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# A BI-AVERAGE TREE SOLUTION FOR PROBABILISTIC COMMUNICATION SITUATIONS WITH FUZZY COALITION

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A probabilistic communication structure considers the setting with communication restrictions in which each pair of players has a probability to communicate directly. In this paper, we consider a more general framework, called a probabilistic communication structure with fuzzy coalition, that allows any player to have a participation degree to cooperate within a coalition. A maximal product spanning tree, indicating a way of the greatest possibility to communicate among the players, is introduced where the unique path from one player to another is optimal. We present a feasible procedure to find the maximal product spanning trees. Furthermore, for games under this model, a new solution concept in terms of the average tree solution is proposed and axiomatized by defining a restricted game in Choquet integral form.

*Keywords:* probabilistic communication situation, fuzzy coalition, average tree solution, maximal product spanning tree

*Classification:* 05C57, 05C72, 91A12

## 1. INTRODUCTION

A cooperative game over a finite set of players is a function assigning to any coalition a profit achieved by cooperation. It is generally supposed that there are no restrictions on communications between players. This leads us to capture the fact that there may be a given structure of communication in various economical and social issues.

Myerson [11] considered a kind of communication restriction and introduced the communication structure represented by an undirected graph. The nodes are players and one link between any two players is established if they can communicate each other. The famous solutions for games with communication structure are Myerson value proposed by Myerson [11] and position value by Borm et al. [3].

Later, Herings et al. [8] proposed an average tree solution for cycle-free communication situations which is an average of specific marginal contribution vectors defined by a rooted tree. The marginal contribution of every player in a rooted tree is equal to the worth of the coalition consisting of this player and his all subordinates minus the sum of worths of the coalitions consisting of one of his successors and all subordinates of this successor.

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Calvo et al. [5] extended the model of Myerson and introduced the probabilistic version of communication structure. A probabilistic communication structure assigns to each link a probability of realization and these probabilities are independent each other. Gómez et al. [7] dropped the independence assumption about the probabilities and consider the generalized communication structure where a probabilistic distribution over all possible communication structures is given.

Aubin [1] considered the uncertainty about the participation levels of players and introduced the fuzzy cooperative game. Two particular forms for fuzzy cooperative games are the proportional form defined by Butnariu [4] and Choquet integral form defined by Tsurumi et al. [12]. Xu et al. [14] and Li et al. [10] considered the participation levels of players under communication structures and studied the Myerson value and position value respectively. Jiménez-Losada et al. [9] introduced the fuzzy communication structure described by a fuzzy graph where the participation levels and communication levels for players are allowed to be not full. The communication level between any two players is limited to be not greater than their own participation level.

Assume that the likelihood of existence of a communication link is represented by a probability assigned to it and greater formation probability of a communication structure among some players reflects its stronger stability than other communication possibilities. In this paper, we consider that any two players can have different preferences for all possible paths through them and in the cooperation they only choose the most stable one as the final communication channel. For example, for the purpose of saving construction cost, cities that plan to build pipelines together to transport natural gas will prefer to choose the path covering all cities with the greatest communication possibility. Synthesizing the uncertainty of players' participation levels in the process of cooperation, it is meaningful to research a new framework of probabilistic communication structure with fuzzy coalition in which every player is permitted to join in a coalition with a certain participation level and any two of them communicate by mutually independent probabilities.

Under this model, a maximal product spanning tree, in which all players communicate in the most likely way and the unique path between any two players is optimal, is introduced. In order to find a reasonable solution in this setting, we make an assumption that every player has a equal opportunity to govern the rest of players and is always rational to take the optimal paths in the process of cooperation. We define a restricted game containing all the information from the probabilistic communication structure with fuzzy coalition and propose a solution concept, called a bi-average tree solution. This new solution is axiomatized following the step of Herings et al. [8]. Meanwhile, it also has a relation with the average tree solution for special cases.

This paper is organized as follows. In Section 2, we get some preliminary knowledge served for the latter contexts. In Section 3, we introduce the probabilistic communication situation with fuzzy coalition arising form the examples in which players cooperate partially and communicate each other with a probability. In Section 4, a bi-average tree solution for this situation is proposed and characterized by defining a restricted game in Choquet integral form. Finally, some conclusions are given in Section 5.

## 2. PRELIMINARIES

### 2.1. Deterministic communication situations

A *cooperative game* can be described by a pair $(N, v)$ where $N = \{1, 2, \ldots, n\}$ denotes the set of players and $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ the corresponding characteristic function. Any subset $S \subseteq N$ is called a *coalition* and $v(S)$ the *worth* of $S$. If there is no ambiguity, we identify the game $(N, v)$ with its characteristic function $v$. The set of all cooperative games over $N$ is denoted by $G^N$. Throughout this paper, any game $v \in G^N$ is thought to be *zero-normalized*, i. e., $v(\{i\}) = 0$ for all $i \in N$.

The *restriction* of a game $v$ to a coalition $S \subseteq N$, denoted by $v_S$, is defined by $v_S(T) = v(S \cap T)$ for all $T \subseteq N$. We say a cooperative game $v$ is *superadditive* if for all $S, T \subseteq N$ such that $S \cap T = \emptyset$, $v(S \cup T) \geq v(S) + v(T)$.

A *solution* for cooperative games is a mapping that assigns to any game a set of payoff vectors. The core of a game $v \in G^N$ is a set-valued solution consisting of all efficient and coalition rational payoffs,

$$C(v) = \{x \in \mathbb{R}^n | \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for any } S \subseteq N\}.$$

In the classical cooperative game theory models, it is often assumed that any two players can communicate freely. Myerson [11] considered that the communication among players can be different and studied the cooperative games with communication structure. A communication structure can be represented by an undirected graph with players as nodes and the bilateral communication relations between them as links. Denote $\mathcal{L} = \big\{\{i, j\} | i, j \in N, i \neq j\big\}$, an undirected graph is a pair $(N, L)$ where $N$ is the set of players and $L \subseteq \mathcal{L}$ is a collection of feasible links among players in $N$. A *subgraph* of $(N, L)$ is another graph $(S, A)$ satisfying that $S \subseteq N$ and $A \subseteq L$. A *path* in $(N, L)$ is a sequence of nodes $(i_1, i_2, \ldots, i_k)$ satisfying that $\{i_h, i_{h+1}\} \in L$ is different for each $h \in \{1, 2, \ldots, k-1\}$. A *cycle* in $(N, L)$ is a sequence of nodes $(i_1, i_2, \ldots, i_k)$ if $k \geq 4$, $(i_1, i_2, \ldots, i_k)$ is a path and $i_k = i_1$. A graph $(N, L)$ is *cycle-free* when it does not contain any cycle. Two nodes $i, j \in N$ are *connected* in $(N, L)$ if there exists a path from $i$ to $j$. $(N, L)$ is *connected* if any two nodes $i, j \in N$ are connected. A connected graph without cycles is named a *tree*. In a connected graph $(N, L)$, a *spanning tree* is a tree subgraph of $(N, L)$ using all its nodes and we denote the set of all spanning trees of $(N, L)$ by $T(N, L)$. For any $S \subseteq N$, the graph $\big(S, L(S)\big)$ is a subgraph of $(N, L)$ with $L(S) = \big\{\{i, j\} \in L | i, j \in S\big\}$. $S \subseteq N$ is a *connected subset* of $(N, L)$ if $\big(S, L(S)\big)$ is connected. $K \subseteq N$ is a *component* of $(N, L)$ if $\big(K, L(K)\big)$ is maximally connected. The set of all components in subgraph $\big(S, L(S)\big)$ for any $S \subseteq N$ is denoted by $C^L(S)$.

A communication situation is a triple $(N, v, L)$ with $(N, v)$ being a cooperative game and $(N, L)$ an undirected graph. Much attention has been paid to the allocation of profit among the players for communication situations, including the Myerson value introduced by Myerson [11] and the average tree solution by Herings et al. [8] Denote the set of all communication situations over $N$ by $CS^N$, a solution for communication situations is a mapping $\psi : CS^N \to \mathbb{R}^n$.

The Myerson value is defined as the Shapley value of a graph game $v^L$,

$$\mu(N, v, L) = Sh(N, v^L),$$

where $v^L(S) = \sum_{C \in C^L(S)} v(C)$ for any $S \subseteq N$.

Before presenting the average tree solution, we first get a knowledge of directed graphs. A directed graph $D = (N, \vec{L})$ is defined by a set of nodes $N$ and a collection of directed links $\vec{L} \subseteq \big\{(i,j) | i,j \in N, i \neq j\big\}$. If $(i,j) \in \vec{L}$, then $j$ is a *successor* of $i$. $j$ is called a *subordinate* of $i$ if there exists a directed sequence of links $(i_h, i_{h+1}) \in \vec{L}$, $h = 1, 2, \dots, k$ such that $i_1 = i$ and $i_{k+1} = j$. $S_D(i)$ is the set of nodes consisting of $i$ and all its subordinates in $D = (N, \vec{L})$. $I_D(i)$ is the set of successors of $i$ in $D = (N, \vec{L})$.

Let $(N, v, L)$ be a cycle-free communication situation. Consider a component $K \in C^L(N)$, every player $i \in K$ can induce a unique directed tree, denoted by $T(i)$, with $i$ being the root. The average tree solution for any $i \in K$ is an average of all marginal payoffs over $|K|$ rooted trees given by

$$AT_i(N, v, L) = \frac{1}{|K|} \sum_{j \in K} \Big[ v\big(S_{T(j)}(i)\big) - \sum_{h \in I_{T(j)}(i)} v\big(S_{T(j)}(h)\big) \Big].$$

For convenience, we also use $S_j^L(i)$ and $I_j^L(i)$ instead of $S_{T(j)}(i)$ and $I_{T(j)}(i)$ respectively.

### 2.2. Fuzzy cooperative games

A *fuzzy coalition* is defined as a fuzzy set $U = (U_1, U_2, \dots, U_n)$ with the $i_{th}$ coordinate being in the interval $[0,1]$ interpreted as the participation level of player $i \in N$. We denote by $\mathcal{F}^N$ the set of fuzzy coalitions. The *support* of $U \in \mathcal{F}^N$ is *supp*$(U) = \{i | U_i \neq 0\}$, representing the set of active players. If $t \in [0,1]$, the *t-level set* of $U \in \mathcal{F}^N$ is $[U]_t = \{i \in N | U_i \geq t\}$. The fuzzy coalition $e^S$ with $e_i^S = 1$ if $i \in S$ and otherwise $e_i^S = 0$ corresponds to the crisp situation where the players within $S$ fully cooperate. We write $e^i$ instead of $e^{\{i\}}$.

A *fuzzy cooperative game* is a function $v^f : \mathcal{F}^N \to \mathbb{R}$ such that $v^f(e^{\emptyset}) = 0$. The set of fuzzy cooperative games is denoted by $FG^N$. The *associated crisp game* $w \in G^N$ of $v^f$ is defined as $w(S) = v^f(e^S)$ for each $S \subseteq N$.

Butnariu [4] studied the fuzzy cooperative game with proportional form. Later, Tsurumi et al. [12] defined the fuzzy characteristic function with Choquet integral form which is continuous with regard to the level of players' participation.

Let $Q(U) = \{U_i | U_i > 0, i \in N\}$ and $q(U) = \big|Q(U)\big|$. A fuzzy cooperative game $v^f$ is said to be with *Choquet integral form* if

$$v^f(U) = \sum_{k=1}^{q(U)} [h_k - h_{k-1}] \cdot w([U]_{h_k})$$

for any $U \in \mathcal{F}^N$, where $Q(U) \cup \{0\} = \{h_0, h_1, h_2, \dots, h_{q(U)}\}$ and $0 = h_0 < h_1 < h_2 < \cdots < h_{q(U)}$.

Yu and Zhang [13] proposed the fuzzy core of the fuzzy cooperative game $v^f$ with a fuzzy coalition $U \in \mathcal{F}^N$,

$$\tilde{C}(v^f)(U) = \{x \in \mathbb{R}^n | \sum_{i \in N} x_i = v^f(U), \sum_{i \in supp(U_T)} x_i \geq v^f(U_T) \text{ for any } T \subseteq N\},$$

where $U_T \in \mathcal{F}^N$ is defined by

$$(U_T)_i = \begin{cases} U_i, & \text{if } i \in T, \\ 0, & \text{otherwise.} \end{cases}$$

### 2.3. Probabilistic communication situations

A *probabilistic communication situation* is a triple $(N, v, p)$ where $(N, v)$ is a cooperative game and $p : \mathcal{L} \to [0, 1]$ is a probabilistic function which assigns to each pair of players a probability of direct communication between them. The probabilities are deemed to be independent. The set of probabilistic links is defined by $supp(p) = \big\{\{i, j\}|p(\{i, j\}) > 0, i, j \in N\big\}$. The *image* of $p$ is $im(p) = \Big\{p\big(\{i, j\}\big)|p\big(\{i, j\}\big) \neq 0\Big\}$. We denote the restriction to any $S \subseteq N$ of the function $p$ by $p_S$ with

$$p_S\big(\{i, j\}\big) = \begin{cases} p\big(\{i, j\}\big), & \text{if } i, j \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Calvo et al. [5] defined a restricted game $v^p$ associated to any probabilistic communication situation integrating both the economic possibilities of players described by the cooperative game $v$ and the possibilities of bilateral communications described by the probabilistic function $p$. Given $S \subseteq N$ and let $\mathcal{L}(S) = \big\{\{i, j\}|i, j \in S, i \neq j\big\}$. The possibility that each link set $L \subseteq \mathcal{L}(S)$ is realized among players in $S$ is

$$p^S(L) = \prod_{l \in L} p_l \prod_{l \in \mathcal{L}(S) \setminus L} (1 - p_l).$$

Since the worth achieved by $S \subseteq N$ if $L \subseteq \mathcal{L}(S)$ is realized is $v^L(S)$, the expected profit of coalition $S$ can be defined by

$$v^p(S) = \sum_{L \subseteq \mathcal{L}(S)} p^S(L) v^L(S).$$

## 3. PROBABILISTIC COMMUNICATION SITUATIONS WITH FUZZY COALITION

In this section, as an extended model of probabilistic communication situation, a probabilistic communication situation with fuzzy coalition is introduced considering the uncertainty of participation levels of players.

A *probabilistic communication structure with fuzzy coalition* can be described by a pair $(U, p)$ where $U = (U_1, U_2, \dots, U_n)$ is a fuzzy coalition with the $i_{th}$ coordinate being the real participation level of player $i$ in a cooperation and $p : \mathcal{L} \to [0, 1]$ is a function with $p\big(\{i, j\}\big)$ being the probability of the formation of communication link $\{i, j\}$ for all $i, j \in N$. A greater probability of one communication link says that it is better recognized by its two endpoints. The *basic graph* of $(U, p)$ is a crisp graph defined by $supp(U, p) = \big(supp(U), supp(p)\big)$. The concepts of a probabilistic communication structure with fuzzy coalition such as "path", "cycle" and "component" coincide with its corresponding basic graph. We denote by $C^p(U)$ the set of components of $(U, p)$. The *restriction* to a subset $S \subseteq N$ of $(U, p)$ is another probabilistic communication structure with fuzzy coalition $(U_S, p_S)$.

Given a cycle-free probabilistic communication structure with fuzzy coalition $(U, p)$ and any $K \in C^p(U)$, every player $i \in K$ induces a unique directed tree $\bar{T}(i)$ with $i$ being the root in *supp*$(U, p)$. With respect to $\bar{T}(i)$ for any $i \in K$, we define $S_i^p(j) = S_{\bar{T}(i)}(j)$ and $I_i^p(j) = I_{\bar{T}(i)}(j)$ for any $j \in K$.

Due to for players in a path, their communication relationship is obviously more stable if the minimal formation probability among them is greater than other communication possibilities, we define a concept of degree of path and take the maximum degree of all paths between any two players as their degree of connectivity.

**Definition 3.1.** Let $(U, p)$ be connected. If there exists a path $P$ connecting $i$ and $j$, the *degree of path* $P$ is defined by

$$d(P) = \min_{e \in E(P)} p(e),$$

where $E(P)$ is the set of all links involved in path $P$. Assume that there are $l$ paths $P_1, P_2, \ldots, P_l$ connecting $i$ and $j$ in $(U, p)$, the *degree of connectivity* $d(i, j)$ between $i$ and $j$ is defined by

$$d(i, j) = \max_{k \in \{1,2,\ldots,l\}} d(P_k), \quad \text{if } i \neq j.$$

Since one link with a lower probability is easier to destroy the communication among its two endpoints, it is very possible to be abandoned in the cooperation. Here we define one of paths from $i$ and $j$ whose degree is exactly the connectivity degree between $i$ and $j$ to be optimal as follows.

**Definition 3.2.** A path $P$ of $(U, p)$ connecting $i$ and $j$ is called the *optimal path* if

$$d(P) = d(i, j).$$

In an optimal path from $i$ to $j$, all of players involved in it are rational to choose their preferable partners. Apparently, the optimal paths are not always unique.

**Definition 3.3.** Let $(U, p)$ be connected. If $T = (V, E)$ is a spanning tree of *supp*$(U, p)$, $\tilde{T} = (U, p_E)$ is called a *spanning tree* of $(U, p)$ where $p_E : \mathcal{L} \to [0, 1]$ is a function with

$$(p_E)\big(\{i, j\}\big) = \begin{cases} p\big(\{i, j\}\big), & \text{if } \{i, j\} \in E, \\ 0, & \text{if } \{i, j\} \notin E. \end{cases}$$

If there is no confusion, we still write $\tilde{T}$ by $T$.

Let $(U, p)$ be connected. If $T$ is a spanning tree of $(U, p)$, then $T$ can be divided into two connected parts after deleting one probabilistic link $e$ in it, denoted by $T_1$ and $T_2$. Let $V(T_1)$ be the set of nodes in $T_1$ and $V(T_2)$ in $T_2$, we denote

$$p^*(T, e) = \big\{\{i, j\} | i \in V(T_1), j \in V(T_2), \{i, j\} \in supp(p)\big\}.$$

**Definition 3.4.** Denote the set of all spanning trees in the connected $(U,p)$ by $T(U,p)$, we say a spanning tree $T^* = (U, p_{E^*})$ is the *maximal product spanning tree* of $(U,p)$ if for any $T = (U, p_E) \in T(U,p)$,

$$\prod_{e\in E} p(e) \leq \prod_{e\in E^*} p(e).$$

A maximal product spanning tree guarantees the communications among all players with the maximal possibility, i. e., all players choose the partners they prefer to cooperate. The set of maximal product spanning trees in a connected $(U,p)$ is denoted by $T^m(U,p)$.

**Lemma 3.5.** Let $(U,p)$ be connected. If $T$ is a spanning tree of $(U,p)$, the following two statements are equivalent.

*(i)* $T$ is the maximal product spanning tree of $(U,p)$;

*(ii)* For all $i,j \in supp(U)$ $(i \neq j)$, the degree of connectivity $d(i,j)$ between $i$ and $j$ is equal to the degree of the unique path $P$ connecting $i$ and $j$ in $T$.

Proof. (i)⇒(ii) For any $i,j \in supp(U)$, let $P$ be a unique path connecting $i$ and $j$ in the spanning tree $T$. Assume that $p(e') = d(P)$, $e' \in E(P)$. Since $T$ is a maximal product spanning tree of $(U,p)$ and $e'$ is a cut edge of $T$, we can obtain that

$$p(e') = \max_{e\in p^*(T,e')} p(e). \tag{1}$$

Meanwhile, on any path $P^*$ connecting $i$ and $j$, there must exist $e^* \in p^*(T,e')$ such that $p(e') \geq p(e^*)$. Then,

$$d(P^*) \leq p(e^*) \leq p(e') = d(P)$$

and further

$$d(i,j) = d(P).$$

(ii)⇒(i) Let $T^*$ be a maximal product spanning tree of $(U,p)$, we assume that the spanning tree $T \neq T^*$ and only these edges $e_1, e_2, \ldots, e_k (k \geq 1)$ are in $T^*$, but not in $T$. Consider $e_k = \{i_k, j_k\}$, in the unique path $P_k$ of $T$ connecting $i_k$ and $j_k$, there must exist $e_k^* \in E(P_k)$ such that $e_k^* \in p^*(T^*, e_k)$ and therefore $p(e_k^*) \leq p(e_k)$ from the equation (1). Due to the result of (i)⇒(ii), we get that $p(e_k) = d(i_k, j_k)$ since $T^*$ is a maximal product spanning tree. In addition, according to the condition that for all $i,j \in supp(U)$ $(i \neq j)$, the degree of connectivity $d(i,j)$ between $i$ and $j$ is equal to the degree of the unique path $P$ connecting $i$ and $j$ in $T$, it is derived that $d(i_k, j_k) = d(P_k) \leq p(e_k^*)$. Hence, $p(e_k^*) = p(e_k)$. Replacing $e_k$ by $e_k^*$, another spanning tree $T_1^*$ is generated and then $\prod_{e\in E(T^*)} p(e) = \prod_{e\in E(T_1^*)} p(e)$ where $E(T^*)$ is the set of edges in $T^*$ and $E(T_1^*)$ in $T_1^*$. Those edges in $T_1^*$ but not in $T$ are $e_1, e_2, \ldots, e_{k-1}$. Repeat this step, we can obtain $T$ after $k$ steps such that $\prod_{e\in E(T^*)} p(e) = \prod_{e\in E(T)} p(e)$, namely, $T$ is also the maximal product spanning tree of $(U,p)$. □

**Remark 3.6.** It is worthwhile to mention that, if the maximal product spanning tree is substituted for the maximal spanning tree which is defined to be that of possessing the maximal sum of probability of each link, the Lemma 3.5 is still true by a similar proof. However, in this paper we use the product of probabilities of communication links, namely, the probability that all of these links are formed, to represent the stability of the generated communication structure because the sum of probabilities of links refers to the probability that at least one of these links is formed.

The Lemma 3.5 indicates that the unique path between any two nodes in a maximal product spanning tree is exactly an optimal path and provides us a procedure to find the maximal product spanning trees as below.

Suppose that $(U,p)$ is connected, $\big|supp(U)\big| = q$.

(1) Take any element in $supp(U)$ as $i_1$. Put $A_1 = \{i_1\}$, $B_1 = supp(U)\backslash A_1$. If there exists $j^* \in B_1$ such that $p\big(\{i_1, j^*\}\big) = \max_{j\in B_1} p\big(\{i_1, j\}\big)$, then let $i_2 = j^*$ and $e_1 = \{i_1, i_2\}$ (If $j^*$ is not unique, one of them can be recorded as $i_2$). Denote $A_2 = A_1 \cup \{i_2\}$, $E_1 = \{e_1\}$.

(2) If $A_k = \{i_1, i_2, \ldots, i_k\}$, $E_{k-1} = \{e_1, e_2, \ldots, e_{k-1}\}$ $(k < q)$, take $B_k = supp(U)\backslash A_k$ and denote

$$E^* = \big\{\{i,j\} | i \in A_k, j \in B_k\big\},$$

$$p\big(\{i^*, j^*\}\big) = \max_{\{i,j\}\in E^*} p\big(\{i,j\}\big).$$

Let $i_{k+1} = j^*$, $e_k = \{i^*, i_{k+1}\}$ (If $j^*$ is not unique, take one of them as $i_{k+1}$). We put $A_{k+1} = A_k \cup \{i_{k+1}\}$, $E_k = E_{k-1} \cup \{e_k\}$.

(3) If $k+1 < q$, let $k = k+1$ and go to the step (2); otherwise if $k+1 = q$, the procedure ends and $T = (U, p_{E_{q-1}})$ is the maximal product spanning tree of $(U,p)$.

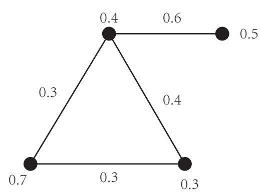


**Fig. 1.** $(U,p)$.

**Example 3.7.** A probabilistic communication structure with fuzzy coalition is depicted in Figure 1, we can easily find its maximal product spanning trees listed in Figure 2.


(a) $(U, \bar{p})$ (b) $(U, \bar{\bar{p}})$

**Fig. 2.** Maximal product spanning trees of $(U, p)$.

## 4. A BI-AVERAGE TREE SOLUTION

A probabilistic communication situation with fuzzy coalition is a triple $(U, v, p)$, where $(U, p)$ is a probabilistic communication structure with fuzzy coalition and $v \in G^N$. The set of probabilistic communication situations with fuzzy coalition is denoted by $PF^N$.

Choquet integrals [12] are used to obtain fuzzy games from their crisp counterparts. The domain of fuzzy games is $\mathcal{F}^N$. For a fuzzy game $\bar{v}$ with the fixed fuzzy coalition $U$, the worth of fuzzy coalition $U_S$ is $\bar{v}(U_S)$. If we regard $\bar{v}(U_S)$ as the worth of crisp coalition $S$ with the participation levels of players in it being $U_i (i \in S)$, $\bar{v}$ with fuzzy coalition $U$ can be transformed into a crisp game with respect to $U$ whose domain is $2^N$, denoted by $v^U$, where $v^U(S) = \bar{v}(U_S)$ for any $S \subseteq N$. Similarly, in the following we define a restricted crisp game $v^{Up}$ on the domain $2^N$ which is essentially the transformation of a fuzzy game $\bar{v^p}$ with fuzzy coalition $U$ and $v^{Up}(S) = \bar{v^p}(U_S)$ for any $S \subseteq N$.

**Definition 4.1.** Given $(U, v, p) \in PF^N$, a *restricted game* in Choquet integral form $v^{Up}$ will be defined by

$$v^{Up}(S) = \sum_{k=1}^{q(U)} [h_k - h_{k-1}] \sum_{L \subseteq \mathcal{L}(S)} p^S(L) v^L([U]_{h_k} \cap S), \quad \text{for any } S \subseteq N,$$

where $Q(U) \cup \{0\} = \{h_0, h_1, h_2, \dots, h_{q(U)}\}$ and $0 = h_0 < h_1 < h_2 < \cdots < h_{q(U)}$.

The restricted game implies that, all players for a probabilistic communication situation with fuzzy coalition take a Choquet behavior, i. e., every of them try their best to pursue the biggest feasible coalition to cooperate.

**Remark 4.2.** If $L \subseteq \mathcal{L}(S)$ is realized, the worth of $U_S$ is $\sum_{k=1}^{q(U)} [h_k - h_{k-1}] v^L([U]_{h_k} \cap S)$. Then, $v^{Up}(S)$ can be seen as the expectation of the worth of $U_S$, that is, $v^{Up}(S) = \sum_{L \subseteq \mathcal{L}(S)} p^S(L) \sum_{k=1}^{q(U)} [h_k - h_{k-1}] v^L([U]_{h_k} \cap S)$. When $U = e^N$, the restricted game $v^{Up}$ is degenerated to $v^p$ having been defined in Calvo et al. [5].

A solution for probabilistic communication situations with fuzzy coalition is a function $\Psi : PF^N \to \mathbb{R}^n$. The real number $\Psi(U, v, p)$ is the final payoff vector in the game $v$ when the communications between players are restricted by $(U, p)$. Now we focus on how to reasonably divide the profit among the players for this situation.

In this paper, we tacitly admit that every player is equally possible to be a leader and tends to choose a optimal channel when they communicate with others. Based on this consideration, we introduce a bi-average tree solution on the class of probabilistic communication situations with fuzzy coalition.

**Definition 4.3.** A *bi-average tree solution* for any $(U,v,p) \in PF^N$ is defined to be

$$BAT_i(U,v,p) = \frac{1}{\left|T^m(U_K,p_K)\right|} \sum_{(\bar{U},\bar{p})\in T^m(U_K,p_K)} \frac{1}{\left|supp(\bar{U})\right|} \cdot \sum_{j\in supp(\bar{U})} \left[v^{\bar{U}\bar{p}}\big(S_j^{\bar{p}}(i)\big) - \sum_{k\in I_j^{\bar{p}}(i)} v^{\bar{U}\bar{p}}\big(S_j^{\bar{p}}(k)\big)\right]$$

if $i \in K$, $K \in C^p(U)$, and $BAT_i(U,v,p) = 0$ if $i \notin supp(U)$.

**Remark 4.4.** Note that $\bar{U} = U_K$, and any $(\bar{U},\bar{p}) \in T^m(U_K,p_K)$ can be found according to the previous Definition 3.3 and 3.4. The Definition 4.3 builds a relation between the bi-average tree solution and the classical average tree solution, i. e., for any $K \in C^p(U)$ and $i \in K$,

$$BAT_i(U,v,p) = \frac{1}{\left|T^m(U_K,p_K)\right|} \sum_{(\bar{U},\bar{p})\in T^m(U_K,p_K)} AT_i\big(supp(\bar{U}), v^{\bar{U}\bar{p}}, supp(\bar{p})\big),$$

and further $BAT_i(U,v,p) = AT_i\big(supp(U_K), v^{Up}, supp(p_K)\big)$ if $(U,p)$ is cycle-free.

Let $\Psi$ be a solution for probabilistic communication situations with fuzzy coalition. We consider some properties described below and aim to characterize the bi-average tree solution.

**Bi-average tree property** For any $(U,v,p) \in PF^N$ and any $K \in C^p(U)$, it holds that

$$\Psi_i(U,v,p) = \frac{1}{\left|T^m(U_K,p_K)\right|} \sum_{(\bar{U},\bar{p})\in T^m(U_K,p_K)} \Psi_i(\bar{U},v,\bar{p}), \quad \text{for any } i \in K.$$

This property says that the payoff to each player $i \in K$ for any $(U,v,p) \in PF^N$ and any $K \in C^p(U)$ is an average of its payoffs associated to all maximal product spanning trees of $(U_K,p_K)$.

**Cycle-free component efficiency** For any $(U,v,p) \in PF^N$ with $(U,p)$ being cycle-free and any $K \in C^p(U)$,

$$\sum_{i\in K} \Psi_i(U,v,p) = v^{Up}(K)$$

and $\Phi_i(U,v,p) = 0$ if $i \notin supp(U)$.

Cycle-free component efficiency states that a solution for a cycle-free probabilistic communication situation with fuzzy coalition $(U,v,p)$ assigns to any component $K$ a payoff $v^{Up}(K)$.

**Cycle-free component fairness** For any $(U, v, p) \in PF^N$ with $(U, p)$ being cycle-free and $\{i, j\} \in supp(p)$,

$$\begin{aligned}&\frac{1}{|K^i|}\sum_{h\in K^i}\big[\Psi_h(U,v,p)-\Psi_h(U,v,p_{-\{i,j\}})\big]\\ =&\frac{1}{|K^j|}\sum_{h\in K^j}\big[\Psi_h(U,v,p)-\Psi_h(U,v,p_{-\{i,j\}})\big],\end{aligned}$$

where $K^i$ and $K^j$ are the components of $(U, p_{-\{i,j\}})$ containing $i$ and $j$ respectively, and $p_{-\{i,j\}}$ is a probabilistic function with

$$p_{-\{i,j\}}\big(\{e,f\}\big)=\begin{cases}p\big(\{e,f\}\big), & \text{if } \{e,f\}\neq\{i,j\},\\ 0, & \text{if } \{e,f\}=\{i,j\}.\end{cases}$$

Cycle-free component fairness requires that deleting a probabilistic link $\{i, j\}$ yields the same average loss of payoff in both $K^i$ and $K^j$.

**Theorem 4.5.** The bi-average tree solution is the unique solution over $PF^N$ if it satisfies the bi-average tree property, cycle-free component efficiency and cycle-free component fairness.

P r o o f. First, we prove that the bi-average tree solution satisfies the properties in this theorem.
*Bi-average tree property.* Since for any $K \in C^p(U)$, $i \in K$,

$$\begin{aligned}BAT_i(U,v,p)=&\frac{1}{\big|T^m(U_K,p_K)\big|}\sum_{(\bar U,\bar p)\in T^m(U_K,p_K)}\frac{1}{\big|supp(\bar U)\big|}\\ &\sum_{j\in supp(\bar U)}\Big[v^{\bar U\bar p}\big(S_j^{\bar p}(i)\big)-\sum_{k\in I_j^{\bar p}(i)}v^{\bar U\bar p}\big(S_j^{\bar p}(k)\big)\Big],\end{aligned}$$

and for any $(\bar U, \bar p) \in T^m(U_K, p_K)$,

$$BAT_i(\bar U,v,\bar p)=\frac{1}{\big|supp(\bar U)\big|}\sum_{j\in supp(\bar U)}\Big[v^{\bar U\bar p}\big(S_j^{\bar p}(i)\big)-\sum_{k\in I_j^{\bar p}(i)}v^{\bar U\bar p}\big(S_j^{\bar p}(k)\big)\Big],$$

it is clear that

$$BAT_i(U,v,p)=\frac{1}{\big|T^m(U_K,p_K)\big|}\sum_{(\bar U,\bar p)\in T^m(U_K,p_K)}BAT_i(\bar U,v,\bar p),\quad \text{for any } i\in K.$$

*Cycle-free component efficiency.* Given $(U, v, p) \in PF^N$ with $(U, p)$ being cycle-free and $K \in C^p(U)$, it is derived that

$$\sum_{i\in K}BAT_i(U,v,p)=\sum_{i\in K}AT_i\big(supp(U),v^{Up},supp(p)\big)=v^{Up}(K)$$

following from the efficiency in Herings et al. [8] of the average tree solution and

$$BAT_i(U,v,p) = \frac{1}{\big|T^m(U_K,p_K)\big|} \sum_{(\bar{U},\bar{p})\in T^m(U_K,p_K)} AT_i\big(supp(\bar{U}), v^{\bar{U}\bar{p}}, supp(\bar{p})\big).$$

*Cycle-free component fairness.* For any $(U,v,p) \in PF^N$ with $(U,p)$ being cycle-free and any $K \in C^p(U)$, let the restriction to $K$ of $supp(p)$ be denoted by $supp(p)(K)$. Assume that $supp(p)(K) = L$, since if $r \in K$,

$$\sum_{h\in K} \Big[v^{Up}\big(S_r^L(h)\big) - \sum_{k\in I_r^L(h)} v^{Up}\big(S_r^L(k)\big)\Big] = v^{Up}(K)$$

holds, we have that if $r \in K^j$,

$$\sum_{h\in K^i} \Big[v^{Up}\big(S_r^L(h)\big) - \sum_{k\in I_r^L(h)} v^{Up}\big(S_r^L(k)\big)\Big] = v^{Up}(K^i),$$

and if $r \in K^i$,

$$\sum_{h\in K^i} \Big[v^{Up}\big(S_r^L(h)\big) - \sum_{k\in I_r^L(h)} v^{Up}\big(S_r^L(k)\big)\Big] = v^{Up}(K) - v^{Up}(K^j).$$

Then,

$$\begin{aligned}
&\sum_{h\in K^i} AT_h\big(K, v^{Up}, supp(p)(K)\big)\\
=&\frac{1}{|K|}\sum_{r\in K}\sum_{h\in K^i} \Big[v^{Up}\big(S_r^L(h)\big) - \sum_{k\in I_r^L(h)} v^{Up}\big(S_r^L(k)\big)\Big]\\
=&\frac{1}{|K|}\Big[|K^j|v^{Up}(K^i) + |K^i|\big(v^{Up}(K) - v^{Up}(K^j)\big)\Big].
\end{aligned}$$

Because for any $h \in K$, $BAT_h(U,v,p) = AT_h\big(K, v^{Up}, supp(p)(K)\big)$, for any $h \in K^i$, $BAT_h(U,v,p_{-\{i,j\}}) = AT_h\big(K^i, v^{Up}, supp(p)(K^i)\big)$ and for any $h \in K^j$, $BAT_h(U,v,p_{-\{i,j\}}) = AT_h\big(K^j, v^{Up}, supp(p)(K^j)\big)$, together with the efficiency of the average tree solution and $|K^i| + |K^j| = |K|$, we derive that

$$\begin{aligned}
&\sum_{h\in K^i} \big[BAT_h(U,v,p) - BAT_h(U,v,p_{-\{i,j\}})\big]\\
=&\sum_{h\in K^i} \Big[AT_h\big(K, v^{Up}, supp(p)(K)\big) - AT_h\big(K^i, v^{Up}, supp(p)(K^i)\big)\Big]\\
=&\frac{1}{|K|}\Big[|K^j|v^{Up}(K^i) + |K^i|\big(v^{Up}(K) - v^{Up}(K^j)\big)\Big] - v^{Up}(K^i)\\
=&\frac{|K^i|}{|K|}\big[v^{Up}(K) - v^{Up}(K^j) - v^{Up}(K^i)\big].
\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{|K^i|}\sum_{h\in K^i}\big[BAT_h(U,v,p)-BAT_h(U,v,p_{-\{i,j\}})\big]\\=&\frac{1}{|K^j|}\sum_{h\in K^j}\big[BAT_h(U,v,p)-BAT_h(U,v,p_{-\{i,j\}})\big].\end{aligned}$$

To prove uniqueness, we suppose that $\Psi$ satisfies the bi-average tree property, cycle-free component efficiency and cycle-free component fairness. In fact, due to the bi-average tree property and cycle-free component efficiency, we only need to check the uniqueness of $\Psi_i(U,v,p)$ for any $(U,v,p)\in PF^N$ when $i\in supp(U)$ and $(U,p)$ is cycle-free.

Let $(U,p)$ be cycle-free. The cycle-free component efficiency implies that for any $K\in C^p(U)$,

$$\sum_{h\in K}\Psi_h(U,v,p)=v^{Up}(K). \tag{2}$$

Also, from the cycle-free component efficiency, we derive that

$$\sum_{h\in K^i}\Psi_h(U,v,p_{-\{i,j\}})=v^{Up}(K^i)\ \text{ and }\ \sum_{h\in K^j}\Psi_h(U,v,p_{-\{i,j\}})=v^{Up}(K^j).$$

Therefore, the cycle-free component fairness can be changed into

$$\frac{1}{|K^i|}\big[\sum_{h\in K^i}\Psi_h(U,v,p)-v^{Up}(K^i)\big]=\frac{1}{|K^j|}\big[\sum_{h\in K^j}\Psi_h(U,v,p)-v^{Up}(K^j)\big]. \tag{3}$$

Since in total there are $|supp(U)|$ equations of type (2) and (3) with $|supp(U)|$ variables and all the equations are linearly independent, $\Psi(U,v,p)$ is uniquely determined. □

Now we show that these three properties mentioned above are independent.

1. The solution $\Psi^1$ defined, for every $(U,v,p)\in PF^N$, by

   $$\Psi^1(U,v,p)=\begin{cases}BAT(U,v,p), & \text{if } (U,p) \text{ is cycle-free},\\ 0, & \text{otherwise}\end{cases}$$

   satisfies cycle-free component efficiency and cycle-free component fairness, but not bi-average tree property.

2. The solution $\Psi^2$ defined, for every $(U,v,p)\in PF^N$, by $\Psi^2(U,v,p)=0$ satisfies bi-average tree property and cycle-free component fairness, but not cycle-free component efficiency.

3. The solution $\Psi^3$ defined, for every $(U,v,p)\in PF^N$ any $K\in C^p(U)$, by

   $$\Psi^3_i(U,v,p)=\frac{1}{\big|T^m(U_K,p_K)\big|}\sum_{(\bar U,\bar p)\in T^m(U_K,p_K)}\frac{v^{\bar U\bar p}(K)}{|K|}$$

   for any $i\in K$ satisfies bi-average tree property and cycle-free component efficiency, but not cycle-free component fairness.

**Remark 4.6.** Jimenez-Losada et al. [9] introduced the fuzzy communication situation, which supposed that the capacity of the players or their communication can be uncertain, and defined the fuzzy Myerson value. Assuming $p(e)$ is a fuzzy degree of edge $e$. The main differences between the bi-average tree solution and the fuzzy Myerson value are that:

1. The fuzzy Myerson value is defined over the fuzzy communication structures where the communication level between any two players is limited to be not greater than their own participation level.

2. The bi-average tree solution focuses on the maximal product spanning trees of the probabilistic communication structure with fuzzy coalition, where the unique path between two players is optimal. However, the fuzzy Myerson value considered all possible communication channels among players.

3. When the probabilistic communication structure with fuzzy coalition is cycle-free, the bi-average tree solution has the component fairness which says that deleting a link between two players yields for both resulting components the same average change in payoff, where the average is taken over the players in the component. However, the fuzzy Myerson value has the fairness which says that the loss of one bilateral communication implies the same loss of payment for the players involved in this link.

Compared to the characterizations of probabilistic Myerson values for probabilistic communication situations studied by Calvo et al. [5] and the average tree solutions for cycle-free communication situations in [8], the characterization of the bi-average tree solutions has two important differences.

1. The probabilistic Myerson value in [5] has the fairness, that is, the loss of one bilateral communication implies the same loss of payment for the players involved in this link. However, the bi-average tree solution for cycle-free case has the component fairness, that is, deleting a probabilistic link between two players yields for both resulting components the same average change in payoff, where the average is taken over the players in the component.

2. The bi-average tree solution is defined on general probabilistic communication situations with fuzzy coalition which only focuses on the induced communication structures with the strongest stability, namely, the maximal product spanning trees and has the bi-average tree property. However, the average tree solution is defined for cycle-free communication situations and has no this property.

**Proposition 4.7.** Given $(U, v, p) \in PF^N$ with $(U, p)$ being cycle-free. If $U = e^N$,

*(i)* $BAT(U, v, p) = \sum_{L \subseteq \mathcal{L}} p^N(L) AT(N, v, L)$;

*(ii)* When $v$ is superadditive, $BAT(U, v, p) \in C(v^p)$.

Proof. (i) For any $(U, v, p) \in PF^N$ with $(U, p)$ being cycle-free and $U = e^N$, it is apparent that the bi-average tree solution can be axiomatized by cycle-free component efficiency and cycle-free component fairness. Put $f(U, v, p) = \sum_{L \subseteq \mathcal{L}} p^N(L) AT(N, v, L)$, this part is verified true if we can prove that $f(U,v,p)$ satisfies the cycle-free component efficiency and cycle-free component fairness. The final result that $BAT(U,v,p) = \sum_{L\subseteq\mathcal{L}} p^N(L)AT(N,v,L)$ can be seen as a expected value of $AT(N,v,L)$ over all possible realized $L \subseteq \mathcal{L}$ among players $N$.

(ii) From Herings et al. [8] we know that $AT(N,v,L) \in C(v^L)$, namely, $\sum_{i\in N} AT_i(N,v,L) = v^L(N)$ and $\sum_{i\in S} AT_i(N,v,L) \geq v^L(S)(\forall S \subseteq N)$ for any cycle-free deterministic communication situation $(N,v,L) \in CS^N$ if $v$ is superadditive. This implies $BAT(U,v,p) \in C(v^p)$ following the fact that

$$\begin{aligned}
\sum_{i\in N} BAT_i(U,v,p) &= \sum_{i\in N}\sum_{L\subseteq\mathcal{L}} p^N(L)AT_i(N,v,L)\\
&= \sum_{L\subseteq\mathcal{L}} p^N(L)\sum_{i\in N} AT_i(N,v,L)\\
&= \sum_{L\subseteq\mathcal{L}} p^N(L)v^L(N)\\
&= v^p(N),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i\in S} BAT_i(U,v,p) &= \sum_{i\in S}\sum_{L\subseteq\mathcal{L}} p^N(L)AT_i(N,v,L)\\
&= \sum_{L\subseteq\mathcal{L}} p^N(L)\sum_{i\in S} AT_i(N,v,L)\\
&\geq \sum_{L\subseteq\mathcal{L}} p^N(L)v^L(S) = v^p(S).
\end{aligned}$$

The last equality holds because

$$\begin{aligned}
&\sum_{L\subseteq\mathcal{L}} p^N(L)v^L(S)\\
=\ &\sum_{L\subseteq\mathcal{L}}\prod_{l\in L} p_l \prod_{l\in\mathcal{L}\setminus L}(1-p_l)v^L(S)\\
=\ &\sum_{A\subseteq\mathcal{L}(S),B\subseteq\mathcal{L}\setminus\mathcal{L}(S)}\prod_{l\in A} p_l \prod_{l\in\mathcal{L}(S)\setminus A}(1-p_l)\frac{\prod_{l\in B} p_l \prod_{l\in\mathcal{L}\setminus\mathcal{L}(S)}(1-p_l)}{\prod_{l\in B}(1-p_l)}v^{A\cup B}(S)\\
=\ &\sum_{A\subseteq\mathcal{L}(S)}\prod_{l\in A} p_l \prod_{l\in\mathcal{L}(S)\setminus A}(1-p_l)\sum_{B\subseteq\mathcal{L}\setminus\mathcal{L}(S)}\frac{\prod_{l\in B} p_l \prod_{l\in\mathcal{L}\setminus\mathcal{L}(S)}(1-p_l)}{\prod_{l\in B}(1-p_l)}v^{A}(S)\\
=\ &\sum_{A\subseteq\mathcal{L}(S)}\prod_{l\in A} p_l \prod_{l\in\mathcal{L}(S)\setminus A}(1-p_l)\sum_{B\subseteq\mathcal{L}\setminus\mathcal{L}(S)}\prod_{l\in B} p_l \prod_{l\in(\mathcal{L}\setminus\mathcal{L}(S))\setminus B}(1-p_l)v^{A}(S)\\
=\ &\sum_{A\subseteq\mathcal{L}(S)}\prod_{l\in A} p_l \prod_{l\in\mathcal{L}(S)\setminus A}(1-p_l)v^{A}(S)\\
=\ &v^p(S).
\end{aligned}$$

□

**Proposition 4.8.** Given $(U,v,p) \in PF^N$ with $(U,p)$ being cycle-free. If $v$ is superadditive and $im(p) = \{1\}$, assume that $supp(p) = L$ and define a fuzzy cooperative game $(U,\tilde{v^L})$ with $\tilde{v^L}(U_S) = v^{Up}(S)$ for any $S \subseteq N$, then $BAT(U,v,L) \in \tilde{C}(\tilde{v^L})(U)$.

Proof. Under the condition of this proposition, we know that $AT(N,v^{L([U]_{h_k})},L) \in C(v^{L([U]_{h_k})})$ from Herings et al. [8], i.e., $\sum_{i\in N} AT_i(N,v^{L([U]_{h_k})},L) = v^{L([U]_{h_k})}(N)$ and $\sum_{i\in S} AT_i(N,v^{L([U]_{h_k})},L) \geq v^{L([U]_{h_k})}(S)$. Then, we have that

$$\begin{aligned}\sum_{i\in N} BAT_i(U,v,p) &= \sum_{k=1}^{q(U)}[h_k-h_{k-1}]\sum_{i\in N} AT_i(N, v^{L([U]_{h_k})}, L)\\ &= \sum_{k=1}^{q(U)}[h_k-h_{k-1}]v^{L([U]_{h_k})}(N) = v^{Up}(N) = \tilde{v^L}(U)\end{aligned}$$

and for any $T \subseteq N$,

$$\begin{aligned}\sum_{i\in supp(U_T)} BAT_i(U,v,p) &= \sum_{k=1}^{q(U)}[h_k-h_{k-1}]\sum_{i\in supp(U_T)} AT_i(N, v^{L([U]_{h_k})}, L)\\ &\geq \sum_{k=1}^{q(U)}[h_k-h_{k-1}]v^{L([U]_{h_k})}\big(supp(U_T)\big)\\ &= v^{Up}(T) = \tilde{v^L}(U_T).\end{aligned}$$

Therefore, $BAT(U,v,L) \in \tilde{C}(\tilde{v^L})(U)$. □

**Example 4.9.** Suppose that there are four cities $N = \{1,2,3,4\}$ ready to transport natural gas by pipelines to benefit the local residents. Certainly, at the same period there may also have other issues for each city to contribute a part of their abilities and resources, so the true levels of their participation in this project cooperation about natural gas are less that 1. Let we assume that they are 0.4, 0.7, 0.3 and 0.5 respectively.

Now, for the purpose of reducing the construction costs of pipelines, these four cities plan to negotiate together about the joint cooperation. Let $p : \mathcal{L}(N) \to [0,1]$ be a probabilistic function with $p(\{i,j\})$ being the probability that cities $i$ and $j$ prefer to reach an agreement on building a pipeline through them, where $p\big(\{1,2\}\big) = p\big(\{2,3\}\big) = 0.3$, $p\big(\{1,3\}\big) = 0.4$, $p\big(\{1,4\}\big) = 0.6$ and $p\big(\{2,4\}\big) = p\big(\{3,4\}\big) = 0$. This scenario can be described by a probabilistic communication structure with fuzzy coalition $(U,p)$ as listed in Figure 1. In order to try to promote cooperation each other, every of cities is rational to select the strongest connection channel to communicate with others and then the final potential network structures, namely the maximal product spanning trees $(U,\bar{p})$ and $(U,\bar{\bar{p}})$ of $(U,p)$ shown in Figure 2 are builded.

Given a cooperative game $v \in G^N$ with the characteristic function $v(S) = 10000|S|$ for any $S \subseteq N$ representing the cost savings of cities that would result from cooperation between the cities of $S$ instead of acting alone if they contribute all their capacities. For this probabilistic communication situation with fuzzy coalition $(U,v,p)$, we are devoted to allocate the cost savings among four cities. First, we calculate the restricted game $v^{U\bar{p}}$ corresponding to $(U,\bar{p})$,

$$v^{U\bar{p}}\big(\{1,2,3\}\big) = 4440,\quad v^{U\bar{p}}\big(\{1,2,4\}\big) = 6840,$$

$$v^{U\bar{p}}\big(\{1,3,4\}\big) = 6480,\quad v^{U\bar{p}}\big(\{1,2,3,4\}\big) = 8016.$$

and $v^{U\bar{\bar{p}}}$ corresponding to $(U,\bar{\bar{p}})$,

$$v^{U\bar{\bar{p}}}\big(\{1,4\}\big) = 4800,\quad v^{U\bar{\bar{p}}}\big(\{2,3\}\big) = 1800,$$

$$v^{U\bar{\bar{p}}}\big(\{1,2,3\}\big) = 2880,\quad v^{U\bar{\bar{p}}}\big(\{1,3,4\}\big) = 6480,\quad v^{U\bar{\bar{p}}}\big(\{1,2,3,4\}\big) = 7500.$$

Eventually, since the network structure $(U,\bar{p})$ or $(U,\bar{\bar{p}})$ are formed with the equal possibility of $\frac{1}{2}$, it is derived that

$$BAT(U,v,\bar{p}) = (4095, 255, 1995, 1155),$$

$$BAT(U,v,\bar{\bar{p}}) = (6444, 384, 294, 1893)$$

and further, the allocation of savings for these four cities is

$$\begin{aligned} BAT(U,v,p) &= \frac{1}{2}\big[BAT(U,v,\bar{p}) + BAT(U,v,\bar{\bar{p}})\big] \\ &= (5269.5, 319.5, 1144.5, 1524). \end{aligned}$$

In this example, in order to save construction cost as much as possible while benefiting residents, four cities choose to build the shortest pipelines which cover all cities by considering the stability of generated communication structure among them. So, naturally the communication channels $(U,\bar{p})$ and $(U,\bar{\bar{p}})$ corresponding to the maximal spanning trees of $(U,p)$ are the optimal choice for four cities.

## 5. CONCLUSIONS

Compared to the previous probabilistic communication situation proposed by Calvo et al. [5], in this paper we have considered a more general framework, called the probabilistic communication situation with fuzzy coalition, in which every of players is permitted to join in a cooperation with some participation level.

A maximal product spanning tree, reflecting the most likely way to communicate among the players, is introduced where the unique path from one player to another is optimal. We provide a feasible procedure to find the maximal product spanning trees. Assume that each player is rational enough to select an optimal path to communicate with others, a solution concept in terms of the average tree solution is given and axiomatized by defining a restricted game in Choquet integral form.

Our new model has showed its prospect in solving real allocation problems along with the uncertainty of players' participation degrees and mutual communication. Other solution concepts for probabilistic communication situations with fuzzy coalition still need further research.

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*Xianghui Li, Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi 710072. P. R. China.*
*e-mail: xianghuili@mail.nwpu.edu.cn*

*Hao Sun, Corresponding author, Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi 710072. P. R. China.*
*e-mail: hsun@nwpu.edu.cn*

*Dongshuang Hou, Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi 710072. P R. China.*
*e-mail: dshhou@126.com*