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TETRAVALENT HALF-ARC-TRANSITIVE GRAPHS OF ORDER p^2q^2

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Abstract. We classify tetravalent G-half-arc-transitive graphs Γ of order p^2q^2, where $G \leqslant \mathsf{Aut}\,\Gamma$ and p, q are distinct odd primes. This result involves a subclass of tetravalent half-arc-transitive graphs of cube-free order.

Keywords: half-arc-transitive graph; normal Cayley graph; cube-free order

MSC 2010: 20B15, 05C25

1. Introduction

Throughout the paper, graphs considered are simple, connected and undirected. For a graph Γ, we denote by $V\Gamma$, $E\Gamma$, $A\Gamma$, $\mathsf{Aut}\,\Gamma$ and $\mathsf{val}(\Gamma)$ the vertex set, edge set, arc set, full automorphism group and the valency of Γ, respectively. A graph Γ is *G-vertex-transitive*, *G-edge-transitive* or *G-arc-transitive* if $G \leqslant \mathsf{Aut}\,\Gamma$ is transitive on $V\Gamma$, $E\Gamma$ or $A\Gamma$, respectively, and Γ is *G-half-arc-transitive* if $G \leqslant \mathsf{Aut}\,\Gamma$ acts transitively on $V\Gamma$ and $E\Gamma$, but not on $A\Gamma$; in particular, when $G = \mathsf{Aut}\,\Gamma$ then Γ is said to be *vertex-transitive*, *edge-transitive*, *arc-transitive* or *half-arc-transitive*, respectively. A graph Γ is a Cayley graph if there exists a group G and a subset $S \subset G$ with $1 \notin S = S^{-1} := \{g^{-1}\colon g \in S\}$ such that the vertices of Γ may be identified with the elements of G in such a way that x is adjacent to y if and only if $yx^{-1} \in S$. The Cayley graph Γ is denoted by $\mathsf{Cay}(G,S)$. A Cayley graph $\Gamma = \mathsf{Cay}(G,S)$ is connected if and only if $G = \langle S\rangle$, that is, S generates G. Let $A = \mathsf{Aut}\,\Gamma$ and $\mathsf{Aut}(G,S) = \{\alpha \in \mathsf{Aut}(G)\colon S^\alpha = S\}$. For each $g \in G$, let $R(g)$ denote the permutation on G defined by $x \mapsto xg$. Then A contains the right regular

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representation $R(G) := \{R(g)\colon\ g \in G\}$ of G, which is regular on $V\Gamma$, and the group $\mathsf{Aut}(G, S)$ is a subgroup of the stabilizer of 1 in A. A Cayley graph $\mathsf{Cay}(G, S)$ is said to be *X-normal* if $X \leqslant A$ contains $R(G)$ as a normal subgroup; in particular, when $G = \mathsf{Aut}\,\Gamma$ then Γ is said to be *normal.*

Let G be a group, N a normal subgroup and H a subgroup of G. Then we use $\mathsf{Aut}(G)$, $\mathsf{Out}(G)$, $Z(G)$, G/N, $\mathrm{C}_G(H)$ and $\mathrm{N}_G(H)$ to denote the automorphism group, outer automorphism group, the center, quotient group of G, the centralizer and the normalizer of H in G, respectively. Let M and N be two groups. Then we use $M : N$, $M \times N$ and $M \cdot N$ to denote a semidirect product, direct product and an extension of M by N. For a positive integer n, we denote by $\mathbb{Z}_n$, D_{2n}, A_n and S_n the cyclic group of order n, the dihedral group of order $2n$, the alternating group and the symmetric group of degree n, respectively.

The investigation of half-arc-transitive graphs was initiated by Tutte, see [25], and he proved that a vertex- and edge-transitive graph with odd valency must be arc-transitive. In 1970, Bouwer constructed the first family of half-arc-transitive graphs in [2]. From then on, half-arc-transitive graphs have been extensively studied over decades and more such graphs were constructed, see for example [1], [7], [8], [9], [12], [13] [16], [19], [24] [26], [27], [28], [29], [30], [32]. In particular, it is proved that for a prime p there is no tetravalent half-arc-transitive graph of order p, p^2, $2p$ and $2p^2$, see [4], [5], [28]. The half-arc-transitive graphs of order $3p$ and $4p$ are classified in [1], [16], respectively. The tetravalent half-arc-transitive graphs of order p^3, p^4 and $2pq$ are classified in [8], [9], [32], respectively. Recently, Pan et al. in [21] classified tetravalent edge-transitive graphs of order p^2q. Wang et al. in [30] studied tetravalent half-arc-transitive graphs of order a product of three primes.

In this paper, we will study tetravalent half-arc-transitive graphs of order p^2q^2 with p, q distinct odd primes. The main result of the paper is the following theorem:

Theorem 1.1. *Let Γ be a tetravalent G-half-arc-transitive graph of order p^2q^2, where $G \leqslant \mathsf{Aut}\,\Gamma$ and p, q are distinct odd primes. Then one of the following statements holds:*

(1) *G is soluble, $\Gamma = \mathsf{Cay}(H, S)$ is a G-normal Cayley graph, $G_1 \leqslant \mathbb{Z}_2^2$ and $S = \{a, a^\tau, a^{-1}, (a^{-1})^\tau\}$, where $a \in H$, and $\tau \in \mathsf{Aut}(H)$ is an involution.*

(2) *G is insoluble, and one of the following holds:*

  (i) *$|V\Gamma| = 225$ or 441, $G \cong F \times \mathrm{A}_5 = \mathbb{Z}_{pq} \times \mathrm{A}_5$ or $\mathbb{Z}_{pq} \times \mathrm{PSL}(2, 7)$, and $|G_\alpha| = 4$ or 8;*

  (ii) *$|V\Gamma| = 225$ or 441, and $\mathrm{soc}(G) \cong \mathrm{A}_5^2$ or $\mathrm{PSL}(2, 7)^2$, where $\mathrm{soc}(G)$ is the socle of G.*

2. Preliminary results

In this section, we will give some necessary preliminary results. The next lemma deals with a basic group-theoretic result.

Lemma 2.1 ([14], Theorem 4.5). *Let H be a subgroup of a group G. Then $\mathrm{C}_G(H)$ is a normal subgroup of $\mathrm{N}_G(H)$, and the quotient $\mathrm{N}_G(H)/\mathrm{C}_G(H)$ is isomorphic to a subgroup of $\mathsf{Aut}(H)$.*

For a graph Γ and a positive integer s, an *s-arc* of Γ is a sequence $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ of vertices such that α_{i-1}, α_i are adjacent for $1 \leqslant i \leqslant s$ and $\alpha_{i-1} \neq \alpha_{i+1}$ for $1 \leqslant i \leqslant s-1$. A graph Γ is said to be (G, s)-arc-transitive, where $G \leqslant \mathsf{Aut}\,\Gamma$, if G is transitive on the set of s-arcs of Γ. If Γ is (G, s)-arc-transitive but not $(G, s+1)$-arc-transitive, then Γ is called a (G, s)-transitive graph. In particular, when $(G, s) = (\mathsf{Aut}\,\Gamma, s)$ then Γ is simply called an s-transitive graph. The following result characterizes the vertex stabilizers of tetravalent edge-transitive graphs of odd order.

Lemma 2.2. *Let Γ be a tetravalent G-edge-transitive graph of odd order, where $G \leqslant \mathsf{Aut}\,\Gamma$. Let $\alpha \in V\Gamma$ and $\{\alpha, \beta\} \in E\Gamma$. Then either*

(1) *G_α is a 2-group, and Γ is G-half-arc-transitive; or*

(2) *Γ is (G, s)-transitive with $1 \leqslant s \leqslant 3$. Furthermore, the pair (s, G_α) satisfies the following Table 1:*

s	G_α
1	2-*group*
2	$\mathrm{A}_4 \leqslant G_\alpha \leqslant \mathrm{S}_4$
3	$\mathrm{A}_4 \times \mathbb{Z}_3 \leqslant G_\alpha \leqslant \mathrm{S}_4 \times \mathrm{S}_3$

Table 1.

P r o o f. Assume that Γ is G-arc-transitive. Then the part (2) can be easily derived from [18], Lemma 2.5. Assume that Γ is not G-arc-transitive. Note that $|V\Gamma|$ is odd, so Γ is G-vertex-transitive. It follows that Γ is G-half-arc-transitive. By [17], Lemma 2.1, $G_\alpha^{\Gamma(\alpha)} \leqslant \mathrm{S}_4$ is a $\{2,3\}$-group. If $3 \mid |G_\alpha^{\Gamma(\alpha)}|$, then $G_\alpha^{\Gamma(\alpha)} = \mathrm{A}_4$ or S_4. It follows that G_α is transitive on $\Gamma(\alpha)$, and so Γ is G-arc-transitive, a contradiction. Thus G_α is a 2-group. This completes the proof of this lemma. □

By [3], page 337, Table 8.1, we give the soluble maximal subgroups of $\mathrm{GL}(2, p)$ in the following lemma.

Lemma 2.3. *Let M be a soluble maximal subgroup of $GL(2, p)$. Then M is isomorphic to one of the following groups:*

(1) *$\mathbb{Z}_{p-1} \times (\mathbb{Z}_p : \mathbb{Z}_{p-1})$;*

(2) $\mathbb{Z}_{p^2-1} : \mathbb{Z}_2$;
(3) $\mathbb{Z}_{p-1} \wr \mathbb{Z}_2$;
(4) $2 \cdot \mathrm{S}_4$.

By [21], we have the following lemma regarding the tetravalent edge-transitive graph with odd but not a prime power order.

Lemma 2.4 ([21], Lemma 4.3). *Let Γ be a tetravalent G-edge-transitive graph with odd but not a prime power order, where $G \leqslant \mathsf{Aut}\,\Gamma$. Suppose that N is a nilpotent normal subgroup of G. Then N is semiregular on $V\Gamma$.*

For a group G, the largest nilpotent normal subgroup of G is called the Fitting subgroup of G.

Lemma 2.5 ([23], page 30, Corollary). *Let F be the Fitting subgroup of a group G. If G is soluble, then $F \neq 1$ and the centralizer $\mathrm{C}_G(F) \leqslant F$.*

The next two lemmas give a characterization and classification for the tetravalent edge-transitive graphs of order p^2q with p, q distinct odd primes.

Lemma 2.6 ([30], Lemma 3.3). *Let p, q be distinct odd primes and Γ a tetravalent half-arc-transitive graph of order p^2q. Then Γ is a normal Cayley graph.*

Lemma 2.7 ([21], Theorem 5.3). *Let Γ be a tetravalent G-edge-transitive graph of order p^2q, where $G \leqslant \mathsf{Aut}\,\Gamma$ and p, q are distinct odd primes. Then one of the following statements holds:*
(1) *Γ is of order 45, 63, 75 or 147, given in [31]. In particular, there are exactly 17 pairwise nonisomorphic graphs in this case;*
(2) *$\Gamma \cong \mathcal{G}_{153}$ is a tetravalent arc-transitive graph of order 153 with $\mathsf{Aut}\,\Gamma \cong \mathrm{PSL}(2,17)$;*
(3) *$\Gamma = \mathsf{Cay}(H,S)$ is a G-normal edge-transitive Cayley graph, and either*
    (i) *Γ is $(G,1)$-transitive, and $S = \{a, a^\sigma, a^{\sigma^2}, a^{\sigma^3}\}$, where $\sigma \in \mathsf{Aut}(H)$ is of order 4; or*
    (ii) *$G_1 \leqslant \mathbb{Z}_2^2$ and $S = \{a, a^\tau, a^{-1}, (a^{-1})^\tau\}$, where $\tau \in \mathsf{Aut}(H)$ is an involution.*

Remark on Lemma 2.7. For the cases (1) and (2), G is insoluble; and for the case (3), G is soluble.

For a tetravalent G-edge-transitive graph Γ of odd order, where $G \leqslant \mathsf{Aut}\,\Gamma$ is a insoluble group, we have the following lemma.

Lemma 2.8 ([21], Corollary 2.4). *Let Γ be a tetravalent G-edge-transitive graph of odd order, where $G \leqslant \mathsf{Aut}\,\Gamma$. If G is insoluble, then Γ is not a G-normal edge-transitive Cayley graph.*

Let G be a finite group and let $\pi(G) = \{p\colon p \text{ is a prime divisor of } |G|\}$. Herzog in [11] and Huppert et al. in [15] classified nonabelian finite simple groups G for $|\pi(G)| = 3$, from which we may deduce the following lemma.

Lemma 2.9. *Let G be a nonabelian simple group, if $|\pi(G)| = 3$. Then $(G, |G|, \mathsf{Out}(G))$ lies in Table 2:*

G	$\|G\|$	$\mathsf{Out}(G)$	G	$\|G\|$	$\mathsf{Out}(G)$
A_5	$2^2{\cdot}3{\cdot}5$	$\mathbb{Z}_2$	A_6	$2^3{\cdot}3^2{\cdot}5$	$\mathbb{Z}_2^2$
$\mathrm{PSp}(4,3)$	$2^6{\cdot}3^4{\cdot}5$	$\mathbb{Z}_2$	$\mathrm{PSL}(2,7)$	$2^3{\cdot}3{\cdot}7$	$\mathbb{Z}_2$
$\mathrm{PSL}(2,8)$	$2^3{\cdot}3^2{\cdot}7$	$\mathbb{Z}_3$	$\mathrm{PSL}(2,17)$	$2^4{\cdot}3^2{\cdot}17$	$\mathbb{Z}_2$
$\mathrm{PSL}(3,3)$	$2^4{\cdot}3^3{\cdot}13$	$\mathbb{Z}_2$	$\mathrm{PSU}(3,3)$	$2^5{\cdot}3^3{\cdot}7$	$\mathbb{Z}_2$

Table 2. Nonabelian simple $\{2, q, p\}$-groups

Regarding the Cayley graph $\Gamma = \mathsf{Cay}(G, S)$, we have the following basic result.

Lemma 2.10 ([10], Lemma 2.1)**.** *Let $\Gamma = \mathsf{Cay}(G, S)$ be a Cayley graph. Then the normalizer $\mathrm{N}_{\mathsf{Aut}\,\Gamma}(G) = G : \mathsf{Aut}(G, S)$.*

Lemma 2.11 ([21], Lemma 2.10)**.** *Let $G \leqslant \mathrm{Sym}(\Omega)$ be a transitive permutation group on Ω, and let p^m be a divisor of $|\alpha^G|$, where $\alpha \in \Omega$ and p is a prime. If G has a subgroup H such that $(p, |G : H|) = 1$, then p^m divides $|\alpha^H|$. In particular, if $(|\Omega|, |G : H|) = 1$, then H is transitive on Ω.*

Let Γ be a vertex-transitive graph, and let N be a subgroup of $\mathsf{Aut}\,\Gamma$. Denote by Γ_N the quotient graph corresponding to the orbits of N, that is, the graph having the orbits of N as vertices with two orbits adjacent in Γ_N if there is an edge in Γ between those orbits. Let $\mathcal{B}$ be the set of N-orbits on $V\Gamma$. If for any adjacent orbits B, C of N, the induced subgraph $[B, C]$ of Γ is regular, then Γ is called a multi-cover of Γ_N. If in addition $[B, C]$ is of valency 1, then Γ is called a normal cover of Γ_N.

Lemma 2.12. *Let Γ be a connected G-half-arc-transitive graph, where $G \leqslant \mathsf{Aut}\,\Gamma$. Let $N \trianglelefteq G$ and let N have more than two orbits on $V\Gamma$. Then Γ is a multi-cover of Γ_N, and $G/K \leqslant \mathsf{Aut}\,\Gamma_N$, where K is the kernel of the action of the set of N-orbits on $V\Gamma$. If $|\Gamma(\alpha) \cap B| = 0$ or 1 for any N-orbit B and $\alpha \in V\Gamma \setminus B$, then the following statements hold:*
(1) *$G/N \leqslant \mathsf{Aut}\,\Gamma_N$;*
(2) *Γ is a normal cover of Γ_N;*
(3) *Γ_N is a G/N-half-arc-transitive graph.*

P r o o f. Let $\mathcal{B}$ be the set of N-orbits on $V\Gamma$ and let K be the kernel of the action of G on $\mathcal{B}$. Obviuosly, $N \leqslant K$. Since $N \trianglelefteq G$, it is easy to show that the induced subgraph $[B, C]$ of Γ is regular for any adjacent orbits B, C. Hence Γ is a multi-cover or a normal cover of Γ_N and $G/K \leqslant \mathsf{Aut}\,\Gamma_N$

Suppose that $|\Gamma(\alpha) \cap B'| = 1$, where B' is an N-orbit on $V\Gamma$. Since N is transitive on B and B', it follows that the subgraph $[B, B']$ is a perfect matching and so Γ and Γ_N have the same valency. It then follows that Γ is a normal cover of Γ_N. For $\alpha \in V\Gamma$, the stabilizer K_α fixes each member of $\mathcal{B}$ setwise, and since distinct vertices of $\Gamma(\alpha)$ lie in distinct N-orbits, we have that K_α acts trivially on $\Gamma(\alpha)$. Since Γ is connected it follows that K_α fixes all the vertices of Γ, and hence $K_\alpha = 1$. Since this is true for all α, K acts semiregularly on $V\Gamma$, and hence so does N. Furthermore, as $N \leqslant K$ and acts transitively on the orbits of K, we see that $K = N$. Thus $G^{V\Gamma_N} \cong G/N$ and so $G/N \leqslant \mathsf{Aut}\,\Gamma_N$.

For any $(\alpha, \beta), (\gamma, \delta) \in A\Gamma$, we have $(\alpha^N, \beta^N), (\gamma^N, \delta^N) \in A\Gamma_N$, where $\alpha, \beta, \gamma, \delta \in V\Gamma$. If Γ_N is G/N-arc-transitive, then we have $g \in G$ such that $(\alpha^N)^g = \alpha^{gN} = \gamma^N$ and $(\beta^N)^g = \beta^{gN} = \delta^N$. It then follows that $(\alpha, \beta)^g = (\gamma^{n_1}, \delta^{n_2})$ for some $n_1, n_2 \in N$. And for $(\gamma^{n_1}, \delta^{n_2}), (\gamma, \delta) \in A\Gamma$, we have $n \in N$ such that $(\gamma^{n_1}, \delta^{n_2})^n = (\gamma, \delta)$. Hence $(\alpha, \beta)^{gn} = (\gamma, \delta)$. Thus Γ is G-arc-transitive, a contradiction. So Γ_N is G/N-half-arc-transitive. $\square$

For the tetravalent normal half-arc-transitive Cayley graphs, the following proposition gives a general construction.

Proposition 2.13. *Let $\Gamma = \mathsf{Cay}(H, S)$ be a tetravalent G-half-arc-transitive Cayley graph of order p^2q^2, where p, q are distinct odd primes. Let 1 denote the vertex of Γ corresponding to the identity element of H. Assume that $H \lhd G$. Then $G_1 \leqslant \mathbb{Z}_2^2$ and $S = \{a, a^\tau, a^{-1}, (a^{-1})^\tau\}$, where $\tau \in \mathsf{Aut}(H)$ is an involution.*

P r o o f. By Lemma 2.10, $G_1 \leqslant \mathsf{Aut}(H, S)$. Since Γ is connected, $H = \langle S\rangle$ and then G_1 acts faithfully on $\Gamma(1) = S$, which implies $G_1 \leqslant \mathrm{S}_4$. Since G_1 is a 2-group, $G_1 \leqslant \mathrm{D}_8$. Let $a \in S$. If $G_1 \geqslant \langle\sigma\rangle \cong \mathbb{Z}_4$, then $\langle\sigma\rangle$ is regular on S. Hence Γ is G-arc-transitive, a contradiction. Thus $G_1 \leqslant \mathbb{Z}_2^2$. Since Γ is a G-normal half-arc-transitive Cayley graph, $S = T^{-1} \cup T$ by [22], Proposition 1, where T is an orbit of the action of G_1 on S. So there exists an involution $\tau \in G_1$ such that $a^\tau \neq a$ or a^{-1}; it follows that $S = \{a, a^\tau, a^{-1}, (a^{-1})^\tau\}$. $\square$

By Proposition 2.13, more specific constructions of the graph $\Gamma = \mathsf{Cay}(H, S)$ depend on the automorphism group of the group H.

3. Proof of Theorem 1.1

Let Γ be a tetravalent G-half-arc-transitive graph of order p^2q^2, where $G \leqslant \mathsf{Aut}\,\Gamma$ and p, q are distinct odd primes. Let $\alpha \in V\Gamma$. By Lemma 2.2, G_α is a 2-group, and hence G is a $\{2, p, q\}$-group. Obviously, G has no nontrivial normal 2-subgroup.

Now we first consider the case when G is soluble.

Lemma 3.1. *If G is soluble, then Γ is a G-normal Cayley graph.*

P r o o f. Since G_α is a 2-group, $|G| = 2^i p^2 q^2$ for some positive integer i. Let F be the Fitting subgroup of G. By Lemma 2.5, $F \neq 1$, $\mathrm{C}_G(F) \leqslant F$. In particular, $F = \vee_p(G) \times \vee_q(G)$, where $\vee_p(G)$ and $\vee_q(G)$ are the largest normal p-subgroup and q-subgroup of G, respectively. Therefore, F is abelian and $\mathrm{C}_G(F) = F$. Now F is semiregular on $V\Gamma$ and hence $|F| \mid p^2q^2$.

Assume $F \cong \mathbb{Z}_p$. Then by Lemma 2.1 $G/F \leqslant \mathsf{Aut}(F) \cong \mathbb{Z}_{p-1}$, it follows that $p^2 \nmid |G|$, which is not possible. Similarly, we can exclude the cases $F \cong \mathbb{Z}_q$ and $\mathbb{Z}_{pq}$.

Assume $|F| = p^2$. Then we consider the quotient graph Γ_F, induced by F. Let K be the kernel of G acting on $V\Gamma_F$. By Lemma 2.12, $G/K \leqslant \mathsf{Aut}\,\Gamma_F$ and $K = F : K_\alpha$. Suppose that $\mathsf{val}(\Gamma_F) = 4$. Again by Lemma 2.12, we obtain that $K = F$ and Γ is a normal cover of Γ_F. So Γ_F is a G/F-half-arc-transitive graph of order q^2. If $F = \mathbb{Z}_{p^2}$, then $G/F \leqslant \mathsf{Aut}(F)$ is abelian. Thus G/F is regular on $V\Gamma_F$, which is not possible. So $F \cong \mathbb{Z}_p^2$, and $G/F \leqslant \mathsf{Aut}(F) \cong \mathrm{GL}(2, p)$. Note that G/F is soluble, G/F is one of subgroups listed in Lemma 2.3. We consider the candidates one by one.

(1) Suppose that $G/F \leqslant \mathbb{Z}_{p-1} \times (\mathbb{Z}_p : \mathbb{Z}_{p-1})$. Since $p \nmid |G/F|$, hence $G/F = \mathbb{Z}_l \times \mathbb{Z}_m$ for some $l, m \mid p-1$, which is not possible.

(2) Suppose that $G/F \leqslant \mathbb{Z}_{p^2-1} : \mathbb{Z}_2$. Then $G/F = \mathbb{Z}_k : \mathbb{Z}_2$ for some $k \mid p-1$ and $q^2 \mid k$. Let Q be a Sylow q-subgroup of G/F. Then $|Q| = q^2$ and $Q \lhd G/F$. Therefore, G has a normal subgroup isomorphic to $F \cdot Q$ which is regular on $V\Gamma$. That is to say Γ is a G-normal Cayley graph in this case.

(3) Suppose that $G/F \leqslant \mathbb{Z}_p \wr \mathbb{Z}_2$. Then $G/F = (\mathbb{Z}_t \times \mathbb{Z}_t) : \mathbb{Z}_2$ for some $t \mid p-1$ and $q \mid t$. Similarly, the Sylow q-subgroup Q of G/F is normal, and G has a normal subgroup isomorphic to $F \cdot Q$ which is regular on $V\Gamma$. Therefore Γ is also a G-normal Cayley graph.

(4) Suppose that $G/F \leqslant 2 \cdot \mathrm{S}_4$. Obviously, this is not possible since $q^2 \nmid |G/F|$.

Now we consider the case $\mathsf{val}(\Gamma_F) = 2$. Then $\Gamma_F := \{B_1, B_2, \ldots, B_{q^2}\}$ is a cycle of length q^2, where B_i is adjacent to B_{i+1} in Γ_F for $1 \leqslant i \leqslant q^2 - 1$, so the induced subgraph $[B_i, B_{i+1}]$ is a cycle of length $2p^2$. This implies that $K_\alpha \leqslant \mathbb{Z}_2$, $K = F$ or $F : \mathbb{Z}_2$, and $G \leqslant K \cdot \mathsf{Aut}\,\Gamma_F = K \cdot \mathrm{D}_{2q^2}$. It follows that G has a normal Hall $\{p, q\}$-subgroup which is regular on Γ, hence Γ is a G-normal Cayley graph. Similarly, Γ is also a G-normal Cayley graph when $|F| = q^2$.

Assume $|F| = p^2q$. Then $G/K \leqslant \mathsf{Aut}\,\Gamma_F$, where K is the kernel of G acting on $V\Gamma_F$. If $\mathsf{val}(\Gamma_F) = 4$, then $K = F$ and Γ_F is G/F half-arc-transitive of order q. Note that G/F is soluble. It follows that $G/F \leqslant \mathbb{Z}_q : \mathbb{Z}_{q-1}$ from [6], Corollary 3.5B. Thus G has a normal subgroup isomorphic to $F \cdot \mathbb{Z}_q$ which is regular on $V\Gamma$. So Γ is a G-normal half-arc-transitive Cayley graph. For $\mathsf{val}(\Gamma_F) = 2$, $K_\alpha \leqslant \mathbb{Z}_2$, $K = F$ or $F : \mathbb{Z}_2$, and $G \leqslant K \cdot \mathsf{Aut}\,\Gamma_F = K \cdot \mathrm{D}_{2q}$. It follows that G has a normal Hall-$\{p, q\}$-subgroup which is regular on Γ, hence Γ is a G-normal Cayley graph. Similarly, Γ is also a G-normal Cayley graph when $|F| = pq^2$.

Finally, assume $|F| = p^2q^2$. Then F is regular on $V\Gamma$, and so Γ is a G-normal Cayley graph on F. □

Next we consider the case when G is insoluble.

Lemma 3.2. *Let M be the radical of G, and let F be the Fitting subgroup of M. If G is insoluble, then one of the following statements holds:*

(1) $M \neq 1$, $F \cong \mathbb{Z}_{pq}$, $|V\Gamma| = 225$ *or* 441, $G \cong F \times \mathrm{A}_5 = \mathbb{Z}_{pq} \times \mathrm{A}_5$ *or* $\mathbb{Z}_{pq} \times \mathrm{PSL}(2,7)$, *and* $|G_\alpha| = 4$ *or* 8;

(2) $M = 1$ *and* $\mathrm{soc}(G) \cong \mathrm{A}_5, \mathrm{A}_6, \mathrm{PSL}(2,7), \mathrm{PSL}(2,8), \mathrm{PSL}(2,17), \mathrm{A}_5^2$ *or* $\mathrm{PSL}(2,7)^2$.

P r o o f. Let N be the socle of G, that is, the product of all minimal normal subgroups of G. Let M be the radical of G, that is, the largest normal soluble subgroup of G. And let $|G| = 2^ip^2q^2$ for some integer i.

Case 1. Assume $M \neq 1$. Let F be the Fitting subgroup of M. Then $F \trianglelefteq G$ and $F \neq 1$ by Lemma 2.5. We consider Γ_F. Let K be the kernel of G acting on $V\Gamma_F$. Then $K = FK_\alpha$, and hence K is soluble as K_α is soluble by Lemma 2.2. If $\mathsf{val}(\Gamma_F) = 2$, then Γ_F is a cycle and $G/K \leqslant \mathsf{Aut}\,\Gamma_F = \mathrm{D}_{2m}$, where $m = |V\Gamma_F|$. So G is soluble, which is a contradiction. Thus, $\mathsf{val}(\Gamma_F) = 4$. Then $K = F$ and $G/F \leqslant \mathsf{Aut}\,\Gamma_F$. Further, by Lemma 2.4, F is semiregular on $V\Gamma$ and hence $|F|$ divides p^2q^2. Suppose $|F| = p^2q^2$, then Γ is a G-normal half-arc-transitive Cayley graph of F, which is not possible by Lemma 2.8.

Suppose $|F| = p^2$. Then Γ_F is a tetravalent G/F-half-arc-transitive graph of order q^2. If $q \geqslant 5$, then we obtain a contradiction by [21], Lemma 4.2. If $q = 3$ then Γ_F is an edge-transitive graph of order 9. By [20], $\Gamma_F = DW(3,3)$ is a deleted wreath graph, and $\mathsf{Aut}\,\Gamma_F \cong \mathbb{Z}_3^2 \cdot \mathrm{D}_8$. It follows that G is soluble, a contradiction. Similarly, we can exclude the case $|F| = q^2$.

Suppose $|F| = pq^2$. Then Γ_F is a tetravalent G/F-half-arc-transitive graph of order p. Since $|V\Gamma_F| = p$, G/F is almost simple and 2-transitive on $V\Gamma_F$ by [6], page 99. It follows that $\Gamma_F = \mathsf{K}_p$. Since $\mathsf{val}(\Gamma_F) = 4$, $p = 5$. As $G/F \leqslant \mathsf{Aut}\,\mathsf{K}_5 = \mathrm{S}_5$ is insoluble, we have $G = F \cdot \mathrm{A}_5$ or $F \cdot \mathrm{S}_5$, and so $3 \bigm| |G_\alpha|$, which is a contradiction by Lemma 2.2. Similarly, we can exclude the case $|F| = p^2q$.

Suppose $|F| = pq$. Then Γ_F is a tetravalent G/F-half-arc-transitive graph of order pq. But by [1], [26], there is no tetravalent edge-transitive graph of order pq which is half-arc-transitive, so Γ_F is arc-transitive. It follows that $(pq, \Gamma_F, \mathsf{Aut}\,\Gamma_F, (\mathsf{Aut}\,\Gamma_F)_{\overline{\alpha}})$ satisfies Table 1 in [21], Lemma 2.6, where $\overline{\alpha} \in V\Gamma_F$. We first consider rows 1–2 of Table 1. If $pq = 15$ or 21, then $|V\Gamma| = 225$ or 441, $G \cong F \times \mathrm{A}_5 = \mathbb{Z}_{pq} \times \mathrm{A}_5$ or $\mathbb{Z}_{pq} \times \mathrm{PSL}(2,7)$, and $|G_\alpha| = 4$ or 8, respectively. For rows 3–5 of Table 1. If $pq = 35$ as in row 3, then $G/F < \mathsf{Aut}\,\Gamma_F = \mathrm{S}_7$. Note that G/F is insoluble, and since G/F is edge-transitive on $V\Gamma_F$, $70 \bigm| |G/F|$, we conclude that $G/F \geqslant \mathrm{A}_7$. It follows that $|G| \geqslant |F||\mathrm{A}_7|$, and so $3 \bigm| |G_\alpha|$, which is a contradiction by Lemma 2.2. Similarly, we can also exclude the cases where $pq = 55$ or 253, as in rows 4 or 5, respectively.

Finally, suppose $|F| = p$. Then Γ_F is a tetravalent G/F-half-arc-transitive Cayley graph of order pq^2. It follows that $\mathsf{Aut}\,\Gamma_F$ is half arc-transitive or arc-transitive on Γ_F. For convenience, we say $\Gamma_F = \mathsf{Cay}(R, S)$, where $|R| = pq^2$. If $\mathsf{Aut}\,\Gamma_F$ is half arc-transitive on Γ_F; then $R \lhd \mathsf{Aut}\,\Gamma_F$ by Lemma 2.6. That is, $\Gamma_F = \mathsf{Cay}(R, S)$ is a normal edge transitive Cayley graph. Noting that G is insoluble, Γ_F is not normal edge transitive by Lemma 2.8. A contradiction occurs. If $\mathsf{Aut}\,\Gamma_F$ is arc-transitive on Γ_F, by checking the tetravalent edge-transitive graphs of order pq^2 in Lemma 2.7, then $\Gamma_F = \mathcal{G}_{153}$ and $\mathsf{Aut}\,\Gamma_F = \mathrm{PSL}(2,17)$. It follows that $G = F \cdot \mathrm{PSL}(2,17) = F \times \mathrm{PSL}(2,17)$. But there exists no tetravalent half arc-transitive graph of order $3^2 \cdot 17^2$ admitting G as a graph automorphism group by simple computing.

Case 2. Assume $M = 1$. Then each nontrivial normal subgroup of G is insoluble. Let $\mathrm{soc}(G) = M_1 \times \ldots \times M_s$, where M_i $(1 \leqslant i \leqslant s)$ are all minimal normal subgroups of G. Suppose that $M_k = T_k^{d_k}$, where T_k is a nonabelian simple group and $1 \leqslant k \leqslant s$. Since G_α is a 2-group, N is a $\{2, p, q\}$-group. By Lemma 2.9, $\mathrm{soc}(G) \cong \mathrm{A}_5$, A_6, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,17)$, A_5^2 or $\mathrm{PSL}(2,7)^2$. □

P r o o f of Theorem 1.1. Let Γ be G-half-arc-transitive. If G is soluble, then, by Lemma 3.1, Γ is a G-normal half-arc-transitive Cayley graph. Combining Proposition 2.13, we complete the proof of part (1) in Theorem 1.1.

Suppose that G is insoluble. Let $\mathrm{soc}(G) \cong \mathrm{A}_5$, A_6, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,17)$, A_5^2 or $\mathrm{PSL}(2,7)^2$. Let $\alpha \in V\Gamma$. Then $|G| = |G_\alpha| \cdot p^2q^2$. If $N := \mathrm{soc}(G) \cong \mathrm{A}_5$, then $G = \mathrm{A}_5$ or S_5. Since $|\mathrm{A}_5| = 2^2 \cdot 3 \cdot 5$ and $|\mathrm{S}_5| = 2^3 \cdot 3 \cdot 5$, $p^2q^2 \nmid |G|$. Similarly, we can exclude the cases $N \cong \mathrm{A}_6$, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$ and $\mathrm{PSL}(2,17)$.

If $N \cong \mathrm{A}_5^2$, then $|N| = 2^4 \cdot 3^2 \cdot 5^2$. Since $|N| \bigm| |G_\alpha| \cdot p^2q^2$ and G_α is a 2-group, $(p^2q^2, |G : N|) = 1$. By Lemma 2.11, N is transitive on $V\Gamma$. So $|N : N_\alpha| = 3^2 \cdot 5^2$, that is, $|V\Gamma| = 225$. Similarly, we can obtain that $|V\Gamma| = 441$ when $N \cong \mathrm{PSL}(2,7)^2$. Apply Lemma 3.2 (1), we complete the proof of part (2) in Theorem 1.1. □

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