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EXISTENCE RESULTS FOR SYSTEMS OF CONFORMABLE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the existence of solutions to systems of conformable fractional differential equations with periodic boundary value or initial value conditions. where the right member of the system is L^1_α-carathéodory function. We employ the method of solution-tube and Schauder's fixed-point theorem.

1. Introduction

Recently, a new fractional derivative called the conformable fractional derivative, was introduced by Khalil et al. in [23]. For recent results on conformable fractional derivatives we refer the reader to [1, 2, 3, 4, 5, 13, 17, 18, 21, 22]. Furthermore, in [8, 19, 27, 32] the authors introduced a conformable fractional calculus on an arbitrary time scale. For some recent contributions on fractional differential equations, see [6, 10, 11, 12, 24, 25, 30, 31, 33, 34].

In this paper, we establish existence results for the following system of conformable fractional differential equations:

$$\begin{cases} x^{(\alpha)}(t) = f\big(t, x(t)\big)\,, & \text{for a.e. } t \in I = [0,b],\ b>0,\\ x \in (\mathfrak{B})\,. \end{cases} \tag{1.1}$$

Where $0<\alpha\le 1$, $f\colon I\times\mathbb{R}^n \to \mathbb{R}^n$ is a L^1_α-carathéodory function, $x^{(\alpha)}(t)$ denotes the conformable fractional derivative of x at t of order α, and $(\mathfrak{B})$ denotes the initial value or the periodic boundary value conditions:

$$x(0) = x_0\,, \tag{1.2}$$

$$x(0) = x(b)\,. \tag{1.3}$$

Existence results for problem (1.1), (1.2) were obtained in [29], by using the Banach fixed point theorem with f a continuous function. In the particular case where $n=1$, existence results for problem (1.1) were obtained in [7] with nonlinear

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functional boundary conditions $B(x(0), x) = 0$ or $H(x, x(b)) = 0$, where B and H are continuous functions that satisfy suitable monotonicity assumptions, their results were established, for the scalar case, with the method of lower and upper solutions and cover, as a particular cases, the boundary conditions (1.2) and (1.3). In [5] the authors solved problem (1.1), (1.2) (for $n = 1$), with f a continuous function by the help of the solution-tube method. As we will see, the used definition is equivalent to the existence of a pair of lower and upper solutions of the considered problem.

In order to obtain the existence results for problem (1.1), we introduce the notion of solution-tube of 1.1 which generalizes the notions of lower and upper solutions given in [7]. It is inspired by a notion of solution tube for first-order systems of differential equations introduced in [26], (see also [14, 15] and [16] on time scales).

This paper is organized as follows. In Section 2, we introduce the definition of conformable fractional calculus and their important properties. In Section 3, we prove the existence and uniqueness of solutions to problem (1.1) by using the method of solution-tube and Schauder's fixed-point theorem.

2. Preliminaries

In this section, we introduce some necessary definitions and properties of the conformable fractional calculus which are used in this paper and can be found in [1, 23, 20, 29] and in [32] (If $\mathbb{T}$ is a real interval $[0,\infty)$) are given:

Definition 2.1 ([23])**.** Given a function $f\colon [0,\infty) \to \mathbb{R}$ and a real constant $\alpha \in (0, 1]$. The conformable fractional derivative of f of order α is defined by,

$$f^{(\alpha)}(t) := \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \tag{2.1}$$

for all $t > 0$.

If $f^{(\alpha)}(t)$ exists and is finite, we say that f is α-differentiable at t.

If f is α-differentiable in some interval $(0, a)$, $a > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then the conformable fractional derivative of f of order α at $t = 0$ is defined as

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t)\,.$$

Example 2.2. Conformable fractional derivatives of certain functions as follow:

(1) $(t^p)^{(\alpha)} = p\, t^{p-\alpha}$, for all $p \in \mathbb{R}$.

(2) $(\lambda)^{(\alpha)} = 0$, for all $\lambda \in \mathbb{R}$.

(3) $(e^{pt})^{(\alpha)} = p\, t^{1-\alpha} e^{pt}$, and $(e^{\frac{p}{\alpha} t^{\alpha}})^{(\alpha)} = p\, e^{\frac{p}{\alpha} t^{\alpha}}$, for all $p \in \mathbb{R}$.

Definition 2.3 ([32])**.** Assume $f\colon [0,\infty) \to \mathbb{R}^n$, $f(t) := (f_1(t), f_2(t), \ldots, f_n(t))$ and let $\alpha \in (0, 1]$ and $t \ge 0$. Then one defines $f^{(\alpha)}(t) = (f_1^{(\alpha)}(t), f_2^{(\alpha)}(t), \ldots, f_n^{(\alpha)}(t))$ (provided it exists). One calls $f^{(\alpha)}(t)$ the conformable fractional derivative of f of order α at $t > 0$. Function f is conformal fractional differentiable of order α provided $f^{(\alpha)}(t)$ exists for all $t > 0$, in such a case, we say that f is α-differentiable at t. We define the conformable fractional derivative at 0 as $f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t)$, provided it exists.

Theorem 2.4 ([32]). *If a function $f\colon [0,\infty) \to \mathbb{R}^n$ is α-differentiable at $t > 0$, $\alpha \in (0,1]$, then f is continuous at t.*

Theorem 2.5 ([32]). *Let $\alpha \in (0,1]$ and assume f, $g\colon [0,\infty) \to \mathbb{R}^n$ are α-differentiable at $t > 0$. Then, by denoting $(fg)(t) = (f_1(t)\, g_1(t), \dots, f_n(t)\, g_n(t))$, we have the following properties:*

(i) $(af + bg)^{(\alpha)} = af^{(\alpha)} + bg^{(\alpha)}$, *for all a, $b \in \mathbb{R}$;*

(ii) $(fg)^{(\alpha)} = fg^{(\alpha)} + gf^{(\alpha)}$;

(iii) $(f/g)^{(\alpha)} = \dfrac{gf^{(\alpha)} - fg^{(\alpha)}}{g^2}$.

(iv) *If, in addition, f is differentiable at a point $t > 0$, then*

$$f^{(\alpha)}(t) = t^{1-\alpha} f'(t)\,.$$

(v) *If f is differentiable at t, then f is α-differentiable at t.*

We introduce the following spaces:

$$\begin{aligned}
C^\alpha(I,\mathbb{R}^n) &= \{f : I \to \mathbb{R}^n, \ \text{ is } \ \alpha\text{-differentiable on } I \ \text{ and } f^{(\alpha)} \in C(I,\mathbb{R}^n)\}\,.\\
C_0^\alpha(I,\mathbb{R}^n) &= \{f \in C^\alpha(I,\mathbb{R}^n) : f(0) = f(b) = 0\}\,.\\
C_{0,b}^\alpha(I,\mathbb{R}^n) &= \{f \in C^\alpha(I,\mathbb{R}^n) : f(0) = f(b)\}\,.
\end{aligned}$$

Definition 2.6 ([23]). Let $\alpha \in (0,1]$ and $f\colon [0,\infty) \to \mathbb{R}$. The conformable fractional integral of f of order α from 0 to t, denoted by $I_\alpha(f)(t)$, is defined by

$$I_\alpha(f)(t) := I_1(t^{\alpha-1} f)(t) = \int_0^t f(s) d_\alpha s := \int_0^t f(s) s^{\alpha-1} ds\,.$$

The considered integral is the usual improper Riemann one.

Definition 2.7 ([32]). Let $f\colon [0,\infty) \to \mathbb{R}^n$ and $\alpha \in (0,1]$. The conformable fractional integral of f of order α from 0 to t, denoted by $I_\alpha(f)(t)$, is defined by

$$I_\alpha(f)(t) = \int_0^t f(s) d_\alpha s = \big(I_\alpha(f_1)(t), I_\alpha(f_2)(t), \dots, I_\alpha(f_n)(t)\big)\,,$$

where $I_\alpha(f_i)(t)$ is the conformable fractional integral of f_i of order α from 0 to t, for $i = 1, \dots, n$.

Lemma 2.8 ([29]). *Let $0 < \alpha \le 1$ and $f\colon [0,\infty) \to \mathbb{R}^n$ be a continuous function in the domain of I_α. Then for all $t \ge 0$ we have*

$$\big(I_\alpha(f)\big)^{(\alpha)}(t) = f(t)\,.$$

Corollary 2.9 ([1, 32]). *Let $f\colon [0,b) \to \mathbb{R}^n$ be such that $I_\alpha\,(f^\alpha)(t)$ exists for $0 < t < b$. Then, f is differentiable on $(0,b)$.*

Lemma 2.10 ([1, 32])**.** *Let $f\colon (0,b) \to \mathbb{R}^n$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > 0$ we have*

$$I_\alpha(f^\alpha)(t) = f(t) - f(0). \tag{2.2}$$

The next result is an adaptation of Lemma 2 in [29].

Proposition 2.11. *Let $0 < \alpha \leq 1$, and W be an open set of $\mathbb{R}^n$. If $g\colon I \to \mathbb{R}^n$ is α-differentiable at $t > 0$ and $f\colon W \to \mathbb{R}^m$ is differentiable at $g(t) \in W$. Then $f \circ g$ is α-differentiable at t and*

$$(f \circ g)^{(\alpha)}(t) = f'\big(g(t)\big)\left(g^{(\alpha)}(t)\right)^T.$$

Here v^T denotes the transpose vector of v.

Example 2.12. Let $\alpha \in (0,1]$, and $x\colon [0,\infty) \to \mathbb{R}^n$ α-differentiable at t. It is not difficult to verify that the Euclidean norm $\|\cdot\|\colon \mathbb{R}^n \setminus \{0\} \to [0,\infty)$ defined as

$$\|x(t)\| =< x(t), x(t) >^{1/2},$$

with $\langle\cdot,\cdot\rangle$ the usual scalar product in $\mathbb{R}^n$, is differentiable.

By the previous Proposition, we have

$$\|x(t)\|^{(\alpha)} = \frac{\langle x(t), x^{(\alpha)}(t)\rangle}{\|x(t)\|}.$$

Next, we develop the fractional Sobolev's spaces via conformable fractional calculus and their important properties. The basic definitions and relations based on [32] (if $\mathbb{T}$ is a real interval $[0,\infty)$) are given:

Definition 2.13. Let $B \subset I$. B is called null set if the measure of B is zero. We say that a property P holds almost everywhere (a.e.) on B, or for almost all (a.a.) $t \in B$ if there is a null set $E_0 \subset B$ such that P holds for all $t \in B \setminus E_0$.

Definition 2.14. Let A be a Lebesgue measurable subset of I. We say that function $f\colon I \to \mathbb{R}$, is a function α-integrable on A if and only if $t^{\alpha-1}f(t)$ is Lebesgue integrable on A. In such a case, we denote

$$\int_A f(t)\, d_\alpha t = \int_A t^{\alpha-1}\, f(t)\, dt.$$

Definition 2.15 ([32])**.** Let $E \subset \mathbb{R}$ be a measurable set, and let $\varphi\colon E \to \mathbb{R}$ be a measurable function. We say that φ belongs to $L^1_\alpha(E,\mathbb{R})$ is the following property is fulfilled

$$\int_E |\varphi(s)|\, d_\alpha s = \int_E |\varphi(s)|\, s^{\alpha-1} ds < +\infty.$$

We say that a measurable function $f\colon E \to \mathbb{R}^n$ is in the set $L^1_\alpha\left(E,\mathbb{R}^n\right)$ provided

$$\int_E \|f(s)\|\, d_\alpha s = \int_E \|f(s)\|\, s^{\alpha-1} ds < +\infty.$$

i.e. $f_i \in L^1_\alpha\left(E,\mathbb{R}\right)$, for each of its components $f_i\colon E \to \mathbb{R}$, $i = 1, \ldots, n$.

Theorem 2.16 ([32]). *The set $L^1_\alpha(I,\mathbb{R}^n)$ is a Banach space together with the norm defined for $\varphi \in L^1_\alpha(I,\mathbb{R}^n)$ as*

$$\|\varphi\|_{L^1_\alpha(I,\mathbb{R}^n)} := \int_I \|\varphi(t)\| d_\alpha t\,.$$

Remark 2.17. It is not difficult to verify the following assertions for all $\alpha \in (0,1]$:

(i) $L^1_\alpha(I,\mathbb{R}^n) \subset L^1(I,\mathbb{R}^n)$.

(ii) For $t \in I$, $t > 0$ and $\varphi\colon I \to \mathbb{R}^n$, it is satisfied that $\varphi^{(\alpha)} \in L^1_\alpha(I,\mathbb{R}^n)$ if and only if $\varphi' \in L^1(I,\mathbb{R}^n)$.

Definition 2.18. A function $f\colon I \to \mathbb{R}^n$ is said to be absolutely continuous on I (i.e., $f \in AC(I,\mathbb{R}^n)$) if for every $\varepsilon > 0$, there exists $\eta > 0$ such that if $\{[a_k,b_k[\}_{k=1}^m$, is a finite pairwise disjoint family of subintervals of I satisfying

$$\sum_{k=1}^{k=m}(b_k - a_k) < \eta\,, \quad \text{then} \quad \sum_{k=1}^{k=m}\|f(b_k) - f((a_k))\| < \varepsilon\,.$$

Theorem 2.19 ([32]). *Assume function $f\colon I \to \mathbb{R}^n$ is absolutely continuous on I, then f is conformable fractional differentiable of order α a.e. on I and the following equality is valid:*

$$f(t) = f(0) + \int_{[0,t]} f^{(\alpha)}(s) d_\alpha s\,, \quad \textit{for all } t \in I.$$

Definition 2.20. Let $\alpha \in (0,1]$ and $f\colon I \to \mathbb{R}^n$. One says that $f \in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)$ if and only if $f \in L^1_\alpha(I,\mathbb{R}^n)$ and there exists $g\colon I \to \mathbb{R}^n$ such that $g \in L^1_\alpha(I,\mathbb{R}^n)$ and

$$\int_I f(t)\phi^{(\alpha)}(t) d_\alpha t = -\int_I g(t)\phi(t) d_\alpha t\,, \quad \text{for all} \quad \phi \in C^\alpha_{0,b}(I,\mathbb{R}^n)\,. \tag{2.3}$$

We denote

$$V^{\alpha,1}_{0,b}(I,\mathbb{R}^n) = \{f \in AC(I,\mathbb{R}^n) : f^{(\alpha)} \in L^1_\alpha(I,\mathbb{R}^n)\,, f(0) = f(b)\}\,.$$

Remark 2.21. We have $V^{\alpha,1}_{0,b}(I,\mathbb{R}^n) \subset W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)$.

Theorem 2.22 ([32]). *Assume that $f \in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)$ and that (2.3) holds for some $g \in L^1_\alpha(I,\mathbb{R}^n)$. Then, there exists a unique function $x \in V^{\alpha,p}_{a,b}([a,b],\mathbb{R}^n)$ such that*

$$x = f,\ x^{(\alpha)} = g \quad \textit{a.e. on } I.$$

Theorem 2.23 ([32]). *The set $W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)$ is a Banach space together with the norm defined as*

$$\|\varphi\|_{W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)} := \int_I \|\varphi(t)\| d_\alpha t + \int_I \|\varphi^{(\alpha)}(t)\| d_\alpha t\,,$$

for every $\varphi \in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)$.

Proposition 2.24. *Let $x \in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)$. Then $\|x\| \in W^{\alpha,1}_{0,b}(I,\mathbb{R})$ and*

$$\|x(t)\|^{(\alpha)} = \frac{< x(t), x^{\alpha}(t) >}{\|x(t)\|}, \quad \textit{a.e. on } \{t \in I : \|x(t)\| > 0\}.$$

Proof. If $x \in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)$. By Theorems 2.22 and 2.19, x is α-differentiable *a.e. on* I. From Example 2.12, we obtain

$$\|x(t)\|^{(\alpha)} = \frac{< x(t), x^{\alpha}(t) >}{\|x(t)\|}, \quad \text{a.e. on } \{t \in I : \|x(t)\| > 0\}.$$

□

We now define a notion of L^1_α-Carathéodory function.

Definition 2.25. A function $f\colon I \times \mathbb{R}^n \to \mathbb{R}^n$ is called a L^1_α-Carathéodory function if the three following conditions hold.

(i) for every $x \in \mathbb{R}^n$, the function $t \mapsto f(t,x)$ is Lebesgue measurable;

(ii) the function $x \mapsto f(t,x)$ is continuous almost every $t \in I$;

(iii) for every $r > 0$, there exists a function $h_r \in L^1_\alpha(I,[0,\infty))$ such that $\|f(t,x)\| \leq h_r(t)$ for almost every $t \in I$ and for all $x \in \mathbb{R}^n$ such that $\|x\| \leq r$.

3. Main result

In this section, we establish an existence result for the problem (1.1). A solution of problem (1.1) will be a function $x \in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)$ for which (1.1) is satisfied. We introduce the notion of solution-tube of this problem as follows.

Definition 3.1. Let $(v,M) \in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n) \times W^{\alpha,1}_{0,b}(I,[0,\infty))$. We say that (v,M) is a solution tube to problem (1.1) if

(i) $\langle x - v(t), f(t,x) - v^{(\alpha)}(t)\rangle \leq M(t)M^{(\alpha)}(t)$ for *a.e.* $t \in I$ and every $x \in \mathbb{R}^n$ such that $\|x - v(t)\| = M(t)$,

(ii) $v^{(\alpha)}(t) = f(t,v(t))$ and $M^{\alpha}(t) = 0$ a.e. on $\{t \in I : M(t) = 0\}$,

(iii) - if $(\mathfrak{B})$ denotes (1.2), then $\|x_0 - v(0)\| \leq M(0)$,
- if $(\mathfrak{B})$ denotes (1.3), then $\|v(b) - v(0)\| \leq M(0) - M(b)$.

If $\alpha = 1$, our definition of solution tube is equivalent to the notion of solution tube introduced in [26] for first order systems of Ordinary Differential Equations.

Now, we introduce the following set

$$\mathbf{T}(v,M) := \left\{x \in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n) : \|x(t) - v(t)\| \leq M(t)\,, \quad \text{for every } t \in I\right\}.$$

Remark 3.2. If $n = 1$, our definition of solution tube is equivalent to the notion of solution tube introduced in [5]. We point out that in this case the solution-tube method is equivalent of the lower and upper solutions one. To this end, we introduce the following definition:

Definition 3.3. A function $\gamma \in W^{\alpha,1}_{a,b}(I)$ is called a lower solution of (1.1), if

(i) $\gamma^{(\alpha)}(t) \geq f(t,\gamma(t)), \quad \textit{for a.e. } t \in I;$

(ii) - if $(\mathfrak{B})$ denotes (1.2), then $\gamma(0) \geq x_0$,
- if $(\mathfrak{B})$ denotes (1.3), then $\gamma(0) \geq \gamma(b)$.

A function $\delta \in W^{\alpha,1}_{0,b}(I)$ is called an upper solution of (1.1) if it satisfies (i),(ii) with the reversed inequalities.

Indeed, we consider the following assumptions:

(A) There exist $\delta \leq \gamma$ respectively upper and lower solutions of (1.1), such that $\delta < \gamma$ a.e. on I.

(B) There exists (v, M) a solution-tube of (1.1).

First, we prove the following assertion

If (B) is satisfied, then (A) is also fulfilled.

Define $\delta = v - M$ and $\gamma = v + M$.

$$\begin{cases} \left(\delta - \frac{\delta+\gamma}{2}(t)\right)\left(f(t,\delta) - \frac{(\gamma+\delta)^{(\alpha)}(t)}{2}\right) \leq \frac{(\gamma-\delta)(t)}{2}\frac{(\gamma-\delta)^{(\alpha)}(t)}{2} & \text{for a.e.} \quad t \in I\,, \\ \left(\gamma - \frac{\delta+\gamma}{2}(t)\right)\left(f(t,\gamma) - \frac{(\gamma+\delta)^{(\alpha)}(t)}{2}\right) \leq \frac{(\gamma-\delta)(t)}{2}\frac{(\gamma-\delta)^{(\alpha)}(t)}{2} & \text{for a.e.} \quad t \in I\,. \end{cases}$$

It is not difficult to verify that, since $\delta < \gamma$ a.e. on I, that

$$\begin{cases} \delta^{(\alpha)}(t) \leq f(t, \delta(t))\,, & \text{for a.e.} \quad t \in I\,, \\ \gamma^{(\alpha)}(t) \geq f(t, \gamma(t))\,, & \text{for a.e.} \quad t \in I\,. \end{cases}$$

Moreover, from condition (iii) it is immediate to conclude that $\delta(0) \leq x_0 \leq \gamma(0)$, provided (1.2) is considered, and $\delta(0) - \delta(b) \leq 0 \leq \gamma(0) - \gamma(b)$ for conditions (1.3).

Now, let's prove the reverse implication, i.e.

If (A) holds, then (B) is satisfied.

To this end, take $v = (\gamma + \delta)/2$ and $M = (\gamma - \delta)/2$, we have $\delta = v - M$ and $\gamma = v + M$.

For $x \in \mathbb{R}$ such that $|x - v(t)| = M(t)$, then $x = \gamma$ or $x = \delta$, and

$$\begin{aligned} (x - v(t))\,\big(f(t,x) - v^{(\alpha)}(t)\big) &= \begin{cases} \left(\delta - \frac{\delta+\gamma}{2}(t)\right)\left(f(t,\delta) - \frac{(\delta+\gamma)^{(\alpha)}}{2}(t)\right) & \text{for a.e. } t \in I\,, \\ \left(\gamma - \frac{\delta+\gamma}{2}(t)\right)\left(f(t,\gamma) - \frac{(\delta+\gamma)^{(\alpha)}}{2}(t)\right) & \text{for a.e. } t \in I\,, \end{cases} \\ &\leq \begin{cases} \left(\frac{\delta-\gamma}{2}(t)\right)\left(\delta^{(\alpha)}(t) - \frac{(\delta+\gamma)^{(\alpha)}}{2}(t)\right) & \text{for a.e. } t \in I\,, \\ \left(\frac{\gamma-\delta}{2}(t)\right)\left(\gamma^{(\alpha)}(t) - \frac{(\delta+\gamma)^{(\alpha)}}{2}(t)\right) & \text{for a.e. } t \in I\,, \end{cases} \\ &= M(t)M^{(\alpha)}(t) \quad \text{for a.e. } t \in I\,. \end{aligned}$$

We consider the following modified problem:

$$\begin{cases} x^{(\alpha)}(t) + \alpha\, x(t) = f\big(t, \overline{x}(t)\big) + \alpha\, \overline{x}(t)\,, & \text{for a.e. } t \in I\,, \\ x \in (\mathfrak{B})\,. \end{cases} \tag{3.1}$$

where

$$\overline{x}(t)=\begin{cases}\frac{M(t)}{\|x-v(t)\|}(x-v(t))+v(t)\,, & \text{if } \|x-v(t)\|>M(t),\\ x(t)\,, & \text{if } \|x-v(t)\|\le M(t).\end{cases}\tag{3.2}$$

We need the following auxiliary lemmas, which are direct generalizations of [7, Corollary 3.3 and Corollary 3.6], and we omit the proofs.

Lemma 3.4. *For every $g\in L^1_\alpha(I,\mathbb{R}^n)$, $x_0\in\mathbb{R}^n$, $0<\alpha\le 1$ and $p\in\mathbb{R}$, problem*

$$\begin{cases}x^{(\alpha)}(t)+px(t)=g(t)\,, & a.e. \quad t\in I\,,\\ x(0)=x_0\,,\end{cases}\tag{3.3}$$

has a unique solution $x\in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)$ given by the expression:

$$x(t):=\int_0^b G_{In}(t,s)g(s)d_\alpha s+x_0e^{-\frac{p}{\alpha}t^\alpha}\,,\tag{3.4}$$

where

$$G_{In}(t,s)=e^{\frac{p}{\alpha}(s^\alpha-t^\alpha)}\begin{cases}1\,, & 0\le s\le t\le b\,,\\ 0\,, & 0\le t\le s\le b\,,\end{cases}\tag{3.5}$$

Lemma 3.5. *For every $g\in L^1_\alpha(I,\mathbb{R}^n)$, $\lambda\in\mathbb{R}^n$, $0<\alpha\le 1$ and $p\in\mathbb{R}\backslash\{0\}$, problem*

$$\begin{cases}x^{(\alpha)}(t)+px(t)=g(t)\,, & a.e. \quad t\in I],\,,\\ x(0)-x(b)=\lambda\,,\end{cases}\tag{3.6}$$

has a unique solution $x\in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)$ given by the following expression:

$$x(t):=\int_0^b G_{Pe}(t,s)g(s)d_\alpha s+\lambda\,\frac{e^{-\frac{p}{\alpha}t^\alpha}}{1-e^{-\frac{p}{\alpha}b^\alpha}}\,,\tag{3.7}$$

where

$$G_{Pe}(t,s)=\frac{e^{\frac{p}{\alpha}(s^\alpha-t^\alpha)}}{1-e^{-\frac{p}{\alpha}b^\alpha}}\begin{cases}1\,, & 0\le s\le t\le b\,,\\ e^{-\frac{p}{\alpha}b^\alpha}\,, & 0\le t<s\le b\,.\end{cases}\tag{3.8}$$

The following lemma can be proved analogously to [5, Lemma 11].

Lemma 3.6. *Let $r\in W^{\alpha,1}_{0,b}(I,\mathbb{R})$, such that $r^{(\alpha)}(t)<0$ a.e. on $\{t\in I: r(t)>0\}$. If one of the two following conditions holds,*

(i) $r(0)\le 0$,

(ii) $r(0)\le r(b)$,

then $r(t)\le 0$ for every $t\in I$.

Let us define the operators $\mathcal{A}_1,\mathcal{A}_2: C(I,\mathbb{R}^n)\to C(I,\mathbb{R}^n)$ by

$$\mathcal{A}_1(x)(t)=\int_0^b G_{In}(t,s)\big(f(s,\overline{x}(s))+\alpha\,\overline{x}(s)\big)s^{\alpha-1}ds+x_0e^{-t^\alpha}$$

and

$$\mathcal{A}_2(x)(t) = \int_0^b G_{Pe}(t,s)\big(f(s,\overline{x}(s)) + \alpha\, \overline{x}(s)\big)s^{\alpha-1}ds\,,$$

where G_{In} (resp., G_{Pe}) is the Green's function related to the initial problem (3.3) (resp., periodic problem (3.6)) and is given by expression (3.5) (resp., (3.8)) with $p=\alpha$.

Clearly, from Lemma 3.4 (resp. Lemma 3.5) with $p=\alpha$, the solutions of problem (3.1), (1.2) (resp. (3.1), (1.3) coincide with the fixed points of operator $\mathcal{A}_1$ (resp. $\mathcal{A}_2$).

Proposition 3.7. *Let $f\colon I\times\mathbb{R}^n \to \mathbb{R}^n$ be a L^1_α-Carathéodory function. Assume there exists $(v,M)\in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)\times W^{\alpha,1}_{0,b}(I,[0,\infty))$ a solution tube of problem* (1.1), (1.3)*, then operator $\mathcal{A}_2$ is compact.*

Proof. We first observe that, from Definitions 2.25 and 3.1, there exists a function $h\in L^1_\alpha(I,[0,\infty))$ such that

$$\|f(t,\overline{x}(t)) + \alpha\,\overline{x}(t)\| \le h(t), \quad \text{for a.e. } \ t\in I \ \text{ and all } \ x\in C(I,\mathbb{R}^n)\,.$$

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of $C(I,\mathbb{R}^n)$ converging to $x\in C(I,\mathbb{R}^n)$. In this case, it is clear that

$$\begin{aligned}\big\|\mathcal{A}_2(x_n(t)) - \mathcal{A}_2(x(t))\big\| &\le \int_0^b s^{\alpha-1}|G_{Pe}(t,s)|\big\|\big(f(s,\overline{x_n}(s)) + \alpha\,\overline{x_n}(s)\big)\\ &\qquad - \big(f(s,\overline{x}(s)) + \alpha\,\overline{x}(s)\big)\big\|\,ds\\ &\le M\int_0^b s^{\alpha-1}\big\|\big(f(s,\overline{x_n}(s)) + \alpha\,\overline{x_n}(s)\big)\\ &\qquad - \big(f(s,\overline{x}(s)) + \alpha\,\overline{x}(s)\big)\big\|\,ds\,.\end{aligned}$$

where $M := \max_{s,t\in I}|G_{Pe}(t,s)|$.

The continuity of operator $\mathcal{A}_2$ follows from the continuous dependence with respect to x of function f, the definition of $\overline{x}$ and the Lebesgue's dominated convergence theorem.

To see that $\mathcal{A}_2(C(I,\mathbb{R}^n))$ is relatively compact set on $C(I,\mathbb{R}^n)$, consider $x\in C(I,\mathbb{R}^n)$. Therefore,

$$\big\|\mathcal{A}_2(x)(t)\big\| \le M\ \|h\|_{L^1_\alpha(I,\mathbb{R}^n)}\,.$$

So, $\mathcal{A}_2(C(I,\mathbb{R}^n))$ is uniformly bounded.

This set is also equicontinuous since for every $t_1<t_2\in I$,

$$\begin{aligned}&\big\|\mathcal{A}_2\left(x\right)\left(t_2\right) - \mathcal{A}_2\left(x\right)\left(t_1\right)\big\|\\ &\quad= \Big\|\int_0^{t_2} G_{Pe}(t_2,s)\big(f(s,\overline{x}(s))+\alpha\,\overline{x}(s)\big)\,d_\alpha s+\int_{t_2}^b G_{Pe}(t_2,s)\big(f(s,\overline{x}(s))+\alpha\,\overline{x}(s)\big)d_\alpha s\\ &\qquad-\int_0^{t_1} G_{Pe}(t_1,s)\big(f(s,\overline{x}(s))+\alpha\,\overline{x}(s)\big)d_\alpha s-\int_{t_1}^b G_{Pe}(t_1,s)\big(f(s,\overline{x}(s))+\alpha\,\overline{x}(s)\big)d_\alpha s\Big\|\end{aligned}$$

$$\leq \frac{|e^{-t_2^\alpha} - e^{-t_1^\alpha}|}{1-e^{-b^\alpha}}\Big(\int_0^{t_1} e^{s^\alpha}\|f\big(s,\overline{x}(s)\big)+\alpha\overline{x}(s)\|d_\alpha s+\int_{t_2}^{b} e^{s^\alpha-b^\alpha}\|f\big(s,\overline{x}(s)\big)+\alpha\overline{x}(s)\|d_\alpha s\Big)$$
$$+\int_{t_1}^{t_2}|G_{Pe}(t_2,s)-G_{Pe}(t_1,s)|\|f\big(s,\overline{x}(s)\big)+\alpha\,\overline{x}(s)\|\,d_\alpha s$$
$$\leq K|e^{-t_2^\alpha} - e^{-t_1^\alpha}|\Big(\int_0^{t_1} h(s)d_\alpha s+\int_{t_2}^{b} h(s)d_\alpha s\Big)+2M\int_{t_1}^{t_2} h(s)d_\alpha s\,,$$

where

$$K := \max_{s\in I}\left\{\frac{e^{s^\alpha}}{1-e^{-b^\alpha}},\frac{e^{s^\alpha-b^\alpha}}{1-e^{-b^\alpha}}\right\} = \frac{1}{1-e^{-b^\alpha}}\,.$$

By Arzelà-Ascoli theorem, we conclude that the set $\mathcal{A}_2(C(I,\mathbb{R}^n))$ is relatively compact in $C(I,\mathbb{R}^n)$. Hence, $\mathcal{A}_2$ is compact. □

The following result can be proved as the previous one.

Proposition 3.8. *Let $f\colon I\times\mathbb{R}^n\to\mathbb{R}^n$ be a L^1_α-Carathéodory function. Assume there exists $(v,M)\in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)\times W^{\alpha,1}_{0,b}(I,[0,\infty))$ a solution tube of* (1.1)*,* (1.2)*, then operator $\mathcal{A}_1$ is compact.*

Now, we can obtain our main theorem. The proof is on the basis on the one given in [16] for first order systems of ordinary differential equations.

Theorem 3.9. *Let $f\colon I\times\mathbb{R}^n\to\mathbb{R}^n$ be a L^1_α-Carathéodory function. Assume there exists $(v,M)\in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)\times W^{\alpha,1}_{0,b}(I,[0,\infty))$ a solution tube of* (1.1)*. Then, problem* (1.1) *has a solution $x\in W^{\alpha,1}_{0,b}(I,\mathbb{R}^n)\cap\mathrm{T}(v,M)$.*

Proof. We will do the proof for the initial case (1.2). As we will see the proof for the periodic problem (1.3) is analogous.
By Proposition 3.8 the operator $\mathcal{A}_1$ is compact. It has a fixed point by the Schauder fixed-point theorem. Lemma 3.4 implies that this fixed point is a solution for the problem (3.1). Then, it suffices to show that for every solution x of (3.1), $x\in\mathbf{T}(v,M)$.
Consider the set $\mathcal{B}:=\{t\in I:\|x(t)-v(t)\|>M(t)\}$. By Proposition 2.24, a.e. on $\mathcal{B}$ we have

$$(\|x(t)-v(t)\|-M(t))^{(\alpha)} = \frac{\langle x(t)-v(t),x^{(\alpha)}(t)-v^{(\alpha)}(t)\rangle}{\|x(t)-v(t)\|}-M^{(\alpha)}(t)\,.$$

Since (v,M) is a solution tube of problem (1.1), we have a.e. on $\{t\in\mathcal{B}:M(t)>0\}$ that

$$\big(\|x(t)-v(t)\|-M(t)\big)^{(\alpha)}$$
$$=\frac{\langle x(t)-v(t),x^{(\alpha)}(t)-v^{(\alpha)}(t)\rangle}{\|x(t)-v(t)\|}-M^{(\alpha)}(t)$$

$$\begin{aligned}
&= \frac{\langle x(t)-v(t), f(t,\bar{x}(t))+\alpha\bar{x}(t)-\alpha x(t)-v^{(\alpha)}(t)\rangle}{\|x(t)-v(t)\|} - M^{(\alpha)}(t) \\
&= \frac{\langle \bar{x}(t)-v(t), f(t,\bar{x}(t))-v^{(\alpha)}(t)\rangle}{M(t)} + \alpha\,\frac{\langle \bar{x}(t)-v(t), \bar{x}(t)-x(t)\rangle}{M(t)} - M^{(\alpha)}(t) \\
&\le \frac{M(t)M^{(\alpha)}(t)}{M(t)} + \alpha\big(M(t)-\|x(t)-v(t)\|\big) - M^{(\alpha)}(t) \\
&< 0\,.
\end{aligned}$$

On the other hand, we have *a.e. on* $\{t \in \mathcal{B} : M(t) = 0\}$ that

$$\begin{aligned}
&\big(\|x(t)-v(t)\| - M(t)\big)^{(\alpha)} \\
&\quad= \frac{\langle x(t)-v(t), f(t,\bar{x}(t))+\alpha\bar{x}(t)-\alpha x(t)-v^{(\alpha)}(t)\rangle}{\|x(t)-v(t)\|} - M^{(\alpha)}(t) \\
&\quad= \frac{\langle x(t)-v(t), f(t,v(t))+\alpha v(t)-\alpha x(t)-v^{(\alpha)}(t)\rangle}{\|x(t)-v(t)\|} - M^{(\alpha)}(t) \\
&\quad\le \frac{\langle x(t)-v(t), f(t,v(t))-v^{(\alpha)}(t)\rangle}{\|x(t)-v(t)\|} - \alpha\,\|x(t)-v(t)\| - M^{(\alpha)}(t) \\
&\quad< 0\,.
\end{aligned}$$

If we set, $r(t) := \|x(t)-v(t)\|-M(t)$, then $r^{(\alpha)} < 0$ a.e. on $\mathcal{B} := \{t \in I : r(t) > 0\}$. Moreover, since (v, M) is a solution tube to problem (1.1) and x satisfies (1.2), then $r(0) \le 0$ and, as consequence, Lemma 3.6 (i) implies that $\mathcal{B} = \emptyset$. So, $x \in T(v, M)$ and the result holds for this case.

When the periodic case is studied, we follow the same steps with operator $\mathcal{A}_2$ and we arrive to the fact that

$$r(0) - r(b) \le \|v(0) - v(b)\| - (M(0) - M(b)) \le 0,$$

and the result is fulfilled from Lemma 3.6 (ii). □

The following example is a modified version, considering a periodic condition, of Example 4.6 in [16]:

Example 3.10. Consider the periodic problem:

$$\begin{cases} x^{(\frac{1}{3})}(t) = a_1\|x(t)\|^2 x(t) - a_2 x(t) + a_3\varphi(t)\,, & \text{a.e. } t \in I = [0,1]\,, \\ x(0) = x(1)\,, \end{cases} \tag{3.9}$$

where $\alpha = 1/3$, a_1, a_2, $a_3 \in \mathbb{R}_+$ such that $a_1 - a_2 + a_3 = 0$, $\varphi\colon I \to \mathbb{R}^n$ is a continuous function satisfying $\|\varphi(t)\| = 1$ for every $t \in I$. Take $v(t) = 0$ and $M(t) = 1$.
So, $v \in W^{\frac{1}{3},1}_{0,1}(I, \mathbb{R}^n)$, $M \in W^{\frac{1}{3},1}_{0,1}(I, [0,\infty[)$, $v^{(\frac{1}{3})}(t) = 0$, $M^{(\frac{1}{3})}(t) = 0$, and

$$\|v(1) - v(0)\| \le M(0) - M(1)\,.$$

For $x \in \mathbb{R}^n$ such that $\|x - v(t)\| = M(t)$, then $\|x\| = 1$, and we have, for a.e. $t \in I$

$$\begin{aligned}\langle x - v(t), f(t,x) - v^{(\frac{1}{3})}(t)\rangle &= \langle x, a_1\|x\|^2 x - a_2 x + a_3\varphi(t)\rangle \\ &= a_1\|x\|^4 - a_2\|x\|^2 + a_3\langle x, \varphi(t)\rangle \\ &\le a_1\|x\|^4 - a_2\|x\|^2 + a_3\|x\|\|\varphi(t)\| \\ &= a_1 - a_2 + a_3 = 0 \\ &\le M(t)M^{(\frac{1}{3})}(t)\,.\end{aligned}$$

Since the set $\{t \in I, M(t) = 0\} = \emptyset$, condition (ii) holds trivially.

So, (v, M) is a solution-tube of (3.9). By Theorem 3.9, problem (3.9) has a solution $x \in W_{0,1}^{\frac{1}{3},1}(I, \mathbb{R}^n)$ such that $\|x(t)\| \le 1$ for every $t \in I$.

Example 3.11. Consider the periodic problem:

$$\begin{cases} x^{(1/2)}(t) = \dfrac{-x^3(t) + 1 - 2t}{\sqrt[4]{t}} & \text{a.e. } \ t \in [0,1]\,, \\ x(0) = x(1)\,. \end{cases} \tag{3.10}$$

This problem is a particular case of (1.1), (1.3), with $n = 1$, $\alpha = 1/2$, and $f(t,x) = \dfrac{-x^3 + 1 - 2t}{\sqrt[4]{t}}$. It is clear that f is a $L^1_{1/2}$-Carathéodory function. Take $v(t) = 0$ and $M(t) = 1$.

So, $v \in W_{0,1}^{\frac{1}{2},1}(I, \mathbb{R})$, $M \in W_{0,1}^{\frac{1}{2},1}(I, [0,\infty[)$, $v^{(\frac{1}{2})}(t) = 0$, $M^{(\frac{1}{2})}(t) = 0$, and

$$|v(1) - v(0)| \le M(0) - M(1).$$

For $x \in \mathbb{R}$ such that $|x - v(t)| = M(t)$, then $x = 1$ or $x = -1$, and we have for a.e. $t \in I$,

$$\begin{aligned}\big\langle x - v(t), f(t,x) - v^{(\frac{1}{2})}(t)\big\rangle &= (x)\Big(\frac{-x^3 + 1 - 2t}{\sqrt[4]{t}}\Big)\,, \\ &= \begin{cases} \dfrac{-2(1-t)}{\sqrt[4]{t}} & \text{if } \ x = -1\,, \\ -2\sqrt[4]{t^3} & \text{if } \ x = 1\,, \end{cases} \\ &\le 0 = M(t)M^{(\frac{1}{2})}(t) \quad \text{for a.e. } \ t \in I\,.\end{aligned}$$

So, (v, M) is a solution-tube of (3.10). By Theorem 3.9, the problem (3.10) has a solution $x \in W_{0,1}^{\frac{1}{2},1}(I)$ such that $|x(t)| \le 1$ for every $t \in I$.

Observe that $\delta = v - M$ and $\gamma = v + M$ are, respectively, upper and lower solutions of (3.10) follows from the fact that

$$\delta^{(\frac{1}{2})}(t) = 0 \le f(t, \delta(t)) = \frac{2(1-t)}{\sqrt[4]{t}}\,, \quad t \in [0,1], \quad \delta(0) \le \delta(1)\,,$$

and

$$\gamma^{(\frac{1}{2})}(t) = 0 \ge f(t, \gamma(t)) = -2\sqrt[4]{t^3}\,, \quad t \in [0,1], \quad \gamma(0) \ge \gamma(1)\,,$$

such that $-1 \leq x(t) \leq 1$, for all $t \in I$.

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