\mathcal{V}' \in [\mathcal{V}]^{\leqslant k}$ such that $X = \bigcup \widetilde{\mathcal{V}'}$. Thus there exists $\mathcal{V}'' \in [\mathcal{U}]^{\leqslant k}$ such that $C \subseteq \bigcup \widetilde{\mathcal{V}''}$. This proves that $\widetilde{L}(C, X) \leqslant k$. □

Definition 3.5. For a space X, the θ-almost Lindelöf degree of X with respect to $\hat{c}$-closed subsets ($= \sup\{\theta\text{-}aL(C, X)\colon\ C$ is $\hat{c}$-closed$\}$) is denoted as $\theta\text{-}aL'(X)$ (instead of $\theta\text{-}aL_{\hat{c}}(X)$).

Note that $\theta\text{-}aL(X) \leqslant \theta\text{-}aL'(X) \leqslant \widetilde{L}(X)$.

Q u e s t i o n 3.5. Does there exist a space X such that $\theta\text{-}aL(X) < \theta\text{-}aL'(X) < \widetilde{L}(X)$?

Now we prove our main result that is a new cardinal bound for Urysohn spaces. To prove this result we use Theorem 3.1 in [9].

Theorem 3.1 (Hodel)**.** *Let X be a set, k an infinite cardinal, $f\colon \mathcal{P}(X) \to \mathcal{P}(X)$ an operator on X, and for each $x \in X$ let $\{V(\alpha, x)\colon\ \alpha < k\}$ be a collection of subsets of X. Assume*

- (T) *(tightness condition) if $x \in f(H)$, then there exists $A \subseteq H$ with $|A| \leqslant k$ such that $x \in f(A)$;*
- (C) *(cardinality condition) if $A \subseteq X$ with $|A| \leqslant k$, then $|f(A)| \leqslant 2^k$;*
- (C-S) *(cover-separation condition) if $H \neq \emptyset$, $f(H) \subseteq H$ and $q \notin H$, then there exists $A \subseteq H$ with $|A| \leqslant k$ and a function $f\colon\ A \to k$ such that $H \subseteq \bigcup\limits_{x \in A} V(f(x), x)$ and $q \notin \bigcup\limits_{x \in A} V(f(x), x)$.*

Then $|X| \leqslant 2^k$.

To prove the next theorem we use Theorem 3.1 and the operator $\tilde{c}(\cdot)$.

Theorem 3.2. *If X is a Urysohn space, then $|X| \leqslant 2^{\theta\text{-}aL'(X)t_{\tilde{c}}(X)\psi_\theta(X)}$.*

P r o o f. Let $k = \theta\text{-}aL'(X)t_{\tilde{c}}(X)\psi_\theta(X)$. As $\psi_\theta(X) \leqslant k$ for every $x \in X$ there exists a family $\mathcal{W}_x = \{W(\alpha, x)\colon\ \alpha < k\}$ of open subsets of X containing x such that $\{x\} = \bigcap\limits_{W \in \mathcal{W}_x} \mathrm{cl}_\theta(\overline{W})$.

For every $x \in X$ and $\alpha < k$, we put $V(\alpha, x) = \mathrm{cl}_\theta(\overline{W(\alpha, x)})$ and prove the three conditions of Theorem 3.1.

For $H \subseteq X$, define $f(H) = \tilde{c}(H)$.

▷ Condition (T) is true because $t_{\tilde{c}} \leqslant k$.

▷ Condition (C) is true by Proposition 3.10.

▷ We prove condition (C-S). Let $\emptyset \neq H \subseteq X$ satisfy $\tilde{c}(H) \subseteq H$. We have that $H \subseteq \tilde{c}(H)$ so $H = \tilde{c}(H)$ and H is $\tilde{c}$-closed. Suppose $q \notin H$. For every $a \in H$ there exists $\alpha_a < k$ such that $q \notin \mathrm{cl}_\theta(\overline{W(\alpha_a, a)}) = V(\alpha_a, a)$. Let $f\colon H \to X$ such that $f(a) = \alpha_a$. The set $\{W(f(a), a)\colon a \in H\}$ is an open cover of H and since H is $\tilde{c}$-closed and θ-$aL'(X) \leqslant k$, there exists $A \in [H]^{\leqslant k}$ such that $H \subseteq \bigcup_{a\in A} V(f(a), a)$ and $q \notin \bigcup_{a\in A} V(f(a), a)$. This proves condition (C-S).

Applying Theorem 3.1 we have that $|X| \leqslant 2^k = 2^{\theta\text{-}aL'(X)t_{\tilde{c}}(X)\psi_\theta(X)}$. □

We can observe that every $\tilde{c}$-closed set is also θ-closed. We also know that the almost Lindelöf degree is hereditary with respect to θ-closed sets, so for every space X we have

$$\theta\text{-}aL'(X) \leqslant \theta\text{-}aL_\theta(X) \leqslant aL_\theta(X) = aL(X).$$

Furthermore, we know that $\psi_\theta(X) \leqslant \chi(X)$ and by Proposition 3.9 we have that $t_{\tilde{c}}(X) \leqslant \chi(X)$.

From these facts we obtain the following:

Corollary 3.1 ([4]). *If X is a Urysohn space, then $|X| \leqslant 2^{aL(X)\chi(X)}$.*

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