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THE STRUCTURES OF HOPF $*$-ALGEBRA ON RADFORD ALGEBRAS

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Abstract. We investigate the structures of Hopf $*$-algebra on the Radford algebras over $\mathbb{C}$. All the $*$-structures on H are explicitly given. Moreover, these Hopf $*$-algebra structures are classified up to equivalence.

Keywords: antilinear map; $*$-structure; Hopf $*$-algebra

MSC 2010: 16G99, 16T05

1. Introduction

Woronowicz studied compact matrix pseudogroup in [14], which is a generalization of compact matrix group. Using the language of C^*-algebra, Woronowicz described compact matrix pseudogroups as C^*-algebras endowed with some comultiplications. This induces the concept of Hopf $*$-algebras. In [14], [15], [16], Woronowicz exhibited Hopf $*$-algebra structures on quantum groups in the framework of C^*-algebras. It was shown that $GL_q(2)$, $SL_q(2)$ and $U_q(sl(2))$ are Hopf $*$-algebras, see [2], [5]. Van Deale [13] studied the Harr measure on a compact quantum group. Podleś [8] studied coquasitriangular Hopf $*$-algebras. Tucker-Simmons [12] studied the $*$-structure of module algebras over a Hopf $*$-algebra. Recently, we investigated the Hopf $*$-algebra structures on $H(1,q)$ over $\mathbb{C}$ and classified these $*$-structures up to equivalence [6].

Radford [9] constructed for every integer $n > 1$ a finite dimensional unimodular Hopf algebra with antipode of order $2n$ and proved that for every even integer there is a finite dimensional Hopf algebra H. For more details, the reader is directed to [3], [9], [10].

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In this paper, we study the structures of Hopf $*$-algebra on the Radford algebra H over the complex number field $\mathbb{C}$. This paper is organized as follow. In Section 2, we recall some basic notions about the Hopf $*$-algebra and the Radford algebra H. In Section 3, we first describe all structures of Hopf $*$-algebra on Radford algebra. It is shown that when $n > 2$, a Hopf $*$-algebra structure on H is uniquely determined by a pair (α, β) of elements in $\mathbb{C}$ with $|\alpha| = |\beta| = 1$, and that when $n = 2$, a Hopf $*$-algebra structure on H is uniquely determined by a 2×2-matrix A over $\mathbb{C}$ with $\bar{A}A = I_2$. Then we classify the Hopf $*$-algebra structures up to equivalence. It is shown that any two $*$-structures on H are equivalent when $n > 2$. When $n = 2$, the two $*$-structures determined by two matrices A and B, respectively, are equivalent if and only if there exists an invertible 2×2-matrix Λ over $\mathbb{C}$ such that $A\bar{\Lambda} = \Lambda B$.

2. Preliminaries

Throughout, let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote all integers, all nonnegative integers, the field of real numbers, and the field of complex numbers, respectively. Let $i \in \mathbb{C}$ be the imaginary unit. For any $\lambda \in \mathbb{C}$ let $\bar{\lambda}$ denote the conjugate complex number of λ, and let $|\lambda|$ denote the norm of λ. For a Hopf algebra H we use $\triangle$, ε and S, respectively, to denote the comultiplication, the counit, and the antipode of H as usual. For the theory of quantum groups and Hopf algebras we refer to [2], [4], [7], [10], [11]. Let $G(H)$ denote the set of group-like elements in a Hopf algebra H, which is a group.

Let V and W be vector spaces over $\mathbb{C}$. A mapping $\psi\colon V \to W$ is said to be conjugate-linear (or antilinear) if

$$\psi(\lambda_1 v_1 + \lambda_2 v_2) = \overline{\lambda_1}\psi(v_1) + \overline{\lambda_2}\psi(v_2) \quad \forall\, v_1, v_2 \in V,\ \forall\, \lambda_1, \lambda_2 \in \mathbb{C}.$$

Let A and B be $\mathbb{C}$-algebras. A conjugate-linear map $\psi\colon A \to B$ (or A) is said to be a conjugate-linear algebra map (or a conjugate-linear algebra endomorphism) if

$$\psi(aa') = \psi(a)\psi(a'), \quad \psi(1) = 1 \quad \forall\, a, a' \in A,$$

and ψ is said to be a conjugate-linear antialgebra map (or a conjugate-linear antialgebra endomorphism) if

$$\psi(aa') = \psi(a')\psi(a), \quad \psi(1) = 1 \quad \forall\, a, a' \in A.$$

Let C and D be two coalgebras over $\mathbb{C}$. A conjugate-linear map $\psi\colon C \to D$ (or C) is said to be a conjugate-linear coalgebra map (or a conjugate-linear coalgebra endomorphism) if

$$\sum \psi(c)_1 \otimes \psi(c)_2 = \sum \psi(c_1) \otimes \psi(c_2), \quad \varepsilon(\psi(c)) = \overline{\varepsilon(c)} \quad \forall\, c \in C,$$

and ψ is said to be a conjugate-linear anticoalgebra map (or a conjugate-linear anticoalgebra endomorphism) if

$$\sum \psi(c)_1 \otimes \psi(c)_2 = \sum \psi(c_2) \otimes \psi(c_1), \quad \varepsilon(\psi(c)) = \overline{\varepsilon(c)} \quad \forall\, c \in C.$$

Definition 2.1. Let H be a Hopf algebra over $\mathbb{C}$. A $*$-structure on H is a conjugate-linear map $*\colon H \to H$ such that the following conditions are satisfied:

$$(h^*)^* = h, \quad (hl)^* = l^* h^*,$$
$$\sum (h^*)_1 \otimes (h^*)_2 = \sum (h_1)^* \otimes (h_2)^*, \quad S(S(h^*)^*) = h,$$

where $h, l \in H$. If H is equipped with a $*$-structure, then we call H a Hopf $*$-algebra. Two $*$-structures $*'$ and $*''$ on H are said to be equivalent if there exists a Hopf algebra automorphism ψ of H such that $\psi(h^{*'}) = \psi(h)^{*''}$ for all $h \in H$.

Let H be a Hopf $*$-algebra. Then it is not difficult to check that

$$\varepsilon(h^*) = \overline{\varepsilon(h)} \quad \forall\, h \in H.$$

Hence, the map $*$ is an antilinear coalgebra endomorphism of H and $\mathbb{C} = \mathbb{C}1_H$ is a subalgebra of H. In this case, $\lambda^* = \bar{\lambda}$ for any $\lambda \in \mathbb{C} \subseteq H$.

Fix a positive integer $n > 1$ and let $\omega \in \mathbb{C}$ be a root of unity of order n. The Radford algebra H over $\mathbb{C}$ is generated, as a $\mathbb{C}$-algebra, by g, x and y subject to the relations:

$$g^n = 1, \quad x^n = y^n = 0, \quad xg = \omega g x, \quad gy = \omega y g, \quad xy = \omega y x.$$

Then H is a Hopf algebra with the coalgebra structure and the antipode given by

$$\triangle(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{n-1},$$
$$\triangle(x) = x \otimes g + 1 \otimes x, \quad \varepsilon(x) = 0, \quad S(x) = -x g^{n-1},$$
$$\triangle(y) = y \otimes g + 1 \otimes y, \quad \varepsilon(y) = 0, \quad S(y) = -y g^{n-1}.$$

Note that H has a canonical basis $\{g^l x^r y^s\colon 0 \leqslant l, r, s < n\}$ over $\mathbb{C}$. For the details, the reader is directed to [3], [9], [10].

3. The structres of Hopf $*$-algebras on H

Throughout this section, let H be the Radford algebra over $\mathbb{C}$ described in the last section. In this section, we study the $*$-structures on the Hopf algebra H. Let $Z(H)$ denote the center of H. Note that H is generated, as an algebra over $\mathbb{R}$, by g, x, y, and i subject to the relations given in the last section together with $i^2 = -1$ and $i \in Z(H)$. In the following, let $H^{\rm op}$ denote the opposite algebra of H. For any $h, l \in H^{\rm op}$, let $h \cdot l$ denote the product of h and l in $H^{\rm op}$, i.e. $h \cdot l = lh$.

Lemma 3.1. *Let $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = |\beta| = 1$. Then H is a Hopf $*$-algebra with the $*$-structure given by*

$$g^* = g, \quad x^* = \alpha x, \quad y^* = \beta y.$$

Proof. We first prove that the relations given in the lemma together with $i^* = -i$ give rise to a real antialgebra endomorphism of H, i.e. a real algebra map from H to $H^{\rm op}$. Since $|\omega|$=1, we have $\omega^* = \overline{\omega} = \omega^{-1}$. Hence in $H^{\rm op}$ we have $(g^*)^n = g^n = 1$, $(x^*)^n = (\alpha x)^n = 0$, $x^* \cdot g^* = \alpha x \cdot g = \alpha g x = \alpha\omega^{-1} x g = \alpha\omega^{-1} g \cdot x = \omega^* g^* \cdot x^*$ and $x^* \cdot y^* = \alpha\beta x \cdot y = \alpha\beta y x = \alpha\beta\omega^{-1} x y = \alpha\beta\omega^{-1} y \cdot x = \omega^* y^* \cdot x^*$. Similarly, one can check that $(y^*)^n = 0$ and $g^* \cdot y^* = \omega^* y^* \cdot g^*$. We also have $i^* = -i \in Z(H^{\rm op})$ and $(i^*)^2 = (-i)^2 = -1$. This shows that the relations given in the lemma together with $i^* = -i$ determine a real algebra map $*\colon H \to H^{\rm op}$. Then it follows that $*$ is a conjugate-linear antialgebra endomorphism of H. Hence, the composition $* \circ *$ is a complex algebra endomorphism of H. It is not difficult to check that $(h^*)^* = h$ for all $h \in \{g, x, y\}$, and so $(h^*)^* = h$ for all $h \in H$. Thus, $*$ is an involution of H. Note that both $\triangle \circ *$ and $(* \otimes *) \circ \triangle$ are conjugate-linear antialgebra maps from H to $H \otimes H$. It is easy check that $\triangle(h^*) = \sum (h_1)^* \otimes (h_2)^*$ for any $h \in \{g, x, y\}$. It follows that $\triangle(h^*) = \sum (h_1)^* \otimes (h_2)^*$ for all $h \in H$. Similarly, we have $\varepsilon(h^*) = \overline{\varepsilon(h)}$ for all $h \in H$. Finally, since S is a complex antialgebra endomorphism of H and $*$ is a conjugate-linear antialgebra endomorphism of H, the map $H \to H$, $h \mapsto S(S(h^*)^*)$ is a complex algebra endomorphism of H. Now we have

$$\begin{aligned}
S(S(g^*)^*) &= S(S(g)^*)) = S((g^{-1})^*) = S((g^*)^{-1}) = S(g^{-1}) = g,\\
S(S(x^*)^*) &= S(S(\alpha x)^*) = S((-\alpha x g^{n-1})^*) = S(-\bar{\alpha}(g^{n-1})^* x^*)\\
&= S(-\bar{\alpha}\alpha g^{n-1} x) = -S(x) S(g^{n-1}) = x g^{n-1} g = x,
\end{aligned}$$

and similarly $S(S(y^*)^*) = y$. It follows that $S(S(h^*)^*) = h$ for all $h \in H$. □

Let $M_2(\mathbb{C})$ be the matrix algebra of all 2×2-matrices over $\mathbb{C}$. For a matrix

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in M_2(\mathbb{C}),$$

let

$$\bar{A} = \begin{pmatrix} \overline{\alpha_{11}} & \overline{\alpha_{12}} \\ \overline{\alpha_{21}} & \overline{\alpha_{22}} \end{pmatrix} \in M_2(\mathbb{C}).$$

Lemma 3.2. *Assume that* $n = 2$ *and let* $A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in M_2(\mathbb{C})$ *with* $\bar{A}A = I_2$, *the* 2×2 *identity matrix. Then* H *is a Hopf* $*$*-algebra with the* $*$*-structure given by*

$$g^* = g, \quad x^* = \alpha_{11}x + \alpha_{12}y, \quad y^* = \alpha_{21}x + \alpha_{22}y.$$

Proof. Assume that $n = 2$. Then $\omega = -1$. We first prove that the relations given in the lemma together with $i^* = -i$ give rise to a real antialgebra endomorphism of H, i.e. a real algebra map from H to H^{op}. In H^{op} we have $(g^*)^2 = g^2 = 1$, $(x^*)^2 = (\alpha_{11}x+\alpha_{12}y)^2 = \alpha_{11}^2x^2+\alpha_{11}\alpha_{12}xy+\alpha_{12}\alpha_{11}yx+\alpha_{12}^2y^2 = 0$ and $x^*\cdot g^* = (\alpha_{11}x+\alpha_{12}y)\cdot g = \alpha_{11}gx + \alpha_{12}gy = -\alpha_{11}xg - \alpha_{12}yg = -g\cdot(\alpha_{11}x+\alpha_{12}y) = -g^*\cdot x^*$. We also have $x^*\cdot y^* = (\alpha_{21}x+\alpha_{22}y)(\alpha_{11}x+\alpha_{12}y) = \alpha_{21}\alpha_{11}x^2+\alpha_{21}\alpha_{12}xy+\alpha_{22}\alpha_{11}yx+\alpha_{22}\alpha_{12}y^2 = (\alpha_{21}\alpha_{12}-\alpha_{22}\alpha_{11})xy$ and $y^*\cdot x^* = (\alpha_{11}x+\alpha_{12}y)(\alpha_{21}x+\alpha_{22}y) = \alpha_{11}\alpha_{21}x^2+\alpha_{11}\alpha_{22}xy+\alpha_{12}\alpha_{21}yx + \alpha_{12}\alpha_{22}y^2 = (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})xy$, which implies that $x^*\cdot y^* = -y^*\cdot x^*$. Similarly, one can check that $(y^*)^2 = 0$ and $g^*\cdot y^* = -y^*\cdot g^*$. We also have $i^* = -i \in Z(H^{\text{op}})$ and $(i^*)^2 = (-i)^2 = -1$. This shows that the relations given in the lemma together with $i^* = -i$ determine a real algebra map $*\colon H \to H^{\text{op}}$. Then it follows that $*$ is a conjugate-linear antialgebra endomorphism of H. Hence, the composition $*\circ*$ is a complex algebra endomorphism of H. Clearly, $(g^*)^* = g$. Since $\bar{A}A = I_2$, $\overline{\alpha_{i1}}\alpha_{1j} + \overline{\alpha_{i2}}\alpha_{2j} = \delta_{ij}$ for $1 \leqslant i,j \leqslant 2$. Hence we have

$$\begin{aligned}(x^*)^* &= (\alpha_{11}x + \alpha_{12}y)^* = \overline{\alpha_{11}}x^* + \overline{\alpha_{12}}y^* \\ &= \overline{\alpha_{11}}(\alpha_{11}x + \alpha_{12}y) + \overline{\alpha_{12}}(\alpha_{21}x + \alpha_{22}y) \\ &= (\overline{\alpha_{11}}\alpha_{11} + \overline{\alpha_{12}}\alpha_{21})x + (\overline{\alpha_{11}}\alpha_{12} + \overline{\alpha_{12}}\alpha_{22})y = x.\end{aligned}$$

Similarly, we also have $(y^*)^* = y$. It follows that $(h^*)^* = h$ for all $h \in H$. Thus, $*$ is an involution of H. Note that both $\triangle\circ*$ and $(*\otimes*)\circ\triangle$ are conjugate-linear antialgebra maps from H to $H\otimes H$. It is easy check that $\triangle(h^*) = \sum(h_1)^*\otimes(h_2)^*$ for any $h \in \{g, x, y\}$. It follows that $\triangle(h^*) = \sum(h_1)^*\otimes(h_2)^*$ for all $h \in H$. Similarly, we have $\varepsilon(h^*) = \overline{\varepsilon(h)}$ for all $h \in H$. Finally, since S is a complex antialgebra endomorphism of H and $*$ is a conjugate-linear antialgebra endomorphism of H, the map $H \to H$, $h \mapsto S(S(h^*)^*)$ is a complex algebra endomorphism of H. Now we have

$$\begin{aligned}S(S(g^*)^*) &= S(S(g)^*)) = S(g^*) = S(g) = g, \\ S(S(x^*)^*) &= S(S(\alpha_{11}x + \alpha_{12}y)^*) = S((-\alpha_{11}xg - \alpha_{12}yg)^*) \\ &= S((\alpha_{11}gx + \alpha_{12}gy)^*) = S(\overline{\alpha_{11}}x^*g^* + \overline{\alpha_{12}}y^*g^*)\end{aligned}$$

$$
\begin{aligned}
&= S(\overline{\alpha_{11}}(\alpha_{11}x + \alpha_{12}y)g + \overline{\alpha_{12}}(\alpha_{21}x + \alpha_{22}y)g) \\
&= S(g)S((\overline{\alpha_{11}}\alpha_{11} + \overline{\alpha_{12}}\alpha_{21})x + (\overline{\alpha_{11}}\alpha_{12} + \overline{\alpha_{12}}\alpha_{22})y) \\
&= S(g)S(x) = g(-xg) = x,
\end{aligned}
$$

and similarly $S(S(y^*)^*) = y$. It follows that $S(S(h^*)^*) = h$ for all $h \in H$. □

The next proposition follows similarly to [1], Lemma 2.7.

Proposition 3.3. *For any $r, s \in \mathbb{N}$ and $l \in \mathbb{Z}$,*

$$\triangle(y^r x^s g^l) = \sum_{i=0}^{r}\sum_{j=0}^{s} \omega^{-(r-i)j} \binom{r}{i}_{\omega} \binom{s}{j}_{\omega^{-1}} y^{r-i}x^{s-j}g^l \otimes y^i x^j g^{l+s-j+r-i}.$$

P r o o f. Since

$$(x \otimes g)(1 \otimes x) = \omega^{-1}(1 \otimes x)(x \otimes g), \quad (y \otimes g)(1 \otimes y) = \omega(1 \otimes y)(y \otimes g),$$

it follows from [2], Proposition IV.2.2 that

$$
\begin{aligned}
\triangle(x)^s &= (1 \otimes x + x \otimes g)^s = \sum_{j=0}^{s} \binom{s}{j}_{\omega^{-1}} x^{s-j} \otimes x^j g^{s-j}, \\
\triangle(y)^r &= (1 \otimes y + y \otimes g)^r = \sum_{i=0}^{r} \binom{r}{i}_{\omega} y^{r-i} \otimes y^i g^{r-i}.
\end{aligned}
$$

Now, since $\triangle$ is an algebra map, we have

$$
\begin{aligned}
\triangle(y^r x^s g^l) &= \triangle(y)^r \triangle(x)^s \triangle(g)^l \\
&= (1 \otimes y + y \otimes g)^r (1 \otimes x + x \otimes g)^s (g \otimes g)^l \\
&= \sum_{i=0}^{r}\sum_{j=0}^{s} \binom{r}{i}_{\omega} \binom{s}{j}_{\omega^{-1}} y^{r-i}x^{s-j}g^l \otimes y^i g^{r-i} x^j g^{l+s-j} \\
&= \sum_{i=0}^{r}\sum_{j=0}^{s} \omega^{-(r-i)j} \binom{r}{i}_{\omega} \binom{s}{j}_{\omega^{-1}} y^{r-i}x^{s-j}g^l \otimes y^i x^j g^{l+s-j+r-i}.
\end{aligned}
$$

□

Note that $\{y^r x^s g^l \colon 0 \leqslant r, s, l < n\}$ is a canonical basis of H over $\mathbb{C}$. Hence,

$$\{y^r x^s g^l \otimes y^{r_1} x^{s_1} g^{l_1} \colon 0 \leqslant r, r_1, s, s_1, l, l_1 < n\}$$

is a basis of $H \otimes H$ over $\mathbb{C}$. For an element

$$h = \sum_{0 \leqslant r,s,l < n} \lambda_{r,s,l} y^r x^s g^l$$

in H, if $\lambda_{r,s,l} \neq 0$, then we say that $y^r x^s g^l$ is a term of h. Moreover, r or s is called the degree of y or x, respectively, in the term $y^r x^s g^l$. Similarly, for an element

$$h = \sum_{0 \leqslant r,s,l,r_1,s_1,l_1 < n} \lambda_{r,s,l,r_1,s_1,l_1} y^r x^s g^l \otimes y^{r_1} x^{s_1} g^{l_1}$$

in $H \otimes H$, if $\lambda_{r,s,l,r_1,s_1,l_1} \neq 0$, then we say that $y^r x^s g^l \otimes y^{r_1} x^{s_1} g^{l_1}$ is a term of h. Moreover, $r + r_1$ or $s + s_1$ is called the total degree of y or x, respectively, in the term

$$y^r x^s g^l \otimes y^{r_1} x^{s_1} g^{l_1}.$$

Lemma 3.4. $G(H) = \{g^l \colon 0 \leqslant l < n\}$.

P r o o f. Obviously, $g^l \in G(H)$ for all $0 \leqslant l < n$. Conversely, let

$$h = \sum_{0 \leqslant r,s,l < n} \lambda_{r,s,l} y^r x^s g^l \in G(H),$$

where $\lambda_{r,s,l} \in \mathbb{C}$. Assume that r_1 is the highest degree of y in the terms of h, that is, there is a nonzero coefficient $\lambda_{r_1,s_1,l_1} \neq 0$ in the above expression of h such that $\lambda_{r,s,l} \neq 0$ implies $r \leqslant r_1$. From Proposition 3.3 one knows that the total degree of y in each term of the expression of $\triangle(y^r x^s g^l)$ is r. Then from

$$\triangle(h) = \sum_{r,s,l} \lambda_{r,s,l} \triangle(y^r x^s g^l)$$

one gets that the highest total degree of y in the terms of $\triangle(h)$ is r_1. However,

$$y^{r_1} x^{s_1} g^{l_1} \otimes y^{r_1} x^{s_1} g^{l_1}$$

is a term of $h \otimes h$ with the nonzero coefficient ${\lambda^2}_{r_1,s_1,l_1} \neq 0$. It follows from $\triangle(h) = h \otimes h$ that $0 \leqslant 2r_1 \leqslant r_1$, which implies $r_1 = 0$. Thus, if $r > 0$, then $\lambda_{r,s,l} = 0$. Similarly, one can show that $\lambda_{r,s,l} = 0$ for any $s > 0$. Therefore $h \in \operatorname{span}\{g^l \colon 0 \leqslant l < n\}$. Since $G(H)$ is linearly independent over $\mathbb{C}$ and $\{g^l \colon 0 \leqslant l < n\} \subseteq G(H)$, we have $h = g^l$ for some $0 \leqslant l < n$. Hence $G(H) \subseteq \{g^l \colon 0 \leqslant l < n\}$, and so $G(H) = \{g^l \colon 0 \leqslant l < n\}$. $\square$

Lemma 3.5. *Let $h \in H$. If $\triangle(h) = h \otimes g + 1 \otimes h$, then $h = \lambda_1 x + \lambda_2 y + \lambda_3(1-g)$ for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$.*

Proof. Let $h = \sum\limits_{0\leqslant r,s,l<n} \lambda_{r,s,l} y^r x^s g^l$ with $\lambda_{r,s,l} \in \mathbb{C}$ such that

$$\triangle(h) = h \otimes g + 1 \otimes h.$$

Then $\varepsilon(h) = 0$. For any $0 \leqslant r, s < n$ let

$$h_{r,s} = \sum_{l=0}^{n-1} \lambda_{r,s,l} y^r x^s g^l.$$

Then by Proposition 3.3 and the proof of Lemma 3.4, one knows that

$$\triangle(h_{r,s}) = h_{r,s} \otimes g + 1 \otimes h_{r,s} \quad \forall\, r, s.$$

Hence, one may assume that

$$h = y^r x^s \sum_{l=0}^{n-1} \lambda_l g^l \neq 0$$

for some $\lambda_l \in \mathbb{C}$, where r and s are fixed integers with $0 \leqslant r, s < n$. Now, by Proposition 3.3 we have

$$\triangle(h) = \sum_{l=0}^{n-1} \sum_{i=0}^{r} \sum_{j=0}^{s} \lambda_l \omega^{-(r-i)j} \binom{r}{i}_{\omega} \binom{s}{j}_{\omega^{-1}} y^{r-i} x^{s-j} g^l \otimes y^i x^j g^{l+s-j+r-i} \tag{3.1}$$

and

$$\triangle(h) = h \otimes g + 1 \otimes h = \sum_{l=0}^{n-1} \lambda_l (y^r x^s g^l \otimes g + 1 \otimes y^r x^s g^l). \tag{3.2}$$

By the paragraph before Lemma 3.4, $H \otimes H$ has a canonical basis over $\mathbb{C}$

$$\{y^r x^s g^l \otimes y^{r_1} x^{s_1} g^{l_1} \colon\ 0 \leqslant r, r_1, s, s_1, l, l_1 < n\}.$$

Now by comparing the coefficients of the basis element $g^l \otimes y^r x^s g^l$ in the two expressions of $\triangle(h)$ given above, one gets that $\lambda_l = 0$ if $l > 1$, and that $\lambda_1 = 0$ if $(r, s) \neq (0, 0)$. Hence, $h = \lambda_0 y^r x^s$ when $r + s \neq 0$, and $h = \lambda_0 + \lambda_1 g$ when $r = s = 0$.

If $h = \lambda_0 + \lambda_1 g$, then $\lambda_0 + \lambda_1 = 0$ by $\varepsilon(h) = 0$, and so $h = \lambda_0(1 - g)$. Now assume $h = \lambda_0 y^r x^s$ with $r + s \neq 0$. Then (3.1) and (3.2) becomes

$$\triangle(h) = \sum_{i=0}^{r} \sum_{j=0}^{s} \lambda_0 \omega^{-(r-i)j} \binom{r}{i}_{\omega} \binom{s}{j}_{\omega^{-1}} y^{r-i} x^{s-j} \otimes y^i x^j g^{s-j+r-i} \tag{3.3}$$

and

$$\triangle(h) = h \otimes g + 1 \otimes h = \lambda_0(y^r x^s \otimes g + 1 \otimes y^r x^s), \tag{3.4}$$

respectively. If both $r > 0$ and $s > 0$, then by comparing the coefficients of the basis element $y^r \otimes x^s g^r$ in the two expressions of $\triangle(h)$ given above, one gets that $\lambda_0 = 0$, and hence $h = 0$, a contradiction. Hence either $r > 0$ and $s = 0$, or $r = 0$ and $s > 0$. If $r > 0$ and $s = 0$, then $h = \lambda_0 y^r$, and (3.3) and (3.4) becomes

$$\triangle(h) = \sum_{i=0}^{r} \lambda_0 \binom{r}{i}_{\omega} y^{r-i} \otimes y^i g^{r-i} \tag{3.5}$$

and

$$\triangle(h) = h \otimes g + 1 \otimes h = \lambda_0(y^r \otimes g + 1 \otimes y^r), \tag{3.6}$$

respectively. If $r > 1$, then by comparing the coefficients of the basis element $y^{r-1} \otimes y g^{r-1}$ in the two expressions of $\triangle(h)$ given above, one gets that $\lambda_0 = 0$, and hence $h = 0$, a contradiction. Hence $r = 1$ and so $h = \lambda_0 y$. Similarly, one can check that if $r = 0$ and $s > 0$, then $h = \lambda_0 x$. This completes the proof. □

Lemma 3.6. *Let $h \in H$ with $\triangle(h) = h \otimes g^w + 1 \otimes h$ for some $1 < w < n$. Then $h = \lambda(1 - g^w)$ for some $\lambda \in \mathbb{C}$.*

Proof. It is similar to the proof of Lemma 3.5. We only need to consider the case

$$h = y^r x^s \sum_{l=0}^{n-1} \lambda_l g^l \neq 0$$

for some $\lambda_l \in \mathbb{C}$, where r and s are fixed integers with $0 \leqslant r, s < n$. Then we have

$$\triangle(h) = h \otimes g^w + 1 \otimes h = \sum_{l=1}^{n-1} \lambda_l (y^r x^s g^l \otimes g^w + 1 \otimes y^r x^s g^l) \tag{3.7}$$

and

$$\triangle(h) = \sum_{l=0}^{n-1} \sum_{i=0}^{r} \sum_{j=0}^{s} \lambda_l \omega^{-(r-i)j} \binom{r}{i}_{\omega} \binom{s}{j}_{\omega^{-1}} y^{r-i} x^{s-j} g^l \otimes y^i x^j g^{l+s-j+r-i}. \tag{3.8}$$

If $r \neq 0$ and $s \neq 0$, then by comparing the coefficients of the basis element $y^r g^l \otimes x^s g^{l+r}$ in the two expressions of $\triangle(h)$ given above, one gets that $\lambda_l = 0$ for all $0 \leqslant l < n$, and hence $h = 0$, a contradiction. So $r = 0$ or $s = 0$. Assume that

$r \neq 0$. Then $s = 0$. In this case, by comparing the coefficients of the basis element $g^l \otimes y^r g^l$ in the two expressions of $\triangle(h)$ given above, one gets that $\lambda_l = 0$ if $l > 0$. Hence $h = \lambda_0 y^r$ and (3.7) and (3.8) become

$$\triangle(h) = h \otimes g^w + 1 \otimes h = \lambda_0 (y^r \otimes g^w + 1 \otimes y^r) \tag{3.9}$$

and

$$\triangle(h) = \sum_{i=0}^{r} \lambda_0 \binom{r}{i}_{\omega} y^{r-i} \otimes y^i g^{r-i}, \tag{3.10}$$

respectively. Then by comparing the coefficients of the basis element $y^r \otimes g^r$ in the both expressions of $\triangle(h)$ given in (3.9) and (3.10) one gets that $r = w > 1$ since $h = \lambda_0 y^r \neq 0$. Now by comparing the coefficients of the basis element $y^{r-1} \otimes y g^{r-1}$ in both expressions of $\triangle(h)$ given in (3.9) and (3.10), one finds that $\lambda_0 = 0$, a contradiction. This shows that $r = 0$. Similarly, one can show that $s = 0$. Hence $h = \sum\limits_l \lambda_l g^l \neq 0$. Then it is easy to see that $h = \lambda(1 - g^w)$ for some $\lambda \in \mathbb{C}$. □

Theorem 3.7. (1) *If $n > 2$, then Lemma* 3.1 *gives all Hopf $*$-algebra structures on H.*

(2) *If $n = 2$, then Lemma* 3.2 *gives all Hopf $*$-algebra structures on H.*

Proof. Assume that H has a Hopf $*$-algebra structure $*$. Then

$$\triangle(g^*) = (* \otimes *)\triangle(g) = g^* \otimes g^* \quad \text{and} \quad \varepsilon(g^*) = \overline{\varepsilon(g)} = 1.$$

Hence $g^* \in G(H)$. By Lemma 3.4, $g^* = g^w$ for some $0 \leqslant w < n$. Since $*$ is an involution and $1^* = 1$, $g^* \neq 1$. Hence $w \neq 0$, and so $0 < w < n$. We also have

$$\triangle(x^*) = (* \otimes *)\triangle(x) = x^* \otimes g^* + 1^* \otimes x^* = x^* \otimes g^w + 1 \otimes x^*.$$

If $w \neq 1$, then it follows from Lemma 3.6 that $x^* = \lambda(1 - g^w)$ for some $\lambda \in \mathbb{C}$. Since $*$ is an involution and a conjugate-linear antialgebra endomorphism of H, we have $x = (x^*)^* = (\lambda(1-g^w))^* = \bar{\lambda}(1-g^{w^2})$. This is impossible. Hence $w = 1$, and so $g^* = g$ and $\triangle(x^*) = x^* \otimes g + 1 \otimes x^*$. Then by Lemma 3.5, $x^* = \alpha_{11}x + \alpha_{12}y + \alpha_{13}(1-g)$ for some $\alpha_{11}, \alpha_{12}, \alpha_{13} \in \mathbb{C}$. Similarly, one can show that $y^* = \alpha_{21}x + \alpha_{22}y + \alpha_{23}(1-g)$ for some $\alpha_{21}, \alpha_{22}, \alpha_{23} \in \mathbb{C}$. Then by $xg = \omega gx$, one gets that $(xg)^* = (\omega gx)^*$. However, $(xg)^* = g^* x^* = g(\alpha_{11}x + \alpha_{12}y + \alpha_{13}(1-g)) = \alpha_{11}gx + \alpha_{12}gy + \alpha_{13}(g - g^2)$ and $(\omega gx)^* = \overline{\omega} x^* g^* = \omega^{-1}(\alpha_{11}x + \alpha_{12}y + \alpha_{13}(1-g))g = \omega^{-1}\alpha_{11}xg + \omega^{-1}\alpha_{12}yg + \omega^{-1}\alpha_{13}(g - g^2) = \alpha_{11}gx + \omega^{-2}\alpha_{12}gy + \omega^{-1}\alpha_{13}(g - g^2)$. It follows that $\alpha_{12} = \omega^{-2}\alpha_{12}$

and $\alpha_{13} = \omega^{-1}\alpha_{13}$. Hence $\alpha_{12}(1-\omega^2) = 0$ and $\alpha_{13} = 0$ by $\omega \neq 1$. Similarly, from $(gy)^* = (\omega yg)^*$ one gets that $\alpha_{21}(1-\omega^2) = 0$ and $\alpha_{23} = 0$.

(1) Assume that $n > 2$. Then $\omega^2 \neq 1$, and hence $\alpha_{12} = \alpha_{21} = 0$ by $\alpha_{12}(1-\omega^2) = 0$ and $\alpha_{21}(1-\omega^2) = 0$. Thus, $x^* = \alpha_{11}x$ and $y^* = \alpha_{22}y$. Then we have $x = (x^*)^* = (\alpha_{11}x)^* = \overline{\alpha_{11}}x^* = \overline{\alpha_{11}}\alpha_{11}x$, which implies that $|\alpha_{11}| = 1$. Similarly, one can show that $|\alpha_{22}| = 1$. This shows Part (1).

(2) Assume that $n = 2$. Then $x^* = \alpha_{11}x + \alpha_{12}y$ and $y^* = \alpha_{21}x + \alpha_{22}y$. Hence we have $x = (x^*)^* = (\alpha_{11}x + \alpha_{12}y)^* = \overline{\alpha_{11}}x^* + \overline{\alpha_{12}}y^* = \overline{\alpha_{11}}(\alpha_{11}x + \alpha_{12}y) + \overline{\alpha_{12}}(\alpha_{21}x + \alpha_{22}y) = (\overline{\alpha_{11}}\alpha_{11} + \overline{\alpha_{12}}\alpha_{21})x + (\overline{\alpha_{11}}\alpha_{12} + \overline{\alpha_{12}}\alpha_{22})y$ and $y = (y^*)^* = (\alpha_{21}x + \alpha_{22}y)^* = \overline{\alpha_{21}}x^* + \overline{\alpha_{22}}y^* = \overline{\alpha_{21}}(\alpha_{11}x + \alpha_{12}y) + \overline{\alpha_{22}}(\alpha_{21}x + \alpha_{22}y) = (\overline{\alpha_{21}}\alpha_{11} + \overline{\alpha_{22}}\alpha_{21})x + (\overline{\alpha_{21}}\alpha_{12} + \overline{\alpha_{22}}\alpha_{22})y$. It follows that

$$\begin{pmatrix} \overline{\alpha_{11}} & \overline{\alpha_{12}} \\ \overline{\alpha_{21}} & \overline{\alpha_{22}} \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This shows Part (2). $\square$

Theorem 3.8. *If $n \geqslant 3$, then up to equivalence, there is a unique Hopf $*$-algebra structure on H given by*

$$g^* = g, \quad x^* = x, \quad y^* = y.$$

Proof. Assume that $n \geqslant 3$. Then by Lemma 3.1, the relations given in the theorem determine a Hopf $*$-algebra structure on H, denoted by $*'$. Now let $*$ be any Hopf $*$-algebra structure on H. Then by Lemma 3.1 and Theorem 3.7 (1) there exist elements $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = |\beta| = 1$ such that

$$g^* = g, \quad x^* = \alpha x, \quad y^* = \beta y.$$

Pick up two elements $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\lambda_1^2 = \alpha$ and $\lambda_2^2 = \beta$. Then $|\lambda_1| = |\lambda_2| = 1$ by $|\alpha| = |\beta| = 1$, and hence $\lambda_1^{-1} = \overline{\lambda_1}$ and $\lambda_2^{-1} = \overline{\lambda_2}$. It is easy to see that there is a Hopf algebra automorphism φ of H such that $\varphi(g) = g$, $\varphi(x) = \lambda_1 x$ and $\varphi(y) = \lambda_2 y$. Then $\varphi(g^{*'}) = \varphi(g) = g = g^* = \varphi(g)^*$, $\varphi(x^{*'}) = \varphi(x) = \lambda_1 x = \lambda_1^{-1}\alpha x = \overline{\lambda_1}x^* = (\lambda_1 x)^* = \varphi(x)^*$ and $\varphi(y^{*'}) = \varphi(y) = \lambda_2 y = \lambda_2^{-1}\beta y = \overline{\lambda_2}y^* = (\lambda_2 y)^* = \varphi(y)^*$. Hence $\varphi(h^{*'}) = \varphi(h)^*$ for all $h \in H$, and so $*$ is equivalent to $*'$. $\square$

Throughout the following, assume that $n = 2$. In this case, $\omega = -1$.

Let $A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$ and $B = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$ be two matrices in $M_2(\mathbb{C})$ with $\bar{A}A = \bar{B}B = I_2$, and let $*_A$ and $*_B$ be the corresponding Hopf $*$-algebra structures on H determined by A and B as in Lemma 3.2, respectively. Then we have the following proposition.

Proposition 3.9. *$*_A$ and $*_B$ are equivalent $*$-structures on H if and only if there exists an invertible matrix Λ in $M_2(\mathbb{C})$ such that $A\Lambda = \bar{\Lambda}B$, i.e. $\bar{\Lambda}^{-1}A\Lambda = B$.*

P r o o f. Suppose that $*_A$ and $*_B$ are equivalent. Then there exists a Hopf algebra automorphism φ of H such that $\varphi(h^{*_A}) = \varphi(h)^{*_B}$ for all $h \in H$. By Lemma 3.4 and $n = 2$, one can see that $\varphi(g) = g$. Then by Lemma 3.5, a straightforward computation shows that there exists a matrix $\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}$ in $M_2(\mathbb{C})$ such that $\varphi(x) = \lambda_{11}x + \lambda_{12}y$ and $\varphi(y) = \lambda_{21}x + \lambda_{22}y$. Since φ is an isomorphism, one can check that Λ is an invertible matrix in $M_2(\mathbb{C})$. Now we have

$$\begin{aligned} \varphi(x^{*_A}) &= \varphi(\alpha_{11}x + \alpha_{12}y) = \alpha_{11}\varphi(x) + \alpha_{12}\varphi(y) \\ &= \alpha_{11}(\lambda_{11}x + \lambda_{12}y) + \alpha_{12}(\lambda_{21}x + \lambda_{22}y) \\ &= (\alpha_{11}\lambda_{11} + \alpha_{12}\lambda_{21})x + (\alpha_{11}\lambda_{12} + \alpha_{12}\lambda_{22})y \end{aligned}$$

and

$$\begin{aligned} \varphi(x)^{*_B} &= (\lambda_{11}x + \lambda_{12}y)^{*_B} = \overline{\lambda_{11}}x^{*_B} + \overline{\lambda_{12}}y^{*_B} \\ &= \overline{\lambda_{11}}(\beta_{11}x + \beta_{12}y) + \overline{\lambda_{12}}(\beta_{21}x + \beta_{22}y) \\ &= (\overline{\lambda_{11}}\beta_{11} + \overline{\lambda_{12}}\beta_{21})x + (\overline{\lambda_{11}}\beta_{12} + \overline{\lambda_{12}}\beta_{22})y. \end{aligned}$$

Hence, it follows from $\varphi(x^{*_A}) = \varphi(x)^{*_B}$ that $\alpha_{11}\lambda_{11} + \alpha_{12}\lambda_{21} = \overline{\lambda_{11}}\beta_{11} + \overline{\lambda_{12}}\beta_{21}$ and $\alpha_{11}\lambda_{12} + \alpha_{12}\lambda_{22} = \overline{\lambda_{11}}\beta_{12} + \overline{\lambda_{12}}\beta_{22}$. Similarly, from $\varphi(y^{*_A}) = \varphi(y)^{*_B}$, one gets that $\alpha_{21}\lambda_{11} + \alpha_{22}\lambda_{21} = \overline{\lambda_{21}}\beta_{11} + \overline{\lambda_{22}}\beta_{21}$ and $\alpha_{21}\lambda_{12} + \alpha_{22}\lambda_{22} = \overline{\lambda_{21}}\beta_{12} + \overline{\lambda_{22}}\beta_{22}$. Thus, we have $A\Lambda = \bar{\Lambda}B$.

Conversely, suppose that there exists an invertible matrix $\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}$ in $M_2(\mathbb{C})$ such that $A\Lambda = \bar{\Lambda}B$. Then it is straightforward to check that there is a Hopf algebra automorphism φ of H uniquely determined by $\varphi(g) = g$, $\varphi(x) = \lambda_{11}x + \lambda_{12}y$ and $\varphi(y) = \lambda_{21}x + \lambda_{22}y$. Obviously, $\varphi(g^{*_A}) = \varphi(g)^{*_B} = g$. From the computation above, one gets that $\varphi(x^{*_A}) = \varphi(x)^{*_B}$ and $\varphi(y^{*_A}) = \varphi(y)^{*_B}$. It follows that $\varphi(h^{*_A}) = \varphi(h)^{*_B}$ for any $h \in H$. This shows that $*_A$ and $*_B$ are equivalent. $\square$

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