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PRESENTATIONS FOR SUBSEMIGROUPS OF PD_n

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Abstract. Let $[n] = \{1, \dots, n\}$ be an n-chain. We give presentations for the following transformation semigroups: the semigroup of full order-decreasing mappings of $[n]$, the semigroup of partial one-to-one order-decreasing mappings of $[n]$, the semigroup of full order-preserving and order-decreasing mappings of $[n]$, the semigroup of partial one-to-one order-preserving and order-decreasing mappings of $[n]$, and the semigroup of partial order-preserving and order-decreasing mappings of $[n]$.

Keywords: presentation; order-decreasing mapping; order-preserving mapping; transformation semigroups

MSC 2010: 20M20, 20M30

1. Introduction and preliminaries

There are many well-known presentations for various finite permutation groups, for example, the famous Coxeter presentation for the symmetric group. After tools of semigroup theory were developed, semigroup theorists became interested in finding presentations for finite transformation semigroups. Some of the early works on this go back to the start of earnest investigation as to what semigroup presentations are, see [24], [25]. Later some papers appeared on presentations for order-preserving mappings, see [13], [14], [23], orientation-preserving mappings, see [12], and for general transformation monoids, see [9]–[11]. Inspired by the Coxeter groups and Brauer-type semigroups, there also appeared presentations for various subsemigroups of the composition monoid in [18] and [21]. A substantial amount of work to find presentations for various transformation-like monoids has been carried out in papers [1]–[8].

In general, the big motivation for finding presentations for semigroups lies in the fact that knowing their presentations is really helpful to construct various types of representations of those semigroups, see [16] and [22]. However, for us the main objective in this paper is purely combinatorial—we are going to provide presentations

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for various subsemigroups of PD_n, which we now define. (An earlier draft of this paper has appeared as [30].)

Consider an n-chain, i.e. $[n] = \{1, \dots, n\}$ together with the the natural linear order on it. Let P_n denote the semigroup of all partial mappings on $[n]$. By an *order-decreasing* mapping we shall mean a partial mapping $\alpha \in P_n$ such that $x\alpha \leqslant x$ for all $x \in \mathrm{dom}(\alpha)$; and by an *order-preserving* mapping we shall mean a partial mapping $\alpha \in P_n$ such that $x\alpha \leqslant y\alpha$ for all $x \leqslant y$ from the domain of α. Let

- ▷ PD_n be the semigroup of all partial order-decreasing mappings of the chain $[n]$;
- ▷ D_n be the semigroup of all full order-decreasing mappings of $[n]$;
- ▷ PC_n be the semigroup of all partial order-preserving and order-decreasing mappings of $[n]$;
- ▷ C_n be the semigroup of all full order-preserving and order-decreasing mappings of $[n]$;
- ▷ IC_n be the semigroup of all partial one-to-one order-preserving and order-decreasing mappings of $[n]$;
- ▷ ID_n be the semigroup of all partial one-to-one order-decreasing mappings of $[n]$.

The subsemigroups of PD_n and D_n are well-studied, one should consult [17], [27]–[29].

Now section by section we will provide the presentations for these semigroups. But prior to that, we recall what semigroup presentations are: Let A be a finite set, and A^* be the free monoid generated by A, i.e. the set of all words over A under concatenation. Let $R \subseteq A^* \times A^*$. Then by the (monoid) *presentation* $\langle A : R \rangle$ we mean the semigroup A^*/ϱ, where ϱ is the congruence on A^*, defined as follows: for $u, v \in A^*$ we have $u\varrho v$ if and only if there exist $x_0, x_1 \dots, x_n \in A^*$ such that $u = x_0 \sim x_1 \sim \dots \sim x_{n-1} \sim x_n = v$, where by $x \sim y$ for $x, y \in A^*$ we mean that there exists $(u, v) \in R$ and $\alpha, \beta \in A^*$ such that either $x = \alpha u \beta$ and $y = \alpha v \beta$, or $x = \alpha v \beta$ and $y = \alpha u \beta$. For more detail on monoid presentations we refer the reader to two theses [25] and [20].

2. Presentation for D_n

A general study of the semigroups D_n and PD_n was initiated in [28] and they arise in language theory [17].

Prior to stating the presentation for D_n, let us discuss its natural generating set. For $1 \leqslant i < j \leqslant n$ let $f_{i,j}$ be the idempotent in D_n which maps j to i and fixes all remaining points. Then one checks that the $f_{i,j}$'s generate D_n (see [27]) and are subject to the following relations (under the mapping $e_{i,j} \mapsto f_{i,j}$):

$$
\begin{aligned}
&(1) \qquad & e_{i,j}^2 &= e_{i,j}, & \\
&(2) \qquad & e_{i,j}e_{k,l} &= e_{k,l}e_{i,j} & \text{if } \{i,j\} \cap \{k,l\} = \emptyset, \\
&(3) \qquad & e_{i,j}e_{i,k} &= e_{j,k}e_{i,j}, & i < j < k, \\
&(4) \qquad & e_{i,k}e_{i,j} &= e_{j,k}e_{i,j}, & i < j < k, \\
&(5) \qquad & e_{i,k}e_{j,k} &= e_{i,k}, & i, j < k.
\end{aligned}
$$

We will prove now that the monoid M_n presented by (1)–(5) is isomorphic to D_n. Firstly, we start with the following lemma.

Lemma 2.1. *Let $i < j$ and $k < l$. If $l < j$, then we can select p, q, r such that $p < q$, $r < j$, $\{p, q, r\} = \{i, k, l\}$ and $e_{i,j}e_{k,l} = e_{p,q}e_{r,j}$.*

Proof. Let $l < j$. If $\{k, l\} \cap \{i, j\} = \emptyset$, then the claim follows from (2). If $\{k, l\} \cap \{i, j\} \neq \emptyset$, then one of k and l must coincide with i. If $l = i$, then $k < l = i < j$ and so $e_{i,j}e_{k,l} = e_{i,j}e_{k,i} = e_{k,i}e_{k,j}$ by (3). If $k = i$, then $i = k < l < j$ and so $e_{i,j}e_{k,l} = e_{i,j}e_{i,l} = e_{i,l}e_{i,j}$ by (3) and (4). The claim follows. $\square$

Now we will show that every element from M_n can be represented in the form

$$(6) \qquad e_{i_1,j_1}e_{i_2,j_2}\dots e_{i_k,j_k}$$

for some $k \geqslant 0$, $2 \leqslant j_1 < j_2 < \ldots < j_k \leqslant n$ and $1 \leqslant i_s < j_s$ for all $s \leqslant k$.

By inductive arguments, to establish this it suffices to show that the product of element (6) and any element $e_{i,j}$ is of the form (6). So, let $\pi = e_{i_1,j_1}e_{i_2,j_2}\dots e_{i_k,j_k}$ with all the above conditions on i_s and j_s. If $j > j_k$, then $\pi e_{i,j}$ is of the required form. If $j = j_k$, then using (5) we have that $\pi e_{i,j} = \pi$ is again of the required form. So, let $j < j_k$. Then by Lemma 2.1 there exist $p, q, r < j_k$ such that $e_{i_k,j_k}e_{i,j} = e_{p,q}e_{r,j_k}$. Repeating this at most $k - 1$ more times will bring $\pi e_{i,j}$ to the required form.

One now notices that there are exactly $n!$ different formal products (6), which yields $|M_n| \leqslant n!$. But $|D_n| = n!$, and since there is an onto homomorphism from M_n onto D_n, this completes the proof that $M_n \cong D_n$.

Remark 2.2. From [28] we know that PD_n is isomorphic to D_{n+1} and so there is no need to give separate presentation for PD_n.

3. Presentation for ID_n

As with D_n and PD_n, a general study of the semigroup ID_n was initiated in [28], see also [29].

Let e_i be the idempotent in ID_n such that $\mathrm{dom}(e_i) = \mathrm{im}(e_i) = \{1, \ldots, n\} \setminus \{i\}$; and for $1 \leqslant i < j \leqslant n$ let $b_{i,j}$ be the element of ID_n with $\mathrm{dom}(b_{i,j}) = \{1, \ldots, n\} \setminus \{i\}$ which maps j to i and fixes all remaining points. One checks that the e_i's together with the $b_{i,j}$'s generate ID_n (see [29]) and that they are subject to the following relations (under the mapping $f_i \mapsto e_i$ and $a_{i,j} \mapsto b_{i,j}$):

$$f_i^2 = f_i, \tag{7}$$

$$f_i f_j = f_j f_i, \qquad i \neq j, \tag{8}$$

$$f_k a_{i,j} = a_{i,j} f_k, \qquad i < j \ \&\ k \notin \{i, j\}, \tag{9}$$

$$f_i a_{i,j} = a_{i,j} f_j = a_{i,j}, \qquad i < j, \tag{10}$$

$$f_j a_{i,j} = a_{i,j} f_i = f_i f_j, \qquad i < j, \tag{11}$$

$$a_{i,j} a_{k,l} = a_{k,l} a_{i,j}, \qquad \{i, j\} \cap \{k, l\} = \emptyset, \tag{12}$$

$$a_{j,k} a_{i,j} = f_j a_{i,k}, \qquad i < j < k. \tag{13}$$

We will prove that the monoid M_n presented by relations (7)–(13) is isomorphic to ID_n. From (9), (10), (11) and by induction we easily see that every element $w \in M_n$ is expressible as

$$f_{i_1} f_{i_2} \ldots f_{i_k} u,$$

where $u \in \{a_{i,j} \colon\ i < j\}^*$, $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$.

Lemma 3.1.

(1) $a_{i,j} a_{k,j} = f_k a_{i,j}$ if $i, k < j$ and $i \neq k$.

(2) $a_{i,j} a_{i,k} = f_j a_{i,k}$ if $i < j, k$ and $j \neq k$.

(3) $a_{i,j} a_{i,j} = f_i f_j$ if $i < j$.

Proof. (1) $a_{i,j} a_{k,j} = a_{i,j} f_j a_{k,j} = a_{i,j} f_j f_k = a_{i,j} f_k = f_k a_{i,j}$.

(2) $a_{i,j} a_{i,k} = a_{i,j} f_i a_{i,k} = f_i f_j a_{i,k} = f_j a_{i,k}$.

(3) $a_{i,j} a_{i,j} = a_{i,j} f_i a_{i,j} = f_i f_j a_{i,j} = f_i f_j$. $\square$

From Lemma 3.1 it follows that any word $w \in M_n$ can be expressed as

$$f_{i_1} \ldots f_{i_k} a_{t_1, j_1} \ldots a_{t_r, j_r} \tag{14}$$

with $2 \leqslant j_1 < \ldots < j_r \leqslant n$ and $t_s < j_s$ for all s.

Now we will prove that additionally we may assume that in (14) all t_i's are pairwise distinct. Indeed, let (14) have a chunk of consecutive letters $a_{t_s,j_s}\ldots a_{t_p,j_p}$ such that $t_s = t_p$ and there is no t_i equal to $t_s = t_p$ for $s < i < p$. Note also that $t_s = t_p \notin \{j_s,\ldots,j_p\}$. Hence by (9), (10) and (11) we have

$$\begin{aligned} a_{t_s,j_s}\ldots a_{t_p,j_p} &= a_{t_s,j_s}\ldots a_{t_{p-1},j_{p-1}} f_{t_p} a_{t_p,j_p} \\ &= a_{t_s,j_s}\ldots a_{t_{p-2},j_{p-2}} f_{t_p} a_{t_{p-1},j_{p-1}} a_{t_p,j_p} \\ &\vdots \\ &= a_{t_s,j_s} f_{t_p} a_{t_{s+1},j_{s+1}}\ldots a_{t_p,j_p} \\ &= f_{t_s} f_{j_s} a_{t_{s+1},j_{s+1}}\ldots a_{t_p,j_p}, \end{aligned}$$

and hence we turn product (14) to one with lesser number of entries of $a_{i,j}$'s, which allows by use of inductive reasoning to deduce that indeed in (14) we may assume that all t_i's are pairwise distinct.

Furthermore, in (14) we may assume that $\{i_1,\ldots,i_k\}\cap\{j_1,\ldots,j_r,t_1,\ldots,t_r\} = \emptyset$. Indeed, otherwise, using relations (8), (9), (10), and (11), we could push the corresponding f_i to the right of the word (14) and either replace some of a_{t_s,j_s} by some f_q and move that newly introduced f_q back to the left, or the corresponding f_i vanishes.

Now, element (14) in M_n with all the above conditions on i_s, t_l and j_l, evaluated in ID_n is the element which maps j_l to t_l for $l \leqslant r$ and any point from $\{1,\ldots,n\}\setminus\{i_1,\ldots,i_k,j_1,\ldots,j_r,t_1,\ldots,t_r\}$ identically. Any such element in ID_n uniquely recovers product (14). Hence $|M_n| \leqslant |ID_n|$, but of course $|M_n| \geqslant |ID_n|$ and so $M_n \cong ID_n$, as required.

4. Presentation for C_n

The semigroup C_n, also known as the Catalan monoid because $|C_n|$ is the nth Catalan number, was first studied by Higgins and it also arose in language theory [17]. We provide for completeness the following result, the proof of which the reader can find in [16] and [26]. The monoid C_n is presented by

$$\langle e_i, 1 \leqslant i \leqslant n-1\colon\ e_i^2 = e_i, \tag{15}$$

$$e_ie_j = e_je_i \quad \text{if } |i-j| \geqslant 2, \tag{16}$$

$$e_ie_{i+1}e_i = e_{i+1}e_i, \tag{17}$$

$$e_{i+1}e_ie_{i+1} = e_{i+1}e_i\rangle. \tag{18}$$

5. Presentation for IC_n

The semigroup IC_n first appeared in [15] and not much is known about it. For $i \leqslant n$ let f_i be the idempotent in IC_n with $\mathrm{dom}(f_i) = \mathrm{im}(f_i) = \{1, \ldots, n\} \setminus \{i\}$; and for $i \leqslant n-1$ let b_i be the element of IC_n which maps $i+1$ to i and fixes all the points from $\{1, \ldots, n\} \setminus \{i, i+1\}$.

Lemma 5.1. $IC_n = \langle f_i,\ i \leqslant n;\ b_i,\ i \leqslant n-1 \rangle$.

Proof. Let $f \in IC_n$. Let F be the set of fixed points of f. Let $\mathrm{dom}(f) \setminus F$ consist of the points $j_1 < \ldots < j_p$ and $\mathrm{im}(f) \setminus F$ consist of the points $t_1 < \ldots < t_p$. Then one has $t_s \leqslant j_s$. Then one easily calculates that

$$f = \prod_{x \in [n] \setminus \mathrm{dom}(f)} f_x \cdot (b_{j_1} \ldots b_{t_1}) \ldots (b_{j_p} \ldots b_{t_p}).$$

□

Also one sees that the f_i's together with the b_i's satisfy the following relations (under the mapping $e_i \mapsto f_i$, $a_i \mapsto b_i$):

$$e_i^2 = e_i, \tag{19}$$

$$e_i e_j = e_j e_i, \tag{20}$$

$$e_i a_j = a_j e_i \qquad \text{if } i < j \text{ or } i > j+1, \tag{21}$$

$$a_i a_j = a_j a_i \qquad \text{if } |i-j| \geqslant 2, \tag{22}$$

$$e_i a_i = a_i e_{i+1} = a_i, \qquad i \leqslant n-1, \tag{23}$$

$$e_{i+1} a_i = a_i e_i = e_i e_{i+1}, \qquad i \leqslant n-1. \tag{24}$$

We will prove that the monoid M_n presented by relations (19)–(24) is isomorphic to IC_n. We proceed with the following lemma.

Lemma 5.2.
(1) $a_{i+1} a_i a_{i+1} = a_{i+1} a_i$.
(2) $a_i a_i = e_i e_{i+1}$.
(3) $a_i a_{i+1} a_i = a_{i+1} a_i$.

Proof. (1) $a_{i+1} a_i a_{i+1} = a_{i+1} e_{i+2} a_i a_{i+1} = a_{i+1} a_i e_{i+2} a_{i+1} = a_{i+1} a_i e_{i+1} e_{i+2} = a_{i+1} a_i e_{i+2} = a_{i+1} e_{i+2} a_i = a_{i+1} a_i$.

(2) $a_i a_i = a_i e_i a_i = e_i e_{i+1} a_i = e_i e_i e_{i+1} = e_i e_{i+1}$.

(3) $a_i a_{i+1} a_i = a_i a_{i+1} e_i a_i = a_i e_i a_{i+1} a_i = e_i e_{i+1} a_{i+1} a_i = e_i a_{i+1} a_i = a_{i+1} e_i a_i = a_{i+1} a_i$. □

Now, one easily sees from (21), (23) and (24) that every element $w \in M_n$ can be expressed as

$$e_{i_1} \dots e_{i_k} a_{j_1} \dots a_{j_p}.$$

Actually, we will prove that any $w \in M_n$ can be represented as

$$\pi = e_{i_1} \dots e_{i_k} (a_{j_1} \dots a_{t_1}) \dots (a_{j_p} \dots a_{t_p}) \tag{25}$$

with $j_1 < j_2 < \dots < j_p$; $t_1 < t_2 < \dots < t_p$; $t_s \leqslant j_s$.

To prove this, we proceed by induction and let i be arbitrary to consider πa_i. If $i > j_p$, then πa_i is of the required form. So let $i \leqslant j_p$. If $t_p \leqslant i$, then by Lemma 5.2 and (22) we can use induction and bring πa_i to the required form. Thus let $i < t_p$. If $i < t_p - 1$, then

$$\pi a_i = e_{i_1} \dots e_{i_k} (a_{j_1} \dots a_{t_1}) \dots (a_{j_{p-1}} \dots a_{t_{p-1}}) a_i (a_{j_p} \dots a_{t_p}),$$

and we further bring πa_i to the required form. So let $i = t_p - 1$. If $t_p - 1 > t_{p-1}$, then πa_i is in the needed form. Thus, let finally $t_p - 1 = t_{p-1}$. Then by Lemma 5.2,

$$\begin{aligned}
\pi a_i &= e_{i_1} \dots e_{i_k} (a_{j_1} \dots a_{t_1}) \dots (a_{j_{p-2}} \dots a_{t_{p-2}})(a_{j_{p-1}} \dots a_{t_{p-1}+1})(a_{j_p} \dots a_{t_p - 1} a_{t_p} a_{t_p - 1}) \\
&= e_{i_1} \dots e_{i_k} (a_{j_1} \dots a_{t_1}) \dots (a_{j_{p-2}} \dots a_{t_{p-2}})(a_{j_{p-1}} \dots a_{t_{p-1}+1})(a_{j_p} \dots a_{t_p - 1}) \\
&= e_{i_1} \dots e_{i_k} (a_{j_1} \dots a_{t_1}) \dots (a_{j_{p-2}} \dots a_{t_{p-2}})(a_{j_p} \dots a_{t_p - 1})
\end{aligned}$$

is in the required form.

Additionally, we may require that in (25) the elements i_s do not coincide with any of the indices i of a_i appearing in (25), and do not coincide with any of $j_1 + 1, \dots, j_p + 1$. Then all such products (25) evaluated in IC_n are pairwise distinct. Hence $M_n \cong IC_n$.

6. Presentation for PC_n

The semigroup PC_n, also known as the Schröder monoid because $|PC_n|$ is the nth (double) Schröder number, first appeared in [19] and it also arose in language theory [17].

Let a_i be the idempotent from PC_n with $\mathrm{dom}(a_i) = \mathrm{im}(a_i) = \{1, \dots, n\} \setminus \{i\}$; and for $i \leqslant n-1$ let b_i be the idempotent from PC_n which maps $i+1$ to i and fixes all remaining points. Then the a_i's together with the b_i's generate PC_n, and are subject to the following relations (under the mapping $f_i \mapsto a_i$, $e_i \mapsto b_i$):

$$e_i^2 = e_i, \tag{26}$$

$$e_i e_j = e_j e_i \quad \text{if } |i-j| \geqslant 2, \tag{27}$$

$$e_i e_{i+1} e_i = e_{i+1} e_i, \tag{28}$$

$$e_{i+1} e_i e_{i+1} = e_{i+1} e_i, \tag{29}$$

$$f_i^2 = f_i, \tag{30}$$

$$f_i f_j = f_j f_i, \tag{31}$$

$$f_j e_i = e_i f_j \quad \text{if } j < i \text{ or } j > i+1, \tag{32}$$

$$f_{i+1} e_i = f_{i+1}, \tag{33}$$

$$e_i f_{i+1} = e_i, \tag{34}$$

$$e_i f_i = f_i f_{i+1}. \tag{35}$$

Similarly to the cases we treated above, one shows that every element of M_n can be expressed as

$$\pi = f_{p_1} \dots f_{p_r} (e_{j_1} e_{j_1 - 1} \dots e_{i_1})(e_{j_2} e_{j_2 - 1} \dots e_{i_2}) \dots (e_{j_k} e_{j_k - 1} \dots e_{i_k}), \tag{36}$$

where $1 \leqslant i_1 < i_2 < \dots < i_k$; $j_1 < j_2 < \dots < j_k$; $i_s \leqslant j_s$ for all s; $k \geqslant 0$; and $\{p_1, \dots, p_r\} \cap \{j_1 + 1, \dots, j_k + 1\} = \emptyset$.

Then distinct words of the form (36) evaluated in PC_n are pairwise distinct and so $M_n \cong PC_n$. Thus, the monoid presented by (26)–(35) is isomorphic to PC_n.

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