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GROUPS WITH ONLY TWO NONLINEAR NON-FAITHFUL IRREDUCIBLE CHARACTERS

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Abstract. We determine the groups with exactly two nonlinear non-faithful irreducible characters whose kernels intersect trivially.

Keywords: non-faithful character; nonlinear character; finite group

MSC 2010: 20C15

1. Introduction

In this paper, all groups are finite. In [4], Seitz determined the groups with exactly one nonlinear irreducible character. Zhang classified in [5] groups with exactly two nonlinear irreducible characters. Iranmanesh and Saeidi in [1] studied groups with exactly one nonlinear non-faithful irreducible character. Furthermore, Saeidi in [3] classified solvable groups with a unique nonlinear non-faithful irreducible character.

In [2], Li, Chen and Li classify the p-groups that have two nonlinear non-faithful irreducible characters. In this paper, we move beyond p-groups to consider all groups with two nonlinear non-faithful irreducible characters. Our goal is modest. In particular, we classify the groups with exactly two nonlinear non-faithful irreducible characters whose kernels intersect trivially.

Theorem 1. *Let G be a group. Then G has exactly two nonlinear non-faithful irreducible characters whose kernels intersect trivially if and only if G is a 2-group with nilpotence class 2 satisfying $|G'| = 2$, and $Z(G)$ is a Klein-4-group.*

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We observe that the groups in the conclusion of Theorem 1 appear in Zhang's list in [5]. Thus, the groups satisfying the conclusion of Theorem 1 have exactly two nonlinear irreducible characters. This gives the following consequence of Theorem 1. Let G be a group with exactly two nonlinear non-faithful irreducible characters χ_1 and χ_2 whose kernels intersection is trivial. Then χ_1 and χ_2 are the only nonlinear irreducible characters of G.

2. Proof

Seitz proved in [4] that G has exactly one nonlinear irreducible character if and only if either G is an extraspecial 2-group or G is a Frobenius group with elementary abelian Frobenius kernel G' and a cyclic Frobenius complement H that satisfies $|G'| - 1 = |H|$. In particular, if G has only one nonlinear irreducible character, then G is solvable.

Proof of Theorem 1. Let χ_1 and χ_2 be the two nonlinear non-faithful irreducible characters of G. Since $\ker\chi_1 \cap \ker\chi_2 = 1$, it follows that G is isomorphic to a subgroup of $G/\ker\chi_1 \times G/\ker\chi_2$. Since the remaining nonlinear irreducible characters are faithful, it follows that χ_i is the only nonlinear irreducible character of $G/\ker\chi_i$ for $i = 1, 2$. Thus, we may apply Seitz's result to both of these quotients. In particular, each of these quotients is solvable, so G is solvable. This implies that $1 < G' < G$, where G' is the derived subgroup of G.

Let $K = \ker\chi_1 \ker\chi_2$. Notice from Sietz's theorem that $G'\ker\chi_i/\ker\chi_i$ is the unique minimal normal subgroup of $G/\ker\chi_i$ for each i. Since $\ker\chi_1 \cap \ker\chi_2 = 1$, it follows that $K > \ker\chi_i$ for each i. This implies that $G'\ker\chi_i \leqslant K$ for each i. We claim that $K = G'\ker\chi_i$ for each i. To prove this claim, suppose it is not true for some i. Without loss of generality, we may assume that $G'\ker\chi_2 < K$. Let $L = \ker\chi_1 \cap G'\ker\chi_2$. Notice that $K/L = \ker\chi_1/L \times G'\ker\chi_2/L$. Thus, $(G/L)'$ is not the unique minimal normal subgroup of G/L, and so G/L does not satisfy Seitz's theorem. On the other hand, it is not difficult to see that χ_1 must be the only nonlinear irreducible character of G/L, so we have a contradiction. Therefore $K = G'\ker\chi_2$. A similar proof can be used to show that $K = G'\ker\chi_1$.

This implies that

$$G/K = (G/\ker\chi_1)/(G/\ker\chi_1)' = (G/\ker\chi_2)/(G/\ker\chi_2)'.$$

Notice that when $G/\ker\chi_1$ is a Frobenius group, G/K is cyclic, and when $G/\ker\chi_2$ is an extraspecial 2-group, then G/K is a noncyclic elementary abelian 2-group. Therefore we have two cases: either both quotients are Frobenius groups or both quotients are extraspecial 2-groups.

1. Suppose that both $G/\ker\chi_i$ are Frobenius groups. Notice that $|\ker\chi_1| = |K/\ker\chi_2| = |G:K|+1$. Similarly, $|\ker\chi_2| = |G:K|+1$. It follows that K is an elementary abelian p-group of order $(|G:K|+1)^2$ for some prime p. In particular, K is a Sylow p-subgroup of G. Let H be a Hall p-complement of G.

We know that $H\ker\chi_1 \cong G/\ker\chi_2$ and $H\ker\chi_2 \cong G/\ker\chi_1$ are Frobenius groups. We have $[\ker\chi_i, H] = \ker\chi_i$. This yields $K = [\ker\chi_1, H][\ker\chi_2, H] \leqslant G'$. Since $G' \leqslant K$, we deduce that $K = G'$. Notice that we can identify the elements in $\ker\chi_1$ and $\ker\chi_2$, and that H will preserve this identification. The resulting diagonal subgroup D will be normalized by H, and so D is normal in G. Since $D < K = G'$, we see that G/D is not abelian. Thus, $\mathrm{Irr}(G/D)$ will contain a nonlinear irreducible character that is not faithful and not equal to χ_1 or χ_2, which is a contradiction. Therefore, we conclude that this case cannot occur.

2. Thus, both $G/\ker\chi_i$ are extraspecial 2-groups. Since $|\ker\chi_1| = |K:\ker\chi_2|$ that G is a 2-group. We now apply Theorem 3.1 of [2] to see that G is as stated.

Conversely, suppose G is a 2-group with $|G'| = 2$ and $\mathbf{Z}(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. By Theorem 3.1 of [2], we see that G has the desired property. $\square$

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