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COMPLETE SOLUTION OF THE DIOPHANTINE EQUATION $x^y + y^x = z^z$

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Abstract. The triples $(x, y, z) = (1, z^z - 1, z)$, $(x, y, z) = (z^z - 1, 1, z)$, where $z \in \mathbb{N}$, satisfy the equation $x^y + y^x = z^z$. In this paper it is shown that the same equation has no integer solution with $\min\{x, y, z\} > 1$, thus a conjecture put forward by Z. Zhang, J. Luo, P. Z. Yuan (2013) is confirmed.

Keywords: exponential Diophantine equation; sieving; modular computations

MSC 2010: 11D61, 11A15

1. Introduction

The title equation is one of the exponential Diophantine equations which were studied in recent years. It is clear that it has solutions of the type $(x, y, z) = (1, z^z - 1, z)$, $(x, y, z) = (z^z - 1, 1, z)$ with $z \in \mathbb{N}$. Zhang, Luo, and Yuan proved in [8] that the equation

$$x^y + y^x = z^z, \quad x, y, z \in \mathbb{N},\ \min\{x, y, z\} > 1, \tag{1.1}$$

has only finitely many solutions and all of them satisfy $z < 2.8 \cdot 10^9$. The same authors put forward a more ambitious statement.

Conjecture 1.1 ([8]). *Equation* (1.1) *has no solution.*

Additional information on hypothetical solutions is provided by subsequent work. Thus, Deng and Zhang [2] excluded the possibility that both x and y be odd primes. More recently Wu showed in [7] that z has to be even. Using this result and bounds on linear forms in 2-adic logarithms due to Bugeaud [1], Du [3] substantially shrinked the region where solutions of (1.1) are confined.

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Theorem 1.2 ([3]). *All solutions* (x, y, z) *of* (1.1) *with* z *even satisfy*

$$\max\{x, y, z\} < 480000.$$

The same author has proved another theorem, according to which both x and y are singular numbers, and Du suggested to verify the above conjecture by combining this result with older computational results found in [4], [6].

The aim of this note is to confirm Conjecture 1.1.

Theorem 1.3. *There are no positive integers satisfying*

$$x^y + y^x = z^z \quad \textit{and} \quad \min\{x, y, z\} > 1.$$

Although our proof is computer-dependent, it is based on a different idea than that suggested by Du. The volume of computations required by our approach is diminished by an elementary observation recorded as Lemma 2.3 below, which allows a relatively fast sieving of integers restricted as in Theorem 1.2. In Section 2 we gather all the knowledge needed in the proof. Section 3 contains the description of the algorithm employed for searching possible solutions to (1.1).

2. Preliminary results

In the rest of the paper, (x, y, z) is a solution of the title equation with $x \leqslant y$ and z even. Then it is known from [8] that the entries x, y, z are pairwise coprime integers greater than 1. In fact, as shown in [2], [8], one has

$$3 < x < z < y. \tag{2.1}$$

These restrictions can be strengthened in various ways. The next lemma shows that x and z cannot be very close to each other.

Lemma 2.1.
(i) *If* $z \leqslant x + 9$, *then* $y \leqslant 2z - x - 2$.
(ii) $x \leqslant z - 5$.

P r o o f. (i) From (1.1) one gets $y < z \ln z / \ln x$. We claim that for $z \leqslant x + 9$ it also holds

$$\frac{z \ln z}{\ln x} < 2z - x. \tag{2.2}$$

Keeping in mind the information on the parities of x, y and z, part (i) follows.

In order to prove (2.2), define a function $f\colon [5,480000] \to \mathbb{R}$ depending on a parameter $d \in [1,9]$ by $f(t) = (t+d)\ln(t+d) - (t+2d)\ln t$. From

$$f'(t) = \ln(t+d) - \ln t - \frac{2d}{t} < \frac{d}{t} - \frac{2d}{t} < 0$$

one obtains $f(t) \leqslant f(5) = (d+5)\ln(d+5) - (2d+5)\ln 5$ for all $t \in [5,480000]$. An elementary study of the auxiliary function $g\colon [1,9] \to \mathbb{R}$ defined by formula $g(d) = (d+5)\ln(d+5) - (2d+5)\ln 5$ shows that g takes only negative values. Hence, $f(t) < 0$ for all $t \in [5,480000]$, so (2.2) holds.

(ii) We establish the second part by reduction to absurd. If $x = z-1$, then from (i) one gets $y < z+1$, in contradiction with (2.1). Suppose now that equation (1.1) has a solution of the form $(x,y,z) = (z-3,y,z)$ for some even integer $z \geqslant 6$ and odd integer y. From (i) one obtains $y < z+3$, so

$$(z-3)^{z+1} + (z+1)^{z-3} = z^z. \tag{2.3}$$

As $\gcd(z, z-3) = 1$, one has z coprime to 3. Since z is even, the right-hand side of (2.3) is congruent to 1 modulo 3 while its left-hand side is a multiple of 3 when $z \equiv 1 \pmod 3$ and congruent to 2 modulo 3 when $z \equiv 2 \pmod 3$. □

The result just proved can be employed to derive an absolute lower bound for z.

Lemma 2.2. *Let (x,y,z) be a solution to (1.1). Then $z \geqslant 18$ if z is divisible by 3 and $z > 30$ otherwise.*

Proof. If z is divisible by 3, then Lemma 2.1 together with (2.1) show, on the one hand, that $z \geqslant 12$ in any solution to (1.1) and, on the other hand, that $x \in \{5,7\}$ when $z = 12$. For $x = 7$ and $z = 12$, Lemma 2.1 yields $y \leqslant 15$, so $y = 13$ is the only possibility not eliminated by restrictions in force. However, the equality $7^{13} + 13^7 = 12^{12}$ is impossible modulo 7. Similarly, for $x = 5$ and $z = 12$ one arrives at one of the equalities $5^{13} + 13^5 = 12^{12}$, $5^{17} + 17^5 = 12^{12}$, either of which is seen to be false modulo 5. Instead, one can invoke the result from [2].

Suppose now that z is not divisible by 3. In order to establish that any hypothetical such solution satisfies $z > 30$, one can proceed similarly to what has been done to eliminate the possibilities $z = 6$, $z = 12$. Now there are more candidates to examine. As the details are more intricate and no new ideas in comparison to the case $3 \mid z$ are involved, we omit detailed exposition. Alternatively, for each z in $\{14,16,20,22,26,28\}$ and for each odd x greater than 3 and less than $z-3$ one can list the odd integers y greater than z and smaller than $z\ln z/\ln x$ such that $\gcd(x,y,z) = 1$. It suffices to let a computer verify that equation (1.1) holds for none of the resulting triples. □

In order to reduce the volume of explicit computations, we slightly improve on Du's bound by noticing that

$$\frac{36.1}{8(\log 2)^4} < 19.5486 \quad \text{and} \quad 0.4 + \log(2\log 2) < 0.7267.$$

Using these values instead of 19.554 and 0.7271 appearing in equation (2.9) from [3], one obtains

$$z < 19.5486(\log x)(\log y)\Big(0.7267 + \log\Big(\frac{x}{\log x} + \frac{y}{\log y}\Big)\Big)^2. \tag{2.4}$$

Proceeding as explained in [3], one readily gets

$$y \leqslant 474421.$$

Further improvements are given by the next elementary observation, based on Chinese Remainder Theorem.

Lemma 2.3. *If* $3 \mid z$, *then* $x + y \equiv 0 \pmod{24}$, *otherwise* $x + y \equiv 16 \pmod{24}$.

P r o o f. Since xy is odd, the left-hand side of the equation of interest satisfies $x^y + y^x \equiv x + y \pmod 8$. The right-hand side is congruent to 0 modulo 8 because z is even and greater than 3.

Using again the fact that both x and y are odd, it readily results that $x^y + y^x \equiv x + y \pmod 3$. This congruence is then compared to $z^z \pmod 3$, which is either 0 or 1, depending on whether 3 divides z or not. □

3. Proof of Theorem 1.3

Our proof relies on a script implementing in the computer algebra system PARI/GP [5] the results mentioned in Section 2.

First we give the details of the search for solutions (x, y, z) with $6 \mid z$ and $z \geqslant 18$. Put $M = 6^4$, $M1 = 6^8$, $M2 = 6^{12}$. We let an integer variable x take a value coprime with 24 and less than $UB := 474500$. Another integer variable y takes a value greater than the current value stored in x yet smaller than UB, and subject to restriction given by Lemma 2.3. We check whether

$$x^y + y^x \equiv 0 \pmod{M}, \quad x^y + y^x \equiv 0 \pmod{M1}, \quad x^y + y^x \equiv 0 \pmod{M2},$$

in this order holds.

Any pair (x, y) failing any of these congruences can be safely ignored as it cannot be prolongated to a solution (x, y, z) to (1.1). The surviving pairs are checked against the necessary condition $\gcd(x, y) = 1$. If this holds, then we compute the expression on the right-hand side of (2.4), calling ZZ the resulting value. If the current values in variables x and ZZ satisfy $x < ZZ$, there is some hope to find a solution, so we print/save the pair (x, y). Next we increase y by 24 if the updated value is still smaller than UB, or increase x by 24 as long as this operation is compatible with the restrictions in force.

This sieving is rather efficient: all pairs have been examined and rejected by the final version of our script in about 390 minutes of computing time on a rather old desktop.

We proceed in a similar way for searching solutions (x, y, z) in which z is coprime to 3. As Lemma 2.2 shows that in any such solution one has $z > 30$, the choice of moduli $M = 2^{10}$, $M1 = 2^{20}$, $M2 = 2^{30}$ is legitimate. There are three surviving pairs:

$$\begin{aligned}(x, y) &= (24795, 273229) \quad \text{for } x \equiv 3 \pmod{24},\\ (x, y) &= (10215, 73897) \quad \text{for } x \equiv 15 \pmod{24},\\ (x, y) &= (24763, 199725) \quad \text{for } x \equiv 19 \pmod{24}.\end{aligned}$$

Elimination of these candidates could be done by choosing either a larger modulus $M2$ or a very small one P with the property that $x^y + y^x \pmod{P}$ is a quadratic non-residue. For instance, the last of the pairs mentioned above satisfies $24763^{199725} + 199725^{24763} \equiv 3 \pmod 5$. Since 3 is quadratic non-residue modulo 5, equation (1.1) has no solutions of the type $(24763, 199725, z)$. The same modulus can serve to eliminate the second candidate pair, while $(24795, 273229)$ is rejected with $P = 23$, for example.

All five tests implemented in the final version of the script have contributed to the reported outcome. For instance, for $x \equiv 7 \pmod{24}$ and $x < 60000$, $3 \nmid z$, there were found 366910 pairs (x, y) with $x^y + y^x$ divisible by 2^{10}, out of which 407 pairs generated an expression divisible by 2^{20}, and for a sole pair, the left-hand side of the title equation is congruent to 0 modulo 2^{30}. The surviving pair has coprime entries which do not pass the test based on (2.4). For $x \equiv 19 \pmod{24}$ and $x < 60000$, $3 \nmid z$, there are two pairs satisfying the three congruence tests and the entries of one of them are not coprime.

We close by noting that the approach employed in this proof can be adapted to the study of other exponential Diophantine equations. But this remains for future work.

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