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ON THE SPECTRUM OF ROBIN LAPLACIAN IN A PLANAR WAVEGUIDE

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Abstract. We consider the Laplace operator in a planar waveguide, i.e. an infinite two-dimensional straight strip of constant width, with Robin boundary conditions. We study the essential spectrum of the corresponding Laplacian when the boundary coupling function has a limit at infinity. Furthermore, we derive sufficient conditions for the existence of discrete spectrum.

Keywords: planar waveguide; discrete spectrum; Robin boundary conditions

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1. Introduction

There are different ways of confining a quantum particle in long and thin structures, the so-called quantum waveguides in suitable subsets Ω of the space $\mathbb{R}^3$ or the plane $\mathbb{R}^2$ [6], [7], [12], [15]. A usual possibility, in two dimensions, is to model the waveguide by a curved strip of constant width which is squeezed between two curves; in this region one considers the Laplacian subject to Dirichlet [8], [10], and combined Dirichlet-Neumann [2], [14] or Robin boundary conditions [11], [13], [16].

Our main interest in this paper is to describe the precise location of the essential spectrum of Robin Laplacian $-\Delta_\alpha^\Omega$, and study the existence of eigenvalues below the essential spectrum, in a straight quantum waveguide Ω; see [4] for related references.

The description of the here studied model is as follows. Given a positive number ε, consider the infinite straight strip $\Omega = \mathbb{R} \times I$, where $I = (0, \varepsilon)$ is a bounded interval. It should be noted that the boundary of Ω is sufficiently regular (e.g., Ω satisfies the segment condition, see [1], Chap. III) and therefore we can verify that the operator $-\Delta_\alpha^\Omega$ acts as the Laplacian in the Hilbert space $L^2(\Omega)$ with Robin conditions at

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the boundary $\partial\Omega$; see Theorem 3.1. More specifically, given a bounded real-valued function on $\partial\Omega$,

$$\alpha(x,y) = \begin{cases} \alpha_0(x) & \text{in } \mathbb{R}\times\{0\}, \\ \alpha_1(x) & \text{in } \mathbb{R}\times\{\varepsilon\}, \end{cases} \tag{1.1}$$

the functions $\psi \in \operatorname{dom}(-\Delta_\alpha^\Omega)$ from the domain of $-\Delta_\alpha^\Omega$ satisfy, in an appropriate sense, Robin boundary conditions

$$\begin{cases} -\dfrac{\partial\psi}{\partial y}(x,0) + \alpha_0(x)\psi(x,0) = 0, \\ \dfrac{\partial\psi}{\partial y}(x,\varepsilon) + \alpha_1(x)\psi(x,\varepsilon) = 0. \end{cases} \tag{1.2}$$

A related type of boundary conditions has been considered in [13]; there the author has investigated spectral properties of the Laplacian by imposing (usual) Robin conditions, i.e.

$$\begin{cases} -\dfrac{\partial\psi}{\partial y}(x,0) + \alpha(x)\psi(x,0) = 0, \\ \dfrac{\partial\psi}{\partial y}(x,\varepsilon) + \alpha(x)\psi(x,\varepsilon) = 0, \end{cases} \tag{1.3}$$

where the Robin parameter $\alpha(x) > 0$ is a real-valued positive bounded function. Considering this case and under the hypothesis that α tends to a constant at infinity, the essential spectrum of the Laplacian was determined and a sufficient condition for the existence of discrete spectrum was given. The strategy in [13] to prove the existence of at least one isolated eigenvalue below the threshold of the essential spectrum, was a variational one based on [8], and the method of Neumann Bracketing was employed to find the location of the essential spectrum.

It is a question whether there are any similar results when one chooses our boundary conditions (1.2). More precisely, under our assumptions on the Robin parameter $\alpha(x,y)$ we get similar results as in [13]. However, the location of the essential spectrum was obtained using similar ideas as in [3].

Let $-\Delta_\alpha^\Omega$ denote the Laplacian with $\operatorname{dom}(-\Delta_\alpha^\Omega) = \{\psi \in H^2(\Omega)\colon\ \psi \text{ satisfies (1.2)}\}$. Concerning the questions of interest presented in this paper, we stress here that in the case of constant boundary conditions, the straight strip has spectrum starting from the first eigenvalue $\lambda_0 = \lambda_0(\alpha_0,\alpha_1) \in \mathbb{R}$ of the transversal Laplacian $-\Delta^I$ in $L^2(I)$ with $\psi(y) \in \operatorname{dom}(-\Delta^I)$ if $\psi \in H^2(I)$ and satisfying

$$\begin{cases} -\psi'(0) + \alpha_0\psi(0) = 0, \\ \psi'(\varepsilon) + \alpha_1\psi(\varepsilon) = 0, \end{cases} \tag{1.4}$$

where α_0 and α_1 are real constants. In what follows we are going to show, under some conditions as

$$\lim_{|x|\to\infty} (\alpha_i(x) - \alpha_i) = 0, \quad i \in \{0, 1\},$$

that

$$\sigma_{\text{ess}}(-\Delta_\alpha^\Omega) = \sigma_{\text{ess}}(-\Delta^I) = [\lambda_0, \infty)$$

and

$$\sigma(-\Delta_\alpha^\Omega) \cap (-\infty, \lambda_0) \neq \emptyset.$$

The operators are introduced as the unique self-adjoint operators associated with appropriate quadratic forms and the boundary conditions should be understood in the sense of traces (see more details in Sections 2 and 3).

The paper is organized as follow. In Section 2 we introduce Robin Laplacian in a bounded interval (transversal section), show that its essential spectrum is empty. In Section 3 we pass to the corresponding study in an infinite straight strip. We show, via quadratic forms, that the operator $-\Delta_\alpha^\Omega$ is self-adjoint (Theorem 3.1). Finally, in Section 4, we find the essential spectrum of such Robin Laplacian operator and give sufficient conditions for the existence of discrete spectrum.

2. Transversal Robin Laplacian

Initially, some results will be presented for our Robin Laplacian in the interval I (transversal section); they will be important later. We find that the Laplacian operator $-\Delta^I$ (classic) in $L^2(I)$ is self-adjoint by using the theory of quadratic forms.

Consider the operator

$$-\Delta^I\colon \operatorname{dom}(-\Delta^I) \to L^2(I)$$

with $\operatorname{dom}(-\Delta^I) = \{\psi \in H^2(I)\colon \psi \text{ satisfies (1.4)}\}$. By b^I denote the corresponding closed and lower bounded sesquilinear form $b^I \geqslant \lambda$ (with λ dependent on the set $\{\alpha_0, \alpha_1\} \subset \mathbb{R}$). In (2.1) we have the action of b^I on its domain $\operatorname{dom} b^I = H^1(I)$,

$$b^I(\phi, \psi) = \int_I \overline{\phi'(y)}\psi'(y)\,\mathrm{d}y + \alpha_1\overline{\phi(\varepsilon)}\psi(\varepsilon) + \alpha_0\overline{\phi(0)}\psi(0). \tag{2.1}$$

Theorem 2.1. *Let α_0 and α_1 belong to $\mathbb{R}$. Then the Laplacian operator $-\Delta^I$ is the unique self-adjoint operator associated with the sesquilinear form b^I, i.e.*

$$b^I(\phi, \psi) = (\phi, -\Delta^I\psi)$$

for each $\phi \in \operatorname{dom} b^I$ and $\psi \in \operatorname{dom}(-\Delta^I)$.

Proof. We will first prove that b^I is lower bounded and we refer the reader to ([5], Section 4.2) to details of how to conclude that b^I is closed. Denote by $\| \, \|$ the norm in $L^2(I)$; we have that

$$b^I(\phi) \geqslant -|\alpha_0 + \alpha_1| \frac{a^2 \varepsilon^{-1}}{2a-1} \|\phi\|^2 \quad \forall \phi \in \operatorname{dom} b^I,$$

where a is large enough. Indeed, first note that for all $a > 1$ we have

$$\begin{aligned} \varepsilon^a \phi(\varepsilon) &= \int_I y^a \phi'(y)\,\mathrm{d}y + \int_I a y^{a-1} \phi(y)\,\mathrm{d}y, \\ \varepsilon^a \phi(0) &= \int_I -(\varepsilon - y^a)\phi'(y)\,\mathrm{d}y + \int_I a(\varepsilon - y)^{a-1} \phi(y)\,\mathrm{d}y. \end{aligned}$$

Then by Cauchy-Schwarz

$$b^I(\phi) \geqslant \Big(1 - \frac{\varepsilon|\alpha_0 + \alpha_1|}{2a+1}\Big)\|\phi'\|^2 - |\alpha_0 + \alpha_1| \frac{a^2 \varepsilon^{-1}}{2a-1} \|\phi\|^2$$

and thus, it suffices to take a large enough so that the coefficient of $\|\phi'\|^2$ becomes positive, consequently we have

$$b^I(\phi) \geqslant -|\alpha_0 + \alpha_1| \frac{a^2 \varepsilon^{-1}}{2a-1} \|\phi\|^2.$$

□

2.1. Absence of essential spectrum. In this section, we discuss the essential spectrum of the Laplace operator $-\Delta^I$ in $L^2(I)$ subject to Robin boundary conditions. It is appointed here that $\sigma_{\text{ess}}(-\Delta^I) = \emptyset$.

Indeed, suppose that $\sigma_{\text{ess}}(-\Delta^I) \neq \emptyset$. Let $\lambda \in \sigma_{\text{ess}}(-\Delta^I)$, it follows that there exists a normalized sequence (ψ_n) in $\operatorname{dom}(-\Delta^I_\alpha)$ (i.e. $\|\psi_n\|_{L^2(I)} = 1$) such that $\psi_n \xrightarrow{w} 0$ in $L^2(I)$; in particular, $\psi_n \xrightarrow{w} 0$ in $H^1(I)$. By recalling the compacteness of the embedding $H^1(I) \hookrightarrow L^2(I)$, there exists a subsequence (ψ_{n_k}) such that $\psi_{n_k} \to 0$ in $L^2(I)$. This is absurd since $\|\psi_n\|_{L^2(I)} = 1$. The result of the present section reads as follows:

Theorem 2.2. *Let α_0 and α_1 belong to $\mathbb{R}$. Then the transversal Robin Laplacian $-\Delta^I$ has purely discrete spectrum and it is the essential spectrum $\sigma_{\text{ess}}(-\Delta^I) = \emptyset$.*

2.2. Point spectrum of the transversal Laplacian. We now investigate the point spectrum of $-\Delta^I$, with Robin boundary conditions. Let us determine $\lambda \in \mathbb{R}$ for which there exists $0 \neq \psi$ in $H^2(I)$, normalized in $L^2(I)$, satisfying

$$-\psi'' = \lambda\psi, \quad \text{in } I = (0, \varepsilon) \tag{2.2}$$

and satisfying the boundary conditions (1.4). Initially, consider that $\alpha_0 \neq -\alpha_1$. If λ is an eigenvalue with $\lambda > 0$, we already know that the general solution (classic) of (2.2) is given by

$$\psi(y) = A\sin\big(\sqrt{\lambda}y\big) + B\cos\big(\sqrt{\lambda}y\big) \tag{2.3}$$

with $A, B \in \mathbb{C}$ determined by the Robin conditions and the normalization condition. Thus, by imposing the Robin conditions on the general solution we obtain the following system:

$$\begin{bmatrix} -\sqrt{\lambda} & \alpha_0 \\ \sqrt{\lambda}\cos\big(\sqrt{\lambda}\varepsilon\big) + \alpha_1\sin\big(\sqrt{\lambda}\varepsilon\big) & \alpha_1\cos\big(\sqrt{\lambda}\varepsilon\big) - \sqrt{\lambda}\sin\big(\sqrt{\lambda}\varepsilon\big) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0. \tag{2.4}$$

Since we are interested in nonzero solutions, we must impose that the determinant of the above matrix is zero. This requirement enables us to obtain λ by means of implicit equation:

$$f(\lambda) = \sin\big(\sqrt{\lambda}\varepsilon\big)(\lambda - \alpha_0\alpha_1) - (\alpha_0 + \alpha_1)\sqrt{\lambda}\cos\big(\sqrt{\lambda}\varepsilon\big) = 0. \tag{2.5}$$

In order to check that equation (2.5) has a solution, it is sufficient to note that f is a continuous function with

$$f\Big(\frac{\pi^2}{\varepsilon^2}\Big) \cdot f\Big(\frac{4\pi^2}{\varepsilon^2}\Big) < 0.$$

Equation (2.4) is equivalent to

$$\sqrt{\lambda}A + \alpha_0 B = 0,$$

so A is given in terms of B, i.e. $A = -\big(\alpha_0/\sqrt{\lambda}\big)B$. The normalization condition of ψ in particular allows one to choose $B \in \mathbb{R}$ and positive. In this case, one has $|\psi(\varepsilon)|^2 + |\psi(0)|^2 > B > 0$.

In particular, if we have $\alpha_0 = -\alpha_1$, then (2.5) is reduced to

$$(\alpha_0^2 + \lambda)\sin\big(\sqrt{\lambda}\varepsilon\big) = 0 \quad \text{and} \quad \lambda = \frac{n^2\pi^2}{\varepsilon^2}, \quad n \in \mathbb{Z} \setminus \{0\}.$$

Therefore the corresponding eigenfunction to $\lambda = n^2\pi^2/\varepsilon^2$, $n \geqslant 1$, is given by

$$\psi_n(y) = \frac{n\pi}{(n^2\pi^2 + \alpha_0^2\varepsilon^2)^{1/2}}\Big(\psi_n^N(y) - \frac{\alpha_0\varepsilon}{n\pi}\psi_n^D(y)\Big)$$

with $\psi_n^N = \sqrt{2/\varepsilon}\cos(n\pi y/\varepsilon)$ and $\psi_n^D = \sqrt{2/\varepsilon}\sin(n\pi y/\varepsilon)$, $n \geqslant 1$, where ψ_n^N and ψ_n^D are the eigenfunctions of $-\Delta_N^I$ and $-\Delta_D^I$, where $-\Delta_N^I$ and $-\Delta_D^I$ are Laplace operators with Neumann and Dirichlet boundary condition, respectively.

Assume now $\lambda < 0$ and set $\mu = -\lambda$, then a classical solution of (2.2) is given by the formula

$$\psi(y) = A\mathrm{e}^{\sqrt{\mu}y} + B\mathrm{e}^{-\sqrt{\mu}y}.$$

In order to satisfy the boundary condition we obtain the system

$$\begin{bmatrix} \alpha_0 - \sqrt{\mu} & \alpha_0 + \sqrt{\mu} \\ \sqrt{\mu}\mathrm{e}^{\sqrt{\mu}\varepsilon} + \alpha_1\mathrm{e}^{\sqrt{\mu}\varepsilon} & \alpha_1\mathrm{e}^{-\sqrt{\mu}\varepsilon} - \sqrt{\mu}\mathrm{e}^{-\sqrt{\mu}\varepsilon} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = 0. \tag{2.6}$$

This system admits a nontrivial solution if and only if

$$f(\mu) = (\mu + \alpha_0\alpha_1)(\mathrm{e}^{-\sqrt{\mu}\varepsilon} - \mathrm{e}^{\sqrt{\mu}\varepsilon}) - \sqrt{\mu}(\alpha_0 + \alpha_1)(\mathrm{e}^{-\sqrt{\mu}\varepsilon} + \mathrm{e}^{\sqrt{\mu}\varepsilon}) = 0. \tag{2.7}$$

In particular, if $\alpha_0 + \alpha_1 = 0$, then (2.7) is reduced to

$$(\alpha_0^2 - \mu)(\mathrm{e}^{-\sqrt{\mu}\varepsilon} - \mathrm{e}^{\sqrt{\mu}\varepsilon}) = 0,$$

and since $\mu \neq 0$, one has $\mu = \alpha_0^2$, i.e. $\lambda = -\alpha_0^2$. It follows from $\psi'(0) + \alpha_0\psi(0) = 0$ that the eigenfunction associated with the negative eigenvalue $\lambda = -\alpha_0^2$ is

$$\psi(y) = c\mathrm{e}^{-\alpha_0 y} \quad \text{with } c^{-1} = \|\mathrm{e}^{-\alpha_0 y}\|_{L^2(I)} > 0.$$

To summarize this discussion we find that if $\alpha_0 = -\alpha_1 \neq 0$, the first eigenvalue is negative and equal to $-\alpha_0^2$ and all the others are positive.

If $\lambda = 0$, then the general solution is $\psi(y) = Ay + B$ and from the boundary conditions it follows that $A = B = 0$. Therefore $\lambda = 0$ is not an eigenvalue.

Returning to (2.6) with $\alpha_0 + \alpha_1 \neq 0$, we have that (2.6) is equivalent to

$$\big(\alpha_0 - \sqrt{\mu}\big)A + \big(\alpha_0 + \sqrt{\mu}\big)B = 0,$$

so A is given, in terms of B, by $A = \big(\alpha_0 + \sqrt{\mu}\big)/\big(\sqrt{\mu} - \alpha_0\big)B$. Note that $f(\alpha_0^2) = 2\mathrm{e}^{\alpha_0\varepsilon}(\alpha_0 + \alpha_1)|\alpha_0|$, so $f(\alpha_0^2) = 0$ if and only if $\alpha_0 = 0$, since $\mu > 0$, the value of the expressions between A and B is well defined. The normalization condition of ψ in particular allows one to choose $B \in \mathbb{R}$ and positive. In this case, one has $|\psi(\varepsilon)|^2 + |\psi(0)|^2 > B > 0$.

Throughout this paper, we denote by λ_0 the first eigenvalue of the Robin Laplacian on I (recall the self-adjointness of $-\Delta^I$) associated with the normalized eigenfunction $\phi_0(y)$.

3. Infinite and straight planar strips

The purpose of this section is to show that the classic Laplacian $-\Delta_\alpha^\Omega$ in $L^2(\Omega)$, with a suitable domain, is self-adjoint. For this purpose a convenient sesquilinear form b_α^Ω will be introduced, whose definition will be made precise later.

Under certain conditions on $\alpha = (\alpha_0(x), \alpha_1(x))$ at infinity, it is possible to prove the existence of isolated bound states, i.e. the existence of eigenvalues (of finite multiplicity) below the essential spectrum $\sigma_{\rm ess}(-\Delta_\alpha^\Omega)$ of the Laplacian. For this purpose, we follow some ideas in [3], [12], [13].

The densely defined closed quadratic form of interest is $b_\alpha^\Omega(\phi)$, $\operatorname{dom} b_\alpha^\Omega = H^1(\Omega) \subset L^2(\Omega)$,

$$b_\alpha^\Omega(\phi) = \int_\Omega |\nabla\phi(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y + \int_\mathbb{R} \alpha_1(x)|\mathrm{tr}(\phi)(x,\varepsilon)|^2 + \alpha_0(x)|\mathrm{tr}(\phi)(x,0)|^2 \, \mathrm{d}x,$$

where $\mathrm{tr}(\phi)$ denotes the range of trace operator $\mathrm{tr}\colon H^1(\Omega) \to L^2(\partial\Omega)$ and $\alpha_0(x)$ and $\alpha_1(x)$ are two given positive functions. A lower bound for b_α^Ω is initially obtained for $\phi|_\Omega$ with $\phi \in C_0^\infty(\mathbb{R}^2)$ and, by density (see [1], Theorem 18), for each $\phi \in H^1(\Omega)$. Note that $\mathrm{tr}(\phi|_\Omega) = \phi|_\Omega$ in $L^2(\partial\Omega)$, consequently

$$b^\Omega(\phi) \geqslant \int_\mathbb{R} \left[\int_I \left|\frac{\partial\phi}{\partial y}\right|^2 \mathrm{d}y - \alpha_1(x)|\phi(x,\varepsilon)|^2 - \alpha_0(x)|\phi(x,0)|^2 \right] \mathrm{d}x$$

since $\phi(x,\cdot) \in H^1(I)$ for a.e. $x \in \mathbb{R}$. By the argument in Theorem 2.1 we obtain

$$b^\Omega(\phi) \geqslant -(\|\alpha_0 + \alpha_1\|_\infty) \frac{a^2\varepsilon^{-1}}{2a-1} \|\phi\|^2_{L^2(\Omega)} \tag{3.1}$$

for each $\phi|_\Omega$ with $\phi \in C_0^\infty(\mathbb{R}^2)$. By standard arguments we can verify that b_α^Ω is closed.

We consider the operator $-\Delta_\alpha^\Omega$ on $L^2(\Omega)$ which acts as the Laplacian on the domain consisting of functions ψ from the Sobolev space $H^2(\Omega)$ satisfying the boundary conditions (1.2), i.e. with $\operatorname{dom}(-\Delta_\alpha^\Omega) = \{\psi \in H^2(\Omega)\colon \psi \text{ satisfies (1.2)}\}$, here we require $\alpha_0(x), \alpha_1(x) \in W^{1,\infty}(\mathbb{R})$.

In the next theorem we show the self-adjointness of $-\Delta_\alpha^\Omega$, where this is the Robin operator associated with the form b_α^Ω.

Theorem 3.1. *Suppose $\alpha_0(x)$ and $\alpha_1(x)$ in $W^{1,\infty}(\mathbb{R})$. Then the Laplacian $-\Delta_\alpha^\Omega$ is the (unique) self-adjoint operator associated with the sesquilinear form b_α^Ω, that is,*

$$b_\alpha^\Omega(\phi,\psi) = (\phi, -\Delta_\alpha^\Omega \psi)_{L^2(\Omega)}$$

for $\phi \in \operatorname{dom} b_\alpha^\Omega$ and $\psi \in \operatorname{dom}(-\Delta_\alpha^\Omega)$.

The proof is presented through Lemmas 3.1 and 3.2. The first lemma gives some information on the domain of $T_{b_\alpha^\Omega}$, associated with b_α^Ω. It is shown actually that $\operatorname{dom} T_{b_\alpha^\Omega} \subset \operatorname{dom}(-\Delta_\alpha^\Omega)$. The second one concludes that $T_{b_\alpha^\Omega}$ is an extension of $-\Delta_\alpha^\Omega$. Therefore we obtain the equality $T_{b_\alpha^\Omega} = -\Delta_\alpha^\Omega$.

Lemma 3.1. *Suppose $\alpha_0(x)$ and $\alpha_1(x)$ in $W^{1,\infty}(\mathbb{R})$. For each $F \in L^2(\Omega)$, every solution $\psi \in H^1(\Omega)$ of the problem*

$$b_\alpha^\Omega(\phi,\psi) = (\phi, F)_{L^2(\Omega)} \quad \forall \phi \in \operatorname{dom} b_\alpha^\Omega = H^1(\Omega), \tag{3.2}$$

belongs to $\operatorname{dom}(-\Delta_\alpha^\Omega)$. *Consequently,* $\operatorname{dom} T_{b_\alpha^\Omega} \subset \operatorname{dom}(-\Delta_\alpha^\Omega)$.

P r o o f. For $\psi \in H^1(\Omega)$ let us introduce the quotient of Newton

$$\psi_\delta(x,y) := \frac{\psi(x+\delta,y) - \psi(x,y)}{\delta}, \quad 0 \neq \delta \in \mathbb{R}.$$

Since

$$|\psi(x+\delta,y) - \psi(x,y)| = \left|\int_0^1 \frac{\partial \psi}{\partial x}(x+\delta t, y)\delta \, \mathrm{d}t\right| \leqslant |\delta| \int_0^1 \left|\frac{\partial \psi}{\partial x}(x+\delta t,y)\right| \mathrm{d}t,$$

we have

$$\int_\Omega |\psi_\delta|^2 \, \mathrm{d}x \, \mathrm{d}y \leqslant \int_0^1 \left[\int_\Omega \left|\frac{\partial \psi}{\partial x}(x+\delta t,y)\right|^2 \mathrm{d}x \, \mathrm{d}y\right] \mathrm{d}t = \int_\Omega \left|\frac{\partial \psi}{\partial x}(x,y)\right|^2 \mathrm{d}x \, \mathrm{d}y.$$

Therefore

$$\int_\Omega |\psi_\delta|^2 \, \mathrm{d}x \, \mathrm{d}y \leqslant \|\psi\|_{1,2}^2 \quad \forall\, 0 \neq \delta \in \mathbb{R}. \tag{3.3}$$

If $\psi \in H^1(\Omega)$ is a solution of (3.2), then ψ_δ is a solution of the problem

$$\begin{aligned} b_\alpha^\Omega(\phi,\psi_\delta) = &-(\phi_{-\delta}, F)_{L^2(\Omega)} \\ &- \int_\mathbb{R} (\alpha_1)_\delta(x)\overline{\phi(x,\varepsilon)}\psi(x+\delta,\varepsilon) + (\alpha_0)_\delta(x)\overline{\phi(x,0)}\psi(x+\delta,0)\, \mathrm{d}x \end{aligned}$$

for each $\phi \in H^1(\Omega)$. By choosing $\phi = \psi_\delta$ we obtain

$$\begin{aligned}(3.4)\quad b_\alpha^\Omega(\psi_\delta) = & -((\psi_\delta)_{-\delta}, F)_{L^2(\Omega)} \\ & - \int_{\mathbb{R}} (\alpha_1)_\delta(x)\overline{\psi_\delta(x,\varepsilon)}\psi(x+\delta,\varepsilon) + (\alpha_0)_\delta(x)\overline{\psi_\delta(x,0)}\psi(x+\delta,0)\,\mathrm{d}x.\end{aligned}$$

For simplicity we write $b_\alpha^\Omega(\psi_\delta) = b_1^\Omega(\psi_\delta) + b_2^\Omega(\psi_\delta)$ with

$$b_1^\Omega(\psi_\delta) = \int_\Omega |\nabla\psi_\delta|^2\,\mathrm{d}x\,\mathrm{d}y \quad \text{and} \quad b_2^\Omega(\psi_\delta) = \int_{\mathbb{R}} \alpha_1(x)|\psi_\delta(x,\varepsilon)|^2 + \alpha_0(x)|\psi_\delta(x,0)|^2\,\mathrm{d}x.$$

By Schwarz inequality, Cauchy inequality, estimate (3.3) and the embedding of $H^1(\Omega)$ in $L^2(\partial\Omega)$, we can produce the following estimates for $t > 0$:

$$\begin{aligned}&|((\psi_\delta)_{-\delta}, F)_{L^2(\Omega)}| \leqslant 2\|F\|_{L^2(\Omega)}\|(\psi_\delta)_{-\delta}\|_{L^2(\Omega)} \leqslant t^{-1}\|F\|^2_{L^2(\Omega)} + t\|\psi_\delta\|^2_{1,2}, \\ &\left|\int_{\mathbb{R}} (\alpha_1)_\delta(x)\overline{\psi_\delta(x,\varepsilon)}\psi(x+\delta,\varepsilon) + (\alpha_0)_\delta(x)\overline{\psi_\delta(x,0)}\psi(x+\delta,0)\,\mathrm{d}x\right| \\ &\qquad\qquad \leqslant C_1\|\psi_\delta\|_{L^2(\partial\Omega)}\|\psi\|_{L^2(\partial\Omega)} \leqslant C\|\psi_\delta\|_{1,2}\|\psi\|_{1,2}\end{aligned}$$

with $C = C(\|\alpha_0\|_{W^{1,\infty}}, \|\alpha_1\|_{W^{1,\infty}}) > 0$ independent of δ,

$$|b_2^\Omega(\psi_\delta)| \leqslant t^{-1}\|\psi\|^2_{1,2} + t b_1^\Omega(\psi_\delta)$$

for $t > 0$ small enough. On one hand, one has

$$b_\alpha^\Omega(\psi_\delta) \geqslant (1-t)b_1^\Omega(\psi_\delta) - t^{-1}\|\psi\|^2_{1,2}.$$

On the other hand, the identity (3.4) produces

$$|b_\alpha^\Omega(\psi_\delta)| \leqslant C\|\psi_\delta\|_{1,2}\|\psi\|_{1,2} + (t^{-1}\|F\|^2_{L^2(\Omega)} + t\|\psi_\delta\|^2_{1,2}).$$

So, we have the following estimate:

$$(1-t)b_1^\Omega(\psi_\delta) - t^{-1}\|\psi\|^2_{1,2} \leqslant C\|\psi_\delta\|_{1,2}\|\psi\|_{1,2} + (t^{-1}\|F\|^2_{L^2(\Omega)} + t\|\psi_\delta\|^2_{1,2}).$$

Now, suppose that $0 < t < 1$ and add $(1-t)\|\psi_\delta\|_2^2$ to both sides of the above inequality to obtain

$$0 \leqslant (2t-1)\|\psi_\delta\|^2_{1,2} + C\|\psi_\delta\|_{1,2}\|\psi\|_{1,2} + (t^{-1}\|F\|^2_{L^2(\Omega)} + t^{-1}\|\psi\|^2_{1,2} + (1-t)\|\psi\|^2_{1,2}).$$

Thus, we assume that $0 < t < 1/2$, so the dominant term of the quadratic function is negative and we get $\|\psi_\delta\|^2_{1,2} \leqslant \widetilde{C}$ with $\widetilde{C}$ independent of δ. But this estimate implies

$$\sup_\delta \|\psi_\delta\|_{1,2} < \infty$$

and since $H^1(\Omega)$ is reflexive, every bounded sequence has a weakly convergent subsequence, then there is $v \in H^1(\Omega)$ and a subsequence $\delta_k \to 0$ such that $\psi_{-\delta_k} \xrightarrow{w} v$ in $H^1(\Omega)$. Hence,

$$\begin{aligned}\int_\Omega \psi \partial_x \phi \,\mathrm{d}x\,\mathrm{d}y &= \int_\Omega \psi \lim_{\delta_k \to 0} \phi_{\delta_k} \,\mathrm{d}x\,\mathrm{d}y = \lim_{\delta_k \to 0} \int_\Omega \psi \phi_{\delta_k} \,\mathrm{d}x\,\mathrm{d}y \\ &= -\lim_{\delta_k \to 0} \int_\Omega \psi_{-\delta_k} \phi \,\mathrm{d}x\,\mathrm{d}y = -\int_\Omega v\phi \,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Therefore $\partial_x \psi = v$ in the weak sense, and so $\partial_x \psi \in H^1(\Omega)$. Consequently, $\partial_{xx}\psi \in L^2(\Omega)$ and $\partial_{yx}\psi \in L^2(\Omega)$. It follows from the standard elliptic regularity theorems (see [9], Theorem 1, Section 6.3.1) that $\psi \in H^2_{\mathrm{loc}}(\Omega)$, so $-\Delta\psi = F$ a.e. in Ω. Hence, $\partial_{yy}\psi = -(F + \partial_{xx}\psi) \in L^2(\Omega)$ and therefore $\psi \in H^2(\Omega)$.

Finally, it remains to verify that ψ satisfies the boundary conditions. After integration by parts,

$$\begin{aligned}(\phi, F)_{L^2(\Omega)} = b^\Omega_\alpha(\phi, \psi) = (\phi, -\Delta\psi)_{L^2(\Omega)} &+ \int_\mathbb{R} \overline{\phi(x,0)}[-\partial_y \psi(x,0) + \alpha_0(x)\psi(x,0)]\,\mathrm{d}x \\ &+ \int_\mathbb{R} \overline{\phi(x,\varepsilon)}[\partial_y \psi(x,\varepsilon) + \alpha_1(x)\psi(x,\varepsilon)]\,\mathrm{d}x\end{aligned}$$

for each $\phi \in H^1(\Omega)$. This implies the boundary conditions, because $-\Delta\psi = F$ a.e. in Ω and ϕ is arbitrary. □

Lemma 3.2. *Suppose $\alpha_0(x)$ and $\alpha_1(x)$ in $W^{1,\infty}(\mathbb{R})$. Then $T_{b^\Omega_\alpha} = -\Delta^\Omega_\alpha$.*

Proof. Let $\psi \in \mathrm{dom}(-\Delta^\Omega_\alpha)$, then $\psi \in H^2(\Omega)$ and it satisfies the boundary conditions (1.2). By integration by parts and (1.2) we obtain for each $\phi \in \mathrm{dom}\, b^\Omega_\alpha$ the identity

$$\begin{aligned}b^\Omega_\alpha(\phi, \psi) &= \int_\mathbb{R} \overline{\phi(x,\varepsilon)}\partial_y\psi(x,\varepsilon)\,\mathrm{d}x - \int_\mathbb{R} \overline{\phi(x,0)}\partial_y\psi(x,0)\,\mathrm{d}x - \int_\Omega \overline{\phi(x,y)}\Delta\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_\mathbb{R} \alpha_1(x)\overline{\phi(x,\varepsilon)}\psi(x,\varepsilon)\,\mathrm{d}x + \int_\mathbb{R} \alpha_0(x)\overline{\phi(x,0)}\psi(x,0)\,\mathrm{d}x \\ &= (\phi, -\Delta\psi)_{L^2(\Omega)}.\end{aligned}$$

Thus, $\psi \in \mathrm{dom}\, T_{b^\Omega_\alpha}$, and it follows that $T_{b^\Omega_\alpha}$ is an extension of $-\Delta^\Omega_\alpha$. Lemma 3.1 yields the desired equality. □

4. The spectrum of the Robin Laplacian in Ω

Here we investigate the spectrum of the operator $-\Delta_\alpha^\Omega$ when the Robin parameter (function) $\alpha(x,y) = (\alpha_0(x), \alpha_1(x)) \in W^{1,\infty}(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})$ satisfies the condition

$$\lim_{|x|\to\infty} (\alpha_0(x) - \alpha_0) = 0 \quad \text{and} \quad \lim_{|x|\to\infty} (\alpha_1(x) - \alpha_1) = 0. \tag{4.1}$$

In the case that (4.1) holds we prove that the essential part $\sigma_{\rm ess}(-\Delta_\alpha^\Omega)$ of the spectrum of $-\Delta_\alpha^\Omega$ is the interval $[\lambda_0, \infty)$, where λ_0 is the first Robin transversal eigenvalue. This statement is contained in Theorem 4.1, whose proof is performed in two steps, that is, Propositions 4.1 and 4.2, whose proofs were inspired in [3]. The proof of Proposition 4.1 makes use of the so-called Weyl criterion for the essential spectrum, which we recall in Lemma 4.1 (see [5], Theorem 11.2.7).

Lemma 4.1 (Weyl criterion)**.** *Let T be a self-adjoint operator in a complex Hilbert space $\mathcal{H}$. Then $\lambda \in \sigma_{\rm ess}(T)$ if and only if there exists a sequence $\{\psi_n\}_{n=1}^\infty \subset$ $\operatorname{dom} T$ such that*

(1) $\|\psi_n\| = 1$ *for all* $n \in \mathbb{N} \setminus \{0\}$;
(2) $\psi_n \xrightarrow{w} 0$ *as* $n \to \infty$ *in* $\mathcal{H}$;
(3) $(T - \lambda)\psi_n \to 0$ *as* $n \to \infty$.

Such a sequence is called a singular Weyl sequence for T at λ.

Initially, we verify by means of Lemma 4.2 that for the operator $T_\alpha = -\Delta$, that is, the operator associated with the form

$$t_\alpha^\Omega(\phi) = \int_\Omega |\nabla\phi|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_\mathbb{R} \alpha_1 |\mathrm{tr}(\phi)(x,\varepsilon)|^2 + \alpha_0 |\mathrm{tr}(\phi)(x,0)|^2 \,\mathrm{d}x$$

which is a special case of b_α^Ω with $\alpha = (\alpha_0(x), \alpha_1(x))$ constant and equal to (α_0, α_1), there is no discrete eigenvalue in its spectrum.

Lemma 4.2. *If $\alpha(x) = (\alpha_0, \alpha_1)$ is a constant function with $\{\alpha_0, \alpha_1\} \subset \mathbb{R}$, then*

$$[\lambda_0, \infty) \subset \sigma_{\rm ess}(T_\alpha).$$

P r o o f. Let $\lambda \in [\lambda_0, \infty)$. So, one can write $\lambda = \lambda_0 + t$ with $t \in [0, \infty)$. Let us introduce the Laplacian operator$-\Delta^\mathbb{R}$ in $L^2(\mathbb{R})$. It is well known that the essential spectrum $\sigma_{\rm ess}(-\Delta^\mathbb{R})$ of operator $-\Delta^\mathbb{R}$ is the set $[0, \infty)$. Hence, there is a singular Weyl sequence $\{\phi_n\}_{n=1}^\infty$ for $-\Delta^\mathbb{R}$ at t. Define the sequence $\{\psi_n\}_{n=1}^\infty$ as $\psi_n(x,y) = \phi_n(x)\phi_0(y)$

with ϕ_0 the eigenfunction (normalized) of self-adjoint transversal operator $-\Delta^I$, associated with the first eigenvalue λ_0. Note that $\{\psi_n\}_{n=1}^{\infty} \subset \mathrm{dom}(T_\alpha)$. It is easy to check that $\|\psi_n\|_{L^2(\Omega)} = 1$ for each $n \geqslant 1$, and $\psi_n \xrightarrow{w} 0$ weakly in $L^2(\Omega)$, and also that $(T_\alpha - \lambda)\psi_n \to 0$ in $L^2(\Omega)$-norm because we have

$$(T_\alpha - \lambda)\psi_n = [(-\Delta^{\mathbb{R}} - t)\phi_n]\phi_0 + [(-\Delta^I - \lambda_0)\phi_0]\phi_n.$$

Hence, $\{\psi_n\}_{n=1}^{\infty}$ is a singular Weyl sequence for T_α at λ. Then by virtue of Lemma 4.1, $\lambda \in \sigma_{\mathrm{ess}}(T_\alpha)$. $\square$

Proposition 4.1. *If* $\alpha(x) = (\alpha_0, \alpha_1)$ *is a constant function with* $\{\alpha_0, \alpha_1\} \subset \mathbb{R}$, *then*

$$\sigma_{\mathrm{ess}}(T_\alpha) = [\lambda_0, \infty).$$

P r o o f. To prove the inverse inclusion it is enough to show that $t_\alpha^\Omega(\phi) \geqslant \lambda_0\|\phi\|^2$ for each ϕ in $H^1(\Omega)$. Since the spectrum of $-\Delta^I$ starts by λ_0, we can write for all $\xi \in H^1(I)$

$$b^I(\xi) = \int_I |\xi|^2 \,\mathrm{d}y + \alpha_1\xi(\varepsilon) + \alpha_0\xi(0) \geqslant \lambda_0\|\xi\|^2. \tag{4.2}$$

Using this inequality together with Fubini's theorem, we get for all $\phi|_\Omega$ with $\phi \in C_0^\infty(\mathbb{R}^2)$, and by density (see [1], Theorem 3.18) for each $\phi \in H^1(\Omega)$,

$$t_\alpha^\Omega(\phi) \geqslant \int_{\mathbb{R}} \left[\int_I \left|\frac{\partial\phi}{\partial y}\right|^2 \mathrm{d}y + \alpha_1|\phi(x,\varepsilon)|^2 + \alpha_0|\phi(x,0)|^2 \,\mathrm{d}x\right] \geqslant \lambda_0\|\phi\|^2.$$

It follows that $\sigma_{\mathrm{ess}}(T_\alpha) \subseteq [\lambda_0, \infty)$. Note that we have used the fact that $\phi(x,\cdot)$ belongs to $H^1(I)$ for a.e. $x \in \mathbb{R}$. $\square$

Next we prove Proposition 4.2 with the help of technical estimate in Lemma 4.3 (see [3], Lemma 5.1). Let us introduce $\beta_i(x) := (\alpha_i(x) - \alpha_i)$ with $i \in \{0,1\}$ and the functions

$$\beta_m(x,y) = \begin{cases} \beta_0(x) & \text{if } |x| < m,\ y = 0, \\ \beta_1(x) & \text{if } |x| < m,\ y = \varepsilon, \\ 0 & \text{if } |x| \geqslant m \end{cases} \tag{4.3}$$

and

$$\beta(x,y) = \begin{cases} \beta_0(x) & \text{in } \mathbb{R} \times \{0\}, \\ \beta_1(x) & \text{in } \mathbb{R} \times \{\varepsilon\}. \end{cases} \tag{4.4}$$

By (4.1)–(4.4) we have that for each integer m the function β_m is bounded with compact support and the sequence $\{\beta_m\}$ converges in $L^\infty(\mathbb{R})$ to β, because given $\delta > 0$ there exists $a > 0$ such that $|\alpha_i(x) - \alpha_i| < \delta$ whenever $|x| > a$ with $i \in \{0, 1\}$.

Lemma 4.3. *Let $\alpha_0, \alpha_1 \in \mathbb{R}$ and $\varphi \in L^2(\partial\Omega)$. Then there exist positive constants c and C, depending on ε and $|\alpha_0 + \alpha_1|$, such that any solution $\psi \in H^2(\Omega)$ of the boundary value problem*

$$\begin{cases} (-\Delta - \lambda)\psi = 0 \quad \text{in } \Omega, \\ -\dfrac{\partial\psi}{\partial y}(x,0) + \alpha_0\psi(x,0) = \varphi(x,0), \\ \dfrac{\partial\psi}{\partial y}(x,\varepsilon) + \alpha_1\psi(x,\varepsilon) = \varphi(x,\varepsilon) \end{cases} \tag{4.5}$$

with any $\lambda < -c$, satisfies the estimate

$$\|\psi\|_{1,2} \leqslant C\|\varphi\|_{L^2(\partial\Omega)}. \tag{4.6}$$

P r o o f. Multiplying the first equation of (4.5) by $\overline{\psi}$ and integrating by parts, one can produce the identity

$$\int_\Omega |\nabla\psi|^2 \,\mathrm{d}x\,\mathrm{d}y + \alpha_0 \int_{\mathbb{R}\times\{0\}} |\psi|^2 \,\mathrm{d}x + \alpha_1 \int_{\mathbb{R}\times\{\varepsilon\}} |\psi|^2 \,\mathrm{d}x - \lambda \int_\Omega |\psi|^2 \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} \varphi\overline{\psi} \,\mathrm{d}\sigma.$$

Using the Schwarz and Cauchy inequalities and the embedding of $H^1(\Omega)$ in $L^2(\partial\Omega)$, we have for $t \in (0,1)$,

$$\begin{aligned} \left|\alpha_0 \int_{\mathbb{R}\times\{0\}} |\psi|^2 \,\mathrm{d}x + \alpha_1 \int_{\mathbb{R}\times\{\varepsilon\}} |\psi|^2 \,\mathrm{d}x\right| & \\ \leqslant \left|\int_{\partial\Omega} (\alpha_1 + \alpha_0)|\psi|^2 \,\mathrm{d}\sigma\right| &\leqslant |\alpha_0 + \alpha_1| \|\psi\|^2_{L^2(\partial\Omega)} \\ &\leqslant t^{-1}\|\psi\|^2_{L^2(\partial\Omega)}|\alpha_0 + \alpha_1|^2 + t\|\psi\|^2_{L^2(\partial\Omega)} \\ &\leqslant t^{-1}|\alpha_0 + \alpha_1|^2\|\psi\|^2_{1,2} + t\widetilde{C}\|\psi\|^2_{1,2}, \\ \left|\int_{\partial\Omega} \varphi\overline{\psi} \,\mathrm{d}\sigma\right| \leqslant \|\psi\|_{L^2(\partial\Omega)}\|\varphi\|_{L^2(\partial\Omega)} &\leqslant t^{-1}\|\varphi\|^2_{L^2(\partial\Omega)} + t\|\psi\|^2_{L^2(\partial\Omega)} \\ &\leqslant t^{-1}\|\varphi\|^2_{L^2(\partial\Omega)} + t\widetilde{C}\|\psi\|^2_{1,2}, \end{aligned}$$

where $\widetilde{C}$ is the constant from the embedding of $H^1(\Omega)$ in $L^2(\partial\Omega)$. By the above estimates, we obtain

$$(1 - \lambda - t^{-1}|\alpha_0 + \alpha_1|^2 - 2t\widetilde{C})\|\psi\|^2_{1,2} \leqslant t^{-1}\|\varphi\|^2_{L^2(\partial\Omega)}.$$

The desired conclusion follows by choosing $t > 0$ small enough and sufficiently large negative λ such that the coefficient of $\|\psi\|^2_{1,2}$ becomes positive. □

Proposition 4.2. *Suppose that* $(\alpha_0(x),\alpha_1(x)) \in W^{1,\infty}(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})$. *In addition, if the difference* $\alpha_i(x)-\alpha_i$ *vanishes at infinity, i.e.* $\lim\limits_{|x|\to\infty}(\alpha_i(x)-\alpha_i)=0$ *for each* $i\in\{0,1\}$, *then for each* $\lambda\in\varrho(-\Delta_\alpha^\Omega)\cap\varrho(T_\alpha)$ *the operator*

$$(-\Delta_\alpha^\Omega-\lambda)^{-1}-(T_\alpha-\lambda)^{-1}$$

is compact in $L^2(\Omega)$.

P r o o f. Due to the first resolvent identity, it is enough to prove the result for a negative λ in the intersection of the respective resolvent sets. Consider a sequence $\{\phi_j\}_{j=1}^\infty\subset L^2(\Omega)$ bounded and let $\psi_j=(-\Delta_\alpha^\Omega-\lambda)^{-1}\phi_j-(T_\alpha-\lambda)^{-1}\phi_j$; note that ψ_j satisfies the first equation in (4.5). Moreover, inserting ψ_j into the second or third equation we obtain

$$\frac{\partial\psi_j}{\partial\vec{n}}+\beta\psi_j=\Big(\frac{\partial}{\partial\vec{n}}+\beta\Big)((-\Delta_\alpha^\Omega-\lambda)^{-1}\phi_j-(T_\alpha-\lambda)^{-1}\phi_j)=\beta\,\mathrm{tr}(-\Delta_\alpha^\Omega-\lambda)^{-1}\phi_j$$

so we now take $\varphi=\beta\,\mathrm{tr}(-\Delta_\alpha^\Omega-\lambda)^{-1}\phi_j$ and by Lemma 4.3 we have

$$\|\psi_j-\psi_k\|_{1,2}\leqslant C\|(\beta\,\mathrm{tr}(-\Delta_\alpha^\Omega-\lambda)^{-1})(\phi_j-\phi_k)\|_{L^2(\partial\Omega)},$$

where tr denotes the trace operator from $H^1(\Omega)\supset\mathrm{dom}(-\Delta_\alpha^\Omega)$ to $L^2(\partial\Omega)$.

Under the assumption that $\beta\,\mathrm{tr}(-\Delta_\alpha^\Omega-\lambda)^{-1}$ is a compact operator, it follows that the sequence $\{\psi_j\}_{j=1}^\infty$ is precompact in the topology of $H^1(\Omega)$, and with the help of the above inequality one can establish that $(-\Delta_\alpha^\Omega-\lambda)^{-1}-(T_\alpha-\lambda)^{-1}$ is a compact operator in $L^2(\Omega)$.

Let us verify the compactness of the operator $\beta\,\mathrm{tr}(-\Delta_\alpha^\Omega-\lambda)^{-1}$. One can show that the sequence of operators $\beta_m\,\mathrm{tr}(-\Delta_\alpha^\Omega-\lambda)^{-1}$ converges to $\beta\ \mathrm{tr}(-\Delta_\alpha^\Omega-\lambda)^{-1}$ in $L^2(\partial\Omega)$-norm, because we have $\|\beta_m-\beta\|_{L^\infty(\mathbb{R})}\to0$ in $L^\infty(\mathbb{R})$. On the other hand, we shall prove that each operator $\beta_m\,\mathrm{tr}(-\Delta_\alpha^\Omega-\lambda)^{-1}$ is compact. Indeed, given the sequence $\{u_n\}_{n=1}^\infty$ bounded in $L^2(\Omega)$ one has $v_n=(-\Delta_\alpha^\Omega-\lambda)^{-1}u_n$ bounded in $H^1(\Omega)$, then there exists a subsequence, which we still denote by v_n, and a function $v\in H^1(\Omega)$ such that $v_n\to v$ weakly in $H^1(\Omega)$. Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega_m)$, where $\Omega_m=(-m,m)\times(0,\varepsilon)\subset\Omega$, due to the Rellich-Kondrachov theorem (see [1], Section VI.) we get $v_n\to v$ in $L^2(\Omega_m)$. Since β_m vanishes identically outside Ω_m, and β on Ω_m, we have

$$\begin{aligned}\|\beta_m\,\mathrm{tr}(v_n)-\beta_m\,\mathrm{tr}(v_l)\|_{L^2(\partial\Omega)}&=\|\beta(\mathrm{tr}(v_n)-\mathrm{tr}(v_l))\|_{L^2(\partial\Omega_m)}\\&\leqslant C_m\|\beta\|_{L^\infty(\mathbb{R})}\|v_n-v_l\|_{L^2(\Omega_m)},\end{aligned}$$

where $C_m>0$ comes from the fact that the operator tr from $H^1(\Omega_m)$ onto $L^2(\partial\Omega_m)$ is bounded.

It follows that $\beta_m \operatorname{tr}(v_n)$ is a Cauchy sequence and thus $\lim_{n\to\infty} \beta_m \operatorname{tr}(v_n)$ exists for each positive integer m. Therefore the operators $\beta_m \operatorname{tr}(-\Delta_\alpha^\Omega - \lambda)^{-1}$ are compact. □

With the results presented up to now we can prove the following theorem:

Theorem 4.1. *Let $\alpha_0(x), \alpha_1(x) \in W^{1,\infty}(\mathbb{R})$. If $\lim_{|x|\to\infty} (\alpha_i(x) - \alpha_i) = 0$, $i \in \{0,1\}$, then*

$$\sigma_{\rm ess}(-\Delta_\alpha^\Omega) = [\lambda_0, \infty).$$

Proof. Proposition 4.2 implies that $(-\Delta_\alpha^\Omega - \lambda)^{-1} - (T_\alpha - \lambda)^{-1}$ is a compact operator in $L^2(\Omega)$; so the essential spectra of $-\Delta_\alpha^\Omega$ and T_α are identical by Theorem XIII.14 in [17]. □

4.1. Existence of discrete spectrum. Now, based on [12], [13], [15] and under appropriate conditions, we shall give a variational argument to conclude that $\sigma(-\Delta_\alpha^\Omega) \cap (-\infty, \lambda_0) \neq \emptyset$. This, together with Theorem 4.2, implies that the spectrum below $-\alpha_0^2$ is nonempty and formed by isolated eigenvalues of finite multiplicity, i.e. $\sigma_{\rm disc}(-\Delta_\alpha^\Omega) \neq \emptyset$; see Corollary 4.1.

Theorem 4.2. *Let $(\alpha_i(x) - \alpha_i) \in W^{1,\infty}(\mathbb{R})$ with $i \in \{0,1\}$. In addition, suppose that the the difference $(\alpha_i(x) - \alpha_i)$ is integrable with $\int_{\mathbb{R}} (\alpha_i(x) - \alpha_i)\,\mathrm{d}x < 0$ for some $i \in \{0,1\}$ and $\int_{\mathbb{R}} (\alpha_j(x) - \alpha_j)\,\mathrm{d}x \leqslant 0$ for $j \neq i$. Then*

$$\inf \sigma(-\Delta_\alpha^\Omega) < \lambda_0.$$

Proof. Following [12] we wish to obtain a trial function ψ from the form domain of $-\Delta_\alpha^\Omega$ such that the quadratic form $Q_\alpha^\Omega(\psi) < 0$, where

$$Q_\alpha^\Omega(\phi) = b_\alpha^\Omega(\phi) - \lambda_0 \|\phi\|_2^2, \quad \operatorname{dom} Q_\alpha^\Omega = \operatorname{dom} b_\alpha^\Omega.$$

Let ζ be a cut-off function, that is, we fix a function $\zeta \in C_0^\infty(\mathbb{R})$ with $0 \leqslant \zeta \leqslant 1$, and $\zeta \equiv 1$ on $(-1/4, 1/4)$, $\zeta \equiv 0$ on $\mathbb{R} \setminus (-1/2, 1/2)$ and $\|\zeta\|_2 = 1$. Given ϕ_0 as defined in the proof of Lemma 4.2, consider the sequence $\{u_n\}_{n=1}^\infty$ of functions into $\operatorname{dom} b_\alpha^\Omega$, defined by $u_n(x,y) = f_n(x)\phi_0(y)$, where $f_n(x) = \zeta(x/n)$. By integration by parts and using the boundary conditions of ϕ_0, we obtain

$$\begin{aligned} Q_\alpha^\Omega(u_n) = n^{-1}\|\zeta'\|_2^2 + \|f_n\|_2^2 \int_I (|\partial_y \phi_0|^2 - \lambda_0 |\phi_0|^2)\,\mathrm{d}y \\ + \int_{\mathbb{R}} |f_n|^2 (\alpha_1(x)|\phi_0(\varepsilon)|^2 + \alpha_0(x)|\phi_0(0)|^2)\,\mathrm{d}x. \end{aligned}$$

Since

$$\int_I (|\partial_y \phi_0|^2 - \lambda_0 |\phi_0|^2)\,\mathrm{d}y = -\alpha_1 |\phi_0(\varepsilon)|^2 - \alpha_0 |\phi_0(0)|^2,$$

we have

$$Q_\alpha^\Omega(u_n) = n^{-1}\|\zeta'\|_2^2 + (|\phi_0(\varepsilon)|^2 + |\phi_0(0)|^2)\sum_{i=0}^{1}\int_{\mathbb{R}} (\alpha_i(x) - \alpha_i)|f_n|^2\,\mathrm{d}x,$$

where $|\phi_0(\varepsilon)|^2 + |\phi_0(0)|^2 > 0$ according to Section 2.2. Taking into account the estimate $|f_n(x)(\alpha_i(x) - \alpha_0)| \leqslant |\alpha_i(x) - \alpha_0| \in L^1(\mathbb{R})$ and the fact that $f_n(x) \to 1$ as $n \to \infty$, we can apply the dominated convergence theorem to get

$$\lim_{n\to\infty} Q_\alpha^\Omega(u_n) = (|\phi_0(\varepsilon)|^2 + |\phi_0(0)|^2)\sum_{i=0}^{1}\int_{\mathbb{R}} (\alpha_i(x) - \alpha_i)\,\mathrm{d}x < 0.$$

It is therefore justified to write that there exists some $u_N \in \operatorname{dom} b_\alpha^\Omega$ such that $b_\alpha^\Omega(u_N) < \lambda_0$. Therefore, by invoking Rayleigh-Ritz Theorem, we can state that $\inf \sigma(-\Delta_\alpha^\Omega) < \lambda_0$. □

Corollary 4.1. *Under the assumptions of Theorem* 4.2, *if condition* (4.1) *holds true, then*

$$\sigma_{\mathrm{disc}}(-\Delta_\alpha^\Omega) \cap (-\infty, \lambda_0) \neq \emptyset.$$

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