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Quanwu Mu; Minhui Zhu; Ping Li

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A DIOPHANTINE INEQUALITY WITH FOUR SQUARES AND ONE kTH POWER OF PRIMES

Quanwu Mu, Minhui Zhu, Ping Li, Xi'an

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Abstract. Let $k \geqslant 5$ be an odd integer and η be any given real number. We prove that if λ_1, λ_2, λ_3, λ_4, μ are nonzero real numbers, not all of the same sign, and λ_1/λ_2 is irrational, then for any real number σ with $0 < \sigma < 1/(8\vartheta(k))$, the inequality

$$|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \mu p_5^k + \eta| < \Big(\max_{1\leqslant j\leqslant 5} p_j\Big)^{-\sigma}$$

has infinitely many solutions in prime variables $p_1, p_2, \ldots, p_5$, where $\vartheta(k) = 3 \times 2^{(k-5)/2}$ for $k = 5, 7, 9$ and $\vartheta(k) = [(k^2 + 2k + 5)/8]$ for odd integer k with $k \geqslant 11$. This improves a recent result in W. Ge, T. Wang (2018).

Keywords: Diophantine inequalities; Davenport-Heilbronn method; prime

MSC 2010: 11D75, 11P55

1. Introduction

In 1937, Vinogradov [23] proved that every sufficiently large odd integer is a sum of three primes. Later, Hua [11] refined Vinogradov's result and showed that all sufficiently large odd integers are sums of two primes and a kth power of a prime, where k is any given positive integer. In [11], Hua also proved that all sufficiently large odd integers satisfying some necessary congruence conditions can be represented

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in the form of four squares of primes and a kth power of a prime. It is of some interest to consider the analogous form for Diophantine inequalities. Some authors obtained many significant results in this direction, see [1], [2], [6], [8], [9], [13], [14], [15], [16], [19], [20], [21] for details. In [14], Li and Wang established the following theorem.

Theorem 1.1. *Let $k \geqslant 3$ be a fixed integer and η be any given real number. Suppose that $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \mu$ are nonzero real numbers, not all of the same sign, and λ_1/λ_2 is irrational. Then the inequality*

$$|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \mu p_5^k + \eta| < \Bigl(\max_{1\leqslant j\leqslant 5} p_j\Bigr)^{-\sigma} \tag{1.1}$$

has infinitely many solutions in prime variables $p_1, p_2, \ldots, p_5$ for $0 < \sigma < 1/(3k2^k)$.

In [17], we improved the above result and showed that under the same assumptions as in Theorem 1.1, inequality (1.1) has infinitely many solutions in prime variables $p_1, p_2, \ldots, p_5$, where $0 < \sigma < 1/16$ for $k = 3$, $0 < \sigma < 5/(3k2^k)$ for $4 \leqslant k \leqslant 5$, and $0 < \sigma < 40/(21k2^k)$ for $k \geqslant 6$. The proof is based on the method of Languasco and Zaccagnini in [12], together with Heath-Brown's improvement on Hua's lemma (see [4], Lemma 5 and [10], Theorem 2). Let

$$s(k) = \Bigl[\frac{k+1}{2}\Bigr], \quad \sigma(k) = \min(2^{s(k)-1}, \tfrac{1}{2}s(k)(s(k)+1)),$$

where $[x]$ denotes the largest integer not exceeding the real number x. Very recently, Ge and Wang [6] refined the result in [17]. They proved that under the same assumptions as in Theorem 1.1, inequality (1.1) has infinitely many solutions in prime variables $p_1, p_2, \ldots, p_5$ for $0 < \sigma < 1/(8\sigma(k))$ (see [6], Theorem 1.3).

The aim of the present paper is to further enlarge the range $0 < \sigma < 1/(8\sigma(k))$ for odd integer k with $k \geqslant 5$. The following theorem is proved.

Theorem 1.2. *Let $k \geqslant 5$ be an odd integer. Suppose that $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \mu$ and η satisfy the same conditions as in Theorem 1.1. Then for any real number σ with $0 < \sigma < 1/(8\vartheta(k))$, inequality (1.1) has infinitely many solutions in prime variables $p_1, p_2, \ldots, p_5$, where*

$$\vartheta(k) = \begin{cases} 3 \times 2^{(k-5)/2} & \text{if } k = 5, 7, 9, \\ [(k^2+2k+5)/8] & \text{if } k \geqslant 11 \text{ and } 2 \nmid k. \end{cases} \tag{1.2}$$

With the help of Corollary 3.2 below, we obtain a wider major arc, this with the very recent work of Bourgain (see [3], Theorem 10) yields the desired conclusion.

The proof of Theorem 1.2 is dependent on the Davenport-Heilbronn circle method (see [22], Chapter 11). For each integer $j \geqslant 2$ set

$$\psi(j) = \begin{cases} 2^j & \text{when } 2 \leqslant j \leqslant 4, \\ j(j+1) & \text{when } j \geqslant 5. \end{cases} \tag{2.1}$$

In what follows, we use ε and δ to denote fixed positive constants which are arbitrarily small. The letter p, with or without subscript, always stands for a prime number. The letter k, except as specially provided, usually denotes an odd integer not less than 5. Since λ_1/λ_2 is irrational, we let q be a large enough denominator of a convergent to λ_1/λ_2. Put

$$X = q^2, \quad \mathcal{N}(X) = \sum_{\substack{\delta X \leqslant p_j^2 \leqslant X,\ 1 \leqslant j \leqslant 4,\ \delta X \leqslant p_5^k \leqslant X \\ |\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \mu p_5^k + \eta| < \tau}} 1,$$

$$\tau = X^{-1/(16\vartheta(k)) + 30\varepsilon}, \quad K_\tau(\alpha) = \begin{cases} \Big(\dfrac{\sin(\pi\tau\alpha)}{\pi\alpha}\Big)^2 & \text{when } \alpha \neq 0, \\ \tau^2 & \text{when } \alpha = 0, \end{cases}$$

$$S_j(\alpha) = \sum_{\delta X \leqslant p^j \leqslant X} (\log p) e(\alpha p^j),$$

$$I(\tau, \eta, \mathfrak{X}) = \int_{\mathfrak{X}} \prod_{j=1}^{4} S_2(\lambda_j \alpha)\, S_k(\mu\alpha) e(\alpha\eta) K_\tau(\alpha)\, \mathrm{d}\alpha,$$

where $e(\alpha) = \mathrm{e}^{2\pi \mathrm{i}\alpha}$, $\mathfrak{X}$ denotes any measurable subset of $\mathbb{R}$ and $\vartheta(k)$ is defined by (1.2). For the Dirichlet kernel $K_\tau(\alpha)$ we have the trivial estimate

$$K_\tau(\alpha) \ll \min(\tau^2,\ |\alpha|^{-2}). \tag{2.2}$$

It follows from Lemma 4 of Davenport and Heilbronn [5] that

$$\int_{-\infty}^{\infty} e(xy) K_\tau(x)\, \mathrm{d}x = \max(0, \tau - |y|). \tag{2.3}$$

Thus,

$$\begin{aligned}
(2.4)\qquad \mathcal{N}(X) &\geqslant \frac{1}{\tau}\sum_{\substack{\delta X\leqslant p_j^2\leqslant X\\ 1\leqslant j\leqslant 4\\ \delta X\leqslant p_5^k\leqslant X}} \max(0,\tau-|\lambda_1p_1^2+\lambda_2p_2^2+\lambda_3p_3^2+\lambda_4p_4^2+\mu p_5^k+\eta|)\\
&\geqslant \frac{1}{\tau(\log X)^5}\sum_{\substack{\delta X\leqslant p_j^2\leqslant X\\ 1\leqslant j\leqslant 4\\ \delta X\leqslant p_5^k\leqslant X}}\prod_{j=1}^{5}\log p_j\\
&\quad\times \max(0,\tau-|\lambda_1p_1^2+\lambda_2p_2^2+\lambda_3p_3^2+\lambda_4p_4^2+\mu p_5^k+\eta|)\\
&= \frac{1}{\tau(\log X)^5}\sum_{\substack{\delta X\leqslant p_j^2\leqslant X\\ 1\leqslant j\leqslant 4\\ \delta X\leqslant p_5^k\leqslant X}}\prod_{j=1}^{5}\log p_j\\
&\quad\times \int_{-\infty}^{\infty} e((\lambda_1p_1^2+\lambda_2p_2^2+\lambda_3p_3^2+\lambda_4p_4^2+\mu p_5^k+\eta)\alpha)K_\tau(\alpha)\,\mathrm{d}\alpha\\
&= \frac{1}{\tau(\log X)^5}\, I(\tau,\eta,\mathbb{R}).
\end{aligned}$$

To prove Theorem 1.2, it suffices to establish the estimate $I(\tau,\eta,\mathbb{R})\gg\tau^2X^{1+1/k}$. For this purpose, we split the real line into three parts

$$\mathfrak{M}=\{\alpha\colon\ |\alpha|\leqslant\varphi\},\quad \mathfrak{m}=\{\alpha\colon\ \varphi<|\alpha|\leqslant\xi\},\quad \mathfrak{t}=\{\alpha\colon\ |\alpha|>\xi\},$$

where $\varphi=X^{-1/(2k)-\varepsilon}$, $\xi=\tau^{-2}X^{3\varepsilon}$. Usually, these sets are called the major arc, the minor arcs and the trivial arcs, respectively. Therefore

$$(2.5)\qquad I(\tau,\eta,\mathbb{R})=I(\tau,\eta,\mathfrak{M})+I(\tau,\eta,\mathfrak{m})+I(\tau,\eta,\mathfrak{t}).$$

It should be noted that the major arc $\mathfrak{M}$ is wider than that of [6]. In what follows, we shall show that

$$|I(\tau,\eta,\mathfrak{M})|\gg\tau^2X^{1+1/k},\quad |I(\tau,\eta,\mathfrak{m})|\ll\tau^2X^{1+1/k-\varepsilon},\quad |I(\tau,\eta,\mathfrak{t})|\ll\tau^2X^{1+1/k-\varepsilon}.$$

3. The major arc

Let $\mathbf{M} = \{\alpha\colon |\alpha| \leqslant X^{-1+5/(6k)-\varepsilon}\}$, then $\mathbf{M} \subset \mathfrak{M}$. In [17], Section 3, we have proved that

$$|I(\tau, \eta, \mathbf{M})| \gg \tau^2 X^{1+1/k}. \tag{3.1}$$

The conditions 'λ_1, λ_2, λ_3, λ_4, μ are nonzero real numbers, not all of the same sign' play an important role in the proof of (3.1), see [17], pages 485–486 for details. It remains to discuss the estimate for $|I(\tau, \eta, \mathfrak{M} \setminus \mathbf{M})|$.

Lemma 3.1 (see [7], Theorem 1). *Let j be an integer with $j \geqslant 2$, and $N \geqslant 2$. Suppose that a and q are integers with*

$$|q\alpha - a| \leqslant \frac{1}{q}, \quad (a,\ q) = 1, \quad q \geqslant 1. \tag{3.2}$$

Then for any $\varepsilon > 0$,

$$\sum_{p \leqslant N} (\log p) e(\alpha p^j) \ll N^{1+\varepsilon} \Big(\frac{1}{q} + \frac{1}{N^{1/2}} + \frac{q}{N^j}\Big)^{4^{1-j}}. \tag{3.3}$$

Corollary 3.2. *Suppose that $X^{-1+5/(6k)-\varepsilon} \leqslant |\alpha| \leqslant X^{-1/(2k)-\varepsilon}$. Then for any given nonzero real μ and $\varepsilon > 0$ we have*

$$|S_k(\mu\alpha)| \ll X^{1/k(1-1/2\times 4^{1-k})+\varepsilon}. \tag{3.4}$$

The implicit constant in the $\ll$ notation depends on k, μ, δ.

P r o o f. Notice that

$$|S_k(\mu\alpha)| \leqslant \Big|\sum_{p \leqslant X^{1/k}} (\log p) e(\mu\alpha p^k)\Big| + \Big|\sum_{p \leqslant (\delta X)^{1/k}} (\log p) e(\mu\alpha p^k)\Big|. \tag{3.5}$$

Similarly to [9], Corollary 2, we take $\mu\alpha$ in place of α in (3.2), and take $q = [1/|\mu\alpha|]$, $a = \pm 1$ (the sign of a is the same as that for $\mu\alpha$), then (3.4) follows from (3.5) and (3.3). □

By Corollary 3.2 and the arithmetic-geometric mean inequality, we get

$$
\begin{aligned}
(3.6)\qquad & |I(\tau,\eta,\mathfrak{M}\setminus\mathbf{M})| \\
&\leqslant \int_{\mathfrak{M}\setminus\mathbf{M}} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_2(\lambda_3\alpha)S_2(\lambda_4\alpha)S_k(\mu\alpha)|K_\tau(\alpha)\,\mathrm{d}\alpha \\
&\ll \tau^2 \max_{\alpha\in\mathfrak{M}\setminus\mathbf{M}} |S_k(\mu\alpha)| \sum_{j=1}^{4} \int_{\mathfrak{M}\setminus\mathbf{M}} \bigl|S_2(\lambda_j\alpha)\bigr|^4\,\mathrm{d}\alpha \\
&\ll \tau^2 X^{1/k(1-1/2\times 4^{1-k})+\varepsilon} \int_0^1 \bigl|S_2(\alpha)\bigr|^4\,\mathrm{d}\alpha \\
&\ll \tau^2 X^{1+1/k-\varepsilon},
\end{aligned}
$$

where (2.2) and Hua's lemma (see [4], page 85) are used. Noting that $I(\tau,\eta,\mathfrak{M}) = I(\tau,\eta,\mathbf{M}) + I(\tau,\eta,\mathfrak{M}\setminus\mathbf{M})$, this with (3.1) and (3.6) implies

$$
(3.7)\qquad |I(\tau,\eta,\mathfrak{M})| \gg \tau^2 X^{1+1/k}.
$$

4. The minor arcs

Let $\widetilde{\mathfrak{m}} = \mathfrak{m}_1 \cup \mathfrak{m}_2$, where

$$
\mathfrak{m}_j = \{\alpha\in\mathfrak{m}\colon\ |S_2(\lambda_j\alpha)| \leqslant X^{7/16+2\varepsilon}\} \quad \text{for } j=1,2.
$$

To estimate the integral $I(\tau,\eta,\mathfrak{m})$, we need several lemmas.

Lemma 4.1. *Let j and s be positive integers with $s\leqslant j$. Then*

$$
(4.1)\qquad \int_0^1 |S_j(\alpha)|^{s(s+1)}\,\mathrm{d}\alpha \ll X^{s^2/j+\varepsilon}
$$

holds for all $\varepsilon>0$.

P r o o f. It follows from [3], Theorem 10 that

$$
(4.2)\qquad \int_0^1 \Bigl|\sum_{\delta X\leqslant x^j\leqslant X} e(\alpha x^j)\Bigr|^{s(s+1)}\,\mathrm{d}\alpha \ll X^{s^2/j+\varepsilon}.
$$

By considering the number of solutions of the underlying Diophantine equation and using (4.2), we obtain (4.1). □

Lemma 4.2. *Let $j \geqslant 2$ be an integer. Suppose that λ and μ are nonzero real constants and k is an odd integer with $k \geqslant 5$. Then for any $\varepsilon > 0$ we have*

$$\int_{\mathbb{R}} |S_j(\lambda\alpha)|^{\psi(j)} K_\tau(\alpha)\,\mathrm{d}\alpha \ll \tau X^{\psi(j)/j-1+\varepsilon}, \tag{4.3}$$

$$\int_{\mathbb{R}} |S_2(\lambda\alpha)|^2 |S_k(\mu\alpha)|^{2\vartheta(k)} K_\tau(\alpha)\,\mathrm{d}\alpha \ll \tau X^{2\vartheta(k)/k+\varepsilon}, \tag{4.4}$$

where $\psi(j)$ and $\vartheta(k)$ are defined by (2.1) *and* (1.2), *respectively. The implicit constant in the $\ll$ notation of* (4.3) *depends on λ, j, and the implicit constant in the $\ll$ notation of* (4.4) *depends on k, λ, μ.*

Proof. For (4.3), see [18], Lemma 4.5 for details. It remains to prove (4.4). Let $a = (k-1)/2$, $b = (k+1)/2$.

We first consider the case of $k \geqslant 11$, $2 \nmid k$, recalling that $\vartheta(k) = [(k^2+2k+5)/8]$ in this case. When $k \equiv 1 \pmod 4$, we have

$$\vartheta(k) = \frac{k^2+2k+5}{8} = \frac{a(a+1)+b(b+1)}{4} + \frac{1}{2}.$$

It follows from the Cauchy-Schwarz inequality and Lemma 4.1 that

$$\begin{aligned}\int_0^1 |S_k(\alpha)|^{2\vartheta(k)}\,\mathrm{d}\alpha &\ll X^{1/k}\int_0^1 |S_k(\alpha)|^{(a(a+1)+b(b+1))/2}\,\mathrm{d}\alpha \\ &\ll X^{1/k}\Big(\int_0^1 |S_k(\alpha)|^{a(a+1)}\Big)^{1/2}\Big(\int_0^1 |S_k(\alpha)|^{b(b+1)}\Big)^{1/2} \\ &\ll X^{1/k}(X^{a^2/k+\varepsilon})^{1/2}(X^{b^2/k+\varepsilon})^{1/2} \\ &\ll X^{(k^2+5)/(4k)+\varepsilon} \ll X^{2\vartheta(k)/k-1/2+\varepsilon},\end{aligned} \tag{4.5}$$

where the trivial upper bound $S_k(\alpha) \ll X^{1/k}$ is used. When $k \equiv 3 \pmod 4$, we have

$$\vartheta(k) = \frac{(k+1)^2}{8} = \frac{a(a+1)+b(b+1)}{4}.$$

By a similar argument as that in (4.5), we also obtain

$$\int_0^1 |S_k(\alpha)|^{2\vartheta(k)}\,\mathrm{d}\alpha \ll X^{2\vartheta(k)/k-1/2+\varepsilon}. \tag{4.6}$$

It follows from (2.3) that

$$\int_{\mathbb{R}} |S_2(\lambda\alpha)|^2 |S_k(\mu\alpha)|^{2\vartheta(k)} K_\tau(\alpha)\,\mathrm{d}\alpha \ll \tau\Sigma, \tag{4.7}$$

where Σ denotes the number of solutions of

$$|\mu(p_1^k + \ldots + p_{\vartheta(k)}^k - p_{\vartheta(k)+1}^k - \ldots - p_{2\vartheta(k)}^k) + \lambda(p_{2\vartheta(k)+1}^2 - p_{2\vartheta(k)+2}^2)| < \tau$$

with $p_i^k \in [\delta X, X]$ for $1 \leqslant i \leqslant 2\vartheta(k)$, and $p_j^2 \in [\delta X, X]$ for $2\vartheta(k) + 1 \leqslant j \leqslant 2\vartheta(k) + 2$. Note that $\tau \to 0$ as $X \to \infty$. When $p_{2\vartheta(k)+1} \neq p_{2\vartheta(k)+2}$, the values of $p_1, p_2, \ldots, p_{2\vartheta(k)}$ determine the values of $p_{2\vartheta(k)+1}$ and $p_{2\vartheta(k)+2}$ with at most X^ε possibilities; these solutions contribute $\ll X^{2\vartheta(k)/k+\varepsilon}$ to Σ. When $p_{2\vartheta(k)+1} = p_{2\vartheta(k)+2}$, we get

$$p_1^k + \ldots + p_{\vartheta(k)}^k - p_{\vartheta(k)+1}^k - \ldots - p_{2\vartheta(k)}^k = 0. \tag{4.8}$$

By (4.5) and (4.6), it follows that equation(4.8) has $O(X^{2\vartheta(k)/k-1/2+\varepsilon})$ solutions in primes $p_1, p_2, \ldots, p_{2\vartheta(k)}$. In this case, these solutions also contribute $\ll X^{2\vartheta(k)/k+\varepsilon}$ to Σ. Thus, we get $\Sigma \ll X^{2\vartheta(k)/k+\varepsilon}$; this with (4.7) yields (4.4).

In the cases of $k = 5, 7, 9$, noting that $\vartheta(k) = 3 \times 2^{(k-5)/2} = 2^{a-2} + 2^{b-2}$, we can also prove (4.6) using the Cauchy-Schwarz inequality and Hua's lemma. In a similar manner as above, we can prove (4.4). This completes the proof of Lemma 4.2. □

From the arithmetic-geometric mean inequality, Hölder's inequality and Lemma 4.2, we get

$$\begin{aligned}
I(\tau, \eta, \mathfrak{m}_1) &\ll \sum_{j=2}^{4} \int_{\mathfrak{m}_1} |S_2(\lambda_1\alpha)||S_2(\lambda_j\alpha)|^3 |S_k(\mu\alpha)| K_\tau(\alpha)\,\mathrm{d}\alpha \\
&\ll \Big(\sup_{\alpha\in\mathfrak{m}_1} |S_2(\lambda_1\alpha)|\Big)^{1/\vartheta(k)} \Big(\int_{\mathbb{R}} |S_2(\lambda_1\alpha)|^4 K_\tau(\alpha)\,\mathrm{d}\alpha\Big)^{1/4-1/(2\vartheta(k))} \\
&\quad \times \Big(\int_{\mathbb{R}} |S_2(\lambda_1\alpha)|^2 |S_k(\mu\alpha)|^{2\vartheta(k)} K_\tau(\alpha)\,\mathrm{d}\alpha\Big)^{1/(2\vartheta(k))} \\
&\quad \times \sum_{j=2}^{4} \Big(\int_{\mathbb{R}} |S_2(\lambda_j\alpha)|^4 K_\tau(\alpha)\,\mathrm{d}\alpha\Big)^{3/4} \\
&\ll (X^{7/16+2\varepsilon})^{1/\vartheta(k)} (\tau X^{1+\varepsilon})^{1/4-1/(2\vartheta(k))} (\tau X^{2\vartheta(k)/k+\varepsilon})^{1/(2\vartheta(k))} (\tau X^{1+\varepsilon})^{3/4} \\
&\ll \tau X^{1+1/k-1/(16\vartheta(k))+4\varepsilon} \ll \tau^2 X^{1+1/k-\varepsilon}.
\end{aligned}$$

By symmetry, the same bound holds for $\mathfrak{m}_2$ in place of $\mathfrak{m}_1$. This implies that

$$I(\tau, \eta, \widetilde{\mathfrak{m}}) \ll \tau^2 X^{1+1/k-\varepsilon}. \tag{4.9}$$

It therefore remains to discuss the set $\mathfrak{m}^* = \mathfrak{m} \setminus \widetilde{\mathfrak{m}}$, in which

$$|S_2(\lambda_1\alpha)| > X^{7/16+2\varepsilon}, \quad |S_2(\lambda_2\alpha)| > X^{7/16+2\varepsilon}, \quad X^{-1/(2k)-\varepsilon} < |\alpha| \leqslant \tau^{-2} X^{3\varepsilon}$$

hold simultaneously. By a familiar dyadic dissection argument, we divide $\mathfrak{m}^*$ into at most $\ll \log^3 X$ disjoint sets $E(Z_1, Z_2, y)$. For $\alpha \in E(Z_1, Z_2, y)$ we have

$$Z_1 < |S_2(\lambda_1\alpha)| \leqslant 2Z_1, \quad Z_2 < |S_2(\lambda_2\alpha)| \leqslant 2Z_2, \quad y < |\alpha| \leqslant 2y,$$

where $Z_1 = 2^{k_1}X^{7/16+2\varepsilon}$, $Z_2 = 2^{k_2}X^{7/16+2\varepsilon}$ and $y = 2^{k_3}X^{-1/(2k)-\varepsilon}$ for some non-negative integers k_1, k_2, k_3.

For simplicity, we take the notation $\mathscr{A}$ as a shortcut for $E(Z_1, Z_2, y)$, and let $m(\mathscr{A})$ denote the Lebesgue measure of $\mathscr{A}$.

Lemma 4.3. *We have*

$$m(\mathscr{A}) \ll yX^{5/2+8\varepsilon}(Z_1Z_2)^{-4}.$$

Proof. See [17], Lemma 6. □

By (2.2), the arithmetic-geometric mean inequality and Hölder's inequality, we have

$$\begin{aligned}
I(\tau,\eta,\mathscr{A}) &\ll \sum_{j=3}^{4}\int_{\mathscr{A}} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)||S_2(\lambda_j\alpha)|^2|S_k(\mu\alpha)|K_\tau(\alpha)\,\mathrm{d}\alpha \\
&\ll \Big(\int_{\mathscr{A}} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)|^4K_\tau(\alpha)\,\mathrm{d}\alpha\Big)^{1/4}\Big(\int_{\mathbb{R}} |S_k(\mu\alpha)|^{\psi(k)}K_\tau(\alpha)\,\mathrm{d}\alpha\Big)^{1/\psi(k)} \\
&\quad\times\Big(\int_{\mathscr{A}} K_\tau(\alpha)\,\mathrm{d}\alpha\Big)^{1/4-1/\psi(k)}\sum_{j=3}^{4}\Big(\int_{\mathbb{R}} |S_2(\lambda_j\alpha)|^4K_\tau(\alpha)\,\mathrm{d}\alpha\Big)^{1/2} \\
&\ll ((Z_1Z_2)^4m(\mathscr{A})\min(\tau^2,y^{-2}))^{1/4}(\tau X^{\psi(k)/k-1+\varepsilon})^{1/\psi(k)} \\
&\quad\times(\min(\tau^2,y^{-2})m(\mathscr{A}))^{1/4-1/\psi(k)}(\tau X^{1+\varepsilon})^{1/2} \\
&\ll \tau^{1/2+1/\psi(k)}(y\min(\tau^2,y^{-2}))^{1/2-1/\psi(k)}X^{7/8+1/k+3\varepsilon} \\
&\ll \tau X^{7/8+1/k+3\varepsilon} \ll \tau^2X^{1+1/k-2\varepsilon},
\end{aligned}$$

where Lemmas 4.2–4.3 and the bounds $Z_j \geqslant X^{7/16+2\varepsilon}$ $(j = 1, 2)$ are used. Thus,

$$I(\tau,\eta,\mathfrak{m}^*) \ll (\log^3 X)\max_{\mathscr{A}}|I(\tau,\eta,\mathscr{A})| \ll \tau^2X^{1+1/k-\varepsilon}. \tag{4.10}$$

It follows from (4.9) and (4.10) that

$$I(\tau,\eta,\mathfrak{m}) \ll \tau^2X^{1+1/k-\varepsilon}. \tag{4.11}$$

5. The trivial arcs

The proof of $|I(\tau,\eta,\mathfrak{t})| \ll \tau^2 X^{1+1/k-\varepsilon}$ is almost identical to that of inequality (25) in [17]. We list it for the sake of completeness.

$$\begin{aligned}
(5.1)\qquad \bigl|I(\tau,\eta,\mathfrak{t})\bigr| &\ll \int_{\xi}^{\infty} \bigl|S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_2(\lambda_3\alpha)S_2(\lambda_4\alpha)S_k(\mu\alpha)\bigr|K_\tau(\alpha)\,\mathrm{d}\alpha \\
&\ll X^{1/k}\sum_{j=1}^{4}\int_{\xi}^{\infty}|S_2(\lambda_j\alpha)|^4 K_\tau(\alpha)\,\mathrm{d}\alpha \\
&\ll X^{1/k}\sum_{j=1}^{4}\int_{|\lambda_j|\xi}^{\infty}\frac{|S_2(\alpha)|^4}{\alpha^2}\,\mathrm{d}\alpha \\
&\ll X^{1/k}\sum_{j=1}^{4}\sum_{n\geqslant|\lambda_j|\xi}\frac{1}{(n-1)^2}\int_{n-1}^{n}|S_2(\alpha)|^4\,\mathrm{d}\alpha \\
&\ll \frac{X^{1/k}X^{1+\varepsilon}}{\xi} \ll \tau^2 X^{1+1/k-\varepsilon}.
\end{aligned}$$

6. Completion of the proof

By (3.7), (4.11), (5.1) and (2.5), we get $I(\tau,\eta,\mathbb{R}) \gg \tau^2 X^{1+1/k}$. It follows from (2.4) that

$$\mathcal{N}(X) \gg \tau X^{1+1/k}(\log X)^{-5} \gg X^{1+1/k-1/(16\vartheta(k))+\varepsilon}.$$

Recalling that λ_1/λ_2 is irrational, q is a large enough denominator of a convergent to λ_1/λ_2 and $X = q^2$. When $q \to \infty$, we have $X \to \infty$; this implies $\mathcal{N}(X) \to \infty$. The value of τ and $\max p_j \asymp X^{1/2}$ give the desired range of σ on the right-hand side of (1.1). This completes the proof of Theorem 1.2.

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Authors' address: Quanwu Mu (corresponding author), Minhui Zhu, Ping Li, School of Science, Xi'an Polytechnic University, No. 19 Jinhua South Road, Xi'an 710048, Shaanxi, P. R. China, e-mail: `muquanwu@163.com`, `xiao-yu123@126.com`, `liping7487@163.com`.